MATHEMATICS
PART - 1

Standard
X

Government of Kerala
Department of General Education

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THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders, respect and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone, lies my happiness.

MATHEMATICS

10

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Dear children,

We have seen in previous classes how measurements are converted to numbers and how problems about measurements are solved by converting them to number problems and then to algebraic problems. Thus we came to understand how relations between operations on pure numbers develop into algebraic principles, which in turn are used to solve problems about measurements. We also saw how special properties of various figures are turned into geometrical theorems.

In this textbook, we move a little more into these various avenues of mathematics. Here we get the first glimpses of how the usually separate disciplines of algebra and geometry merge into a new branch of mathematics. The basic ideas of probability theory, which is used in all sciences and which is an essential component of artificial intelligence used widely now.

The fundamental ideas needed for those who opt for mathematics as a part of their continuing education are all discussed in our textbooks. The inherent value of these books is the culture of mathematical thinking which everyone should be able to acquire as they finish one stage of their basic education. The essence of this culture is the ability to analyze everything logically and form conclusions based only on such reflections.

With love and regards,

Dr. Jayaprakash R.K.
Director
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CONTENTS

Certain icons are used in this textbook for convenience

Let's do problems

Project

ICT possibilities

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

---

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

1

ARITHMETIC SEQUENCES

Number patterns

We have seen many number patterns:

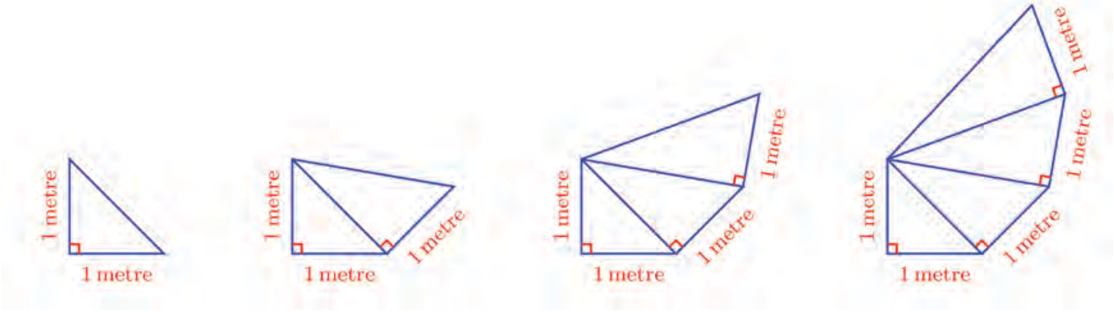
Natural numbers	1, 2, 3, ...
Even numbers	2, 4, 6, ...
Odd numbers	1, 3, 5, ...
Halves of natural numbers	$\frac{1}{2}, 1, 1\frac{1}{2}, \ldots$
Squares of natural numbers	1, 4, 9, ...

A collection of numbers, which are ordered as the first, second, third and so on like this, according to some rule, is called a **number sequence.**

Such sequences often arise in computing various measurements. For example, if we order polygons according to the number of sides, as triangles, quadrilaterals, pentagons and so on and calculate the sum of their inner angles, we get the number sequence

$$180, 360, 540, \ldots$$

As another example, look at these pictures:



If we write down the length of the hypotenuse of the largest triangle in each picture, in metres, the sequence we get is that of square roots of natural numbers from 2

$$\sqrt{2}, \sqrt{3}, 2, \ldots$$

The same sequence can be described in many ways. For example, the sequence of natural numbers ending in 1 is

$$1, 11, 21, 31, \ldots$$

It can also be described as the sequence of natural numbers which leave remainder 1 on division by 10.

All kinds of sequences

The English word 'sequence' comes from the Latin word *sequent*, which means 'following'. In mathematics, we use this word to denote things arranged in definite positions as the first, second, third and so on. The things so arranged can be any objects, not necessarily numbers. For example, here's the sequence of regular polygons:

We can also have a sequence of polynomials like this:

$$1 + x, 1 + x^2, 1 + x^3, \ldots$$

The alphabetical arrangement of words of a language is also a sequence.

(1) We can make triangles by stacking dots:

Write the number of dots in each triangle. Calculate the number of dots needed to make the next three triangles in this pattern.

(2) From the sequence equilateral triangle, square, regular pentagon and so on of regular polygons, form the following sequences

Using GeoGebra, we can draw the sequence of regular polygons with a specified side.

First mark two points A, B. Then type this command in the Input Bar:

```
Sequence[Polygon[A,B,n],n,3,10]
```

This command is the instruction to change the value of the numbers n from 3 to 10, and draw regular polygon of n sides with AB as one side

To draw the polygons one by one, create an integer slider m and change the command as below:

```
Sequence[Polygon[A,B,n+2],n,1,m]
```

This means to draw regular polygons of sides 3, 4, 5, ... and so on, with *AB* as a side, as we change the value of m through 1, 2, 3, ... and so on using the slider

If we change n + 2 to 2n in the above command, what sort of polygons would we get? What if we change it to 2n + 1?

Number of sides	3, 4, 5, 6, ...
Sum of inner angles	
Sum of outer angles	
An inner angle	
An outer angle	

(3) Write the sequence of natural numbers which leave remainder 1 on division by 3, and the sequence of natural numbers which leave remainder 2 on division by 3.

(4) Write in ascending order, the sequence of natural numbers with last digit 1 or 6. Describe this sequence in two other ways.

(5) See these figures:

The first picture shows an equilateral triangle with the smaller triangle got by joining the midpoints of sides cut off. The second picture shows the same thing done on each of the three triangles in the first picture. The third picture shows the same thing done on the second picture.

(i) How many red triangles are there in each picture?

(ii) Taking the area of whole uncut triangle as 1, compute the area of a small triangle in each picture.

(iii) What is the total area of all the red triangles in each picture?

(iv) Write the first five terms of each of the three sequences got by continuing this process.

The numbers in a sequence can be generated using GeoGebra. To get the first ten numbers of the sequence 1, 4, 9, ... of perfect squares, type the command below in the Input Bar:

```
Sequence[n^2,n,1,10]
```

If we create an integer slider m and issue the command

```
Sequence[n^2,n,1,m]
```

then the number of squares in the sequence changes as we change m. The algebra of the sequence 2, 4, 8, ... is 2^n. To get this type

```
Sequence(2^n,n,1,m)
```

Circle division

If we choose any two points on a circle and join them with a line, it divides the circle into two parts:

If we choose three points instead and join them with lines, they divide the circle into four parts:

What if choose four points and join every pair?

How about five points?

How many parts do you expect to get by joining six points? Check your guess by actually drawing a picture.

Arithmetic sequences

The sequence of even numbers goes like this:

2, 4, 6, ...

What is the next number in this?

And the next after?

In general, to go from one number in this sequence to the next, we need only add 2.

Now let's look at the sequence of odd numbers:

1, 3, 5, ...

In this also, we need only add 2 to go from one number to the next.

The difference between these two is the number at which they start: the first sequence starts at 2 and proceeds by adding 2 successively; the second starts at 1 and proceeds the same way.

Now what if we take halves of natural numbers?

$$\frac{1}{2}, 1, 1\frac{1}{2}, \ldots$$

At what number does it start? And what number is successively added?

Let's look at some more sequences:

(i) The sequence of the sum of outer angles of polygons ordered according to the number of sides:

360, 360, 360, ...

Here we can say that the sequence starts at 360 and proceeds by adding 0 at every step

Hasty conclusions

We discussed a problem on the number of parts of a circle got by joining points on it, in the section **Circle division**. For the number of points equal to 2, 3, 4, 5 we get the number of parts as 2, 4, 8, 16. What about 6 points? We tend to guess the number of parts as 32. Do we get this many parts if we actually draw such a picture?

If the points are equally spaced, we get 30 parts:

Otherwise, 31 parts

Either way the maximum number of parts is 31.

It can be proved that in general if we take n points on the circle and join all pairs, the maximum number of parts got is

$$\frac{1}{24}n(n-1)(n-2)(n-3) + \frac{1}{2}n(n-1) + 1$$

The interesting point is that in this expression and in 2^{n-1}, if we take $n = 2, 3, 4, 5$ we get the same numbers 2, 4, 8, 16. From $n = 6$ onwards, the numbers differ.

(ii) Water flows out of a tank containing 100 litres of water, at the rate of 5 litres each hour. If we calculate the water remaining in the tank in litres after each hour, we get the sequence

95, 90, 85, ...

This sequence starts at 95 and proceeds by subtracting 5 at every step.

Instead of saying subtracting 5, we can say adding –5.

Let's have another look at the way the sequences seen now are formed:

Rule	Sequence
Start from 2 and successively add 2	2, 4, 6, ...
Start from 1 and successively add 2	1, 3, 5, ...
Start from $\frac{1}{2}$ and successively add $\frac{1}{2}$	$\frac{1}{2}$, 1, $1\frac{1}{2}$, ...
Start from 360 and successively add 0	360, 360, 360, ...
Start from 95 and successively add –5	95, 90, 85, ...

All such sequences have a common name

A sequence starting with a number and proceeding by adding one number again and again, is called an arithmetic sequence

The numbers in a sequence are called the *terms* of the sequence. For example, the terms of the sequence of half the natural numbers are $\frac{1}{2}$, 1, $1\frac{1}{2}$ and so on; more precisely, $\frac{1}{2}$ is the first term, 1 is the second term, $1\frac{1}{2}$ is the third term and so on.

Thus we can say this about an arithmetic sequence:

> In an arithmetic sequence, the same number is added to go from any term to the next

This can be put like this also:

> In an arithmetic sequence, the same number is subtracted to go from any term to the previous term.

This number, got by subtracting the previous term from any term, is called the **common difference** of the arithmetic sequence.

For example, the arithmetic sequence got by starting with 10 and adding 3 at each step is

10, 13, 16, 19, 22, ...

And the common difference of this sequence is 3

What if we start with 10 and subtract 3 at every step?

We get the sequence

$$10, 7, 4, 1, -2, \ldots$$

And the common difference is

$$7 - 10 = 4 - 7 = 1 - 4 = -2 \ \ - 1 = \cdots \ = \ -3$$

We often test whether a sequence is an arithmetic sequence by checking whether the number got by subtracting the previous term from a term is the same.

For example, let's look at the sequence of natural numbers which leave remainder 1 on division by 3:

Sequence rule

What is the next term of the sequence 3, 5, 7, ...?

Here, it's not said that the sequence is an arithmetic sequence. So, the next term may not be 9. For example, if the sequence is supposed to be that of odd primes, the next number is 11.

What's the moral of this?

From a few numbers written in a certain order, we cannot say for sure what the next numbers are.

For this, the rule of formulation of the sequence or the context in which it arises should be made clear.

$$1, 4, 7, \ldots$$

We can write these numbers in more detail like this:

$$(3 \times 0) + 1, \ (3 \times 1) + 1, \ (3 \times 2) + 1, \ldots$$

Thus if we take any two consecutive terms of this sequence, the first would be 1 added to 3 multiplied by a number and the second, 1 added to 3 multiplied by the next number. So, the first subtracted from the second would give 3.

So, this is an arithmetic sequence with common difference 3.

Next let's look at those natural numbers which do not leave a remainder 1 on division by 3

$$2, 3, 5, 6, \ldots$$

In this we have

$$3 - 2 = 1$$

$$5 - 3 = 2$$

Since these differences are not the same, the sequence is not an arithmetic sequence.

If the remainder on dividing a number by 3 is not 1, it can be either 0 or 2. Write down those leaving remainder 0 and those leaving remainder 2 separately. Are they arithmetic sequences?

(1) Check whether each of the sequences given below are arithmetic sequences. Give reasons also. Find the common differences of the arithmetic sequences:

(i) Natural numbers leaving remainder 1 on division by 4

(ii) Natural numbers leaving remainder 1 or 2 on division by 4

(iii) Squares of natural numbers

(iv) Reciprocals of natural numbers

(v) Powers of 2

(vi) Half of the odd numbers

(2) See these pictures:

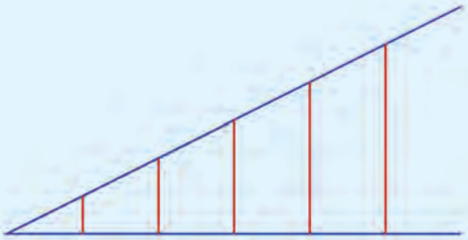
(i) How many small squares are there in each picture?

(ii) How many large squares?

(iii) How many squares in all in each picture?

If we continue the pattern of pictures, are the sequences above arithmetic sequences?

(3) In the picture below, the perpendiculars drawn from the bottom line are equally spaced.



Show that the sequence of the heights of the perpendiculars, on continuing this, form an arithmetic sequence.

(Hint: Draw perpendiculars from the top of each perpendicular to the next perpendicular)

Position and term

Can you write an arithmetic sequence with 1 and 11 as the first and second terms ?

Easy, isn't it ?

To get to 11 from 1, we must add 10; and if we continue adding 10, we get an arithmetic sequence:

1, 11, 21, 31, ...

Now another question: Can you write an arithmetic sequence with 1 and 11 as the first and the third terms ?

How do we find the common difference of such an arithmetic sequence ?

The second number is got by adding the common difference to 1; and we don't know this number. But we do know that adding the common difference again to this number gives the third number 11.

Sequence and remainder

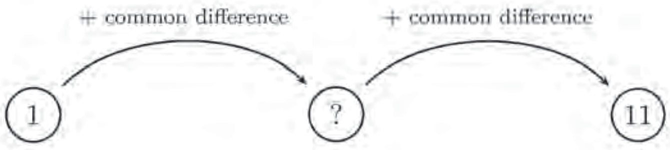
The even numbers 2, 4, 6, ... is an arithmetic sequence; so is the sequence 1, 3, 5, ... of odd numbers. And both have common difference 2.

Now even numbers are numbers divisible by 2 (that is, numbers which leave remainder 0 on division by 2) and the odd numbers are numbers which leave remainder 1 on division by 2.

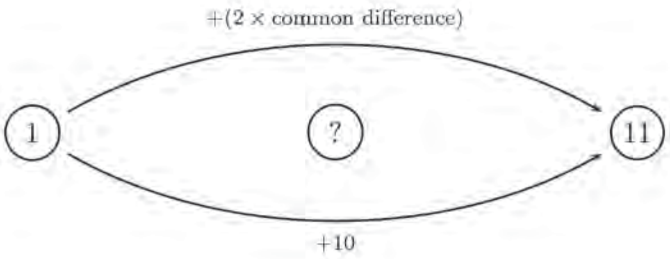
Similarly, we get three arithmetic sequences of natural numbers based on division by 3: those which leave remainder 0, 1 or 2. What is the common difference of each of these three sequences?

What if we divide by 4?

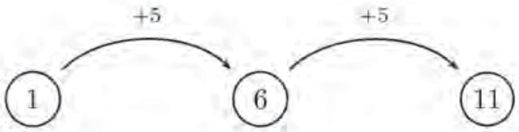
On the other hand, in any arithmetic sequence of natural numbers, the difference between any two terms is the common difference multiplied by a number. So, dividing these numbers by the common difference leave the same remainder (why?).



In other words, we get 11 by adding the common difference twice:



So, twice the common difference is 10 and so the common difference is 5



Now we can write the arithmetic sequence as

1, 6, 11, 16, 21, ...

In each of the arithmetic sequences below, some of the terms are not written, but indicated by ◯. Find out these numbers:

(i) 24, 42, ◯, ◯, ...

(ii) ◯, 24, 42, ◯, ...

(iii) ◯, ◯, 24, 42, ...

(iv) 24, ◯, 42, ◯, ...

(v) ◯, 24, ◯, 42, ...

(vi) 24, ◯, ◯, 42, ...

Let's look at another problem:

> Calculate the first five terms of the arithmetic sequence with 3rd term 37 and 7th term 73.

We think like this:

(i) To go from the 3rd term to the 7th term, we must add the common difference $7 - 3 = 4$ times

(ii) The number added is $73 - 37 = 36$

(iii) So 4 times the common difference is 36

(iv) Common difference is $36 \div 4 = 9$

Next how do we compute the 1st term?

(i) To go from the 3rd term to the 1st term, we must subtract the common difference $3 - 1 = 2$ times

(ii) The 1st term is $37 - (2 \times 9) = 19$

Now can't we write the sequence ?

19, 28, 37, 46, 55, ...

The two terms in specific positions of some arithmetic sequences are given below. Write the first five terms of each:

(i) 3rd term 34	(ii) 3rd term 43	(iii) 3rd term 2	(iv) 5th term 8	(v) 5th term 7
6th term 67	6th term 76	5th term 3	9th term 10	7th term 5

Changes in position and terms

See this arithmetic sequence:

19, 28, 37, ...

In this,

1st term is 19

2nd term is 28

3rd term is 37

....

We can write it like this:

Position	1	2	3	...
Term	19	28	37	...

How does the term change with the position in this?

When the position increases by 1, the term increases by 9

So, how do we compute the 10th term of this sequence?

This can be done in several ways

We can start at the last term written above:

(i) From the 3rd to the 10th, position increases by 10 – 3 = 7

(ii) To get the 10th term from the 3rd, we must add 7 times 9

(iii) The 10th term is $37 + (7 \times 9) = 100$

Or we can start from the first term:

(i) From the 1st to the 10th, position increases by 10 – 1 = 9

(ii) To get the 10th term from the 1st, we must add 9 times 9

(iii) The 10th term is $19 + (9 \times 9) = 100$

Can you compute the 10th term from the 2nd like this?

Let's look at another problem:

> The 4th term of an arithmetic sequence is 45, and the 5th term is 56. What is the first term?

To get the 5th term from the 4th term in this, the number added is 56 – 45 = 11. Since this is an arithmetic sequence, the same number 11 must be added to get any term from the previous term.

What we must compute is the 1st term

We can think like this:

(i) To get to the 1st term from the 4th term, the position must be decreased by 4 – 1 = 3

(ii) To get the 1st term from the 4th term, 11 must be subtracted 3 times

(iii) The 1st term is 45 – (3 × 11) = 12

Now we can write the sequence itself as

$$12, 23, 34, \ldots$$

In both these examples, the terms were increasing.

What about a sequence of decreasing terms?

For example, see this arithmetic sequence:

$$91, 82, 73, \ldots$$

In what all ways can we calculate its 10th term?

In this, the terms decrease as the positions increase. By how much?

For each increase in position by 1, the term decreases by 91 – 82 = 9

So, how do we compute the 10th term ?

(i) To get to the 10th term from the 3rd term, the position must be increased by 10 – 3 = 7

(ii) To get the 10th term from the 3rd term, 9 must be subtracted 7 times

(iii) The 10th term is 73 – (7 × 9) = 10

In such sequences as this, how do we compute a term from a term before it?

For example, see this problem:

> The 4th term of an arithmetic sequence is 65 and its 5th term is 54. What is the first term?

In this sequence also, terms decrease as positions increase.

For each increase in position by 1, the term decreases by $65 - 54 = 11$

On the other hand, what happens for each decrease in position by 1?

The term increases by 11, right?

So how do we compute the 1st term?

(i) To get to the 1st term from the 4th term, the position must be decreased by $4 - 1 = 3$

(ii) To get the 1st term from the 4th term, 11 must be added 3 times

(iii) The 1st term is $65 + (3 \times 11) = 98$

(1) What is the 25th term of the arithmetic sequence 1, 11, 21, ... ?

(2) The 10th term of an arithmetic sequence is 46 and its 11th term is 51

- (i) What is its first term?
- (ii) Write the first five terms of the sequence

(3) What is the 21st term of the arithmetic sequence 100, 95, 90, ...?

(4) The 10th term of an arithmetic sequence is 56 and its 11th term is 51

- (i) What is its first term ?
- (ii) Write the first five terms of the sequence

Let's take a more detailed look at the relation between the changes in the positions and terms of an arithmetic sequence.

For example, consider this arithmetic sequence:

5, 15, 25, ...

We can write the connection between positions and terms like this:

Position	1	2	3	4	5	6	7	8	9	10	...
Term	5	15	25	35	45	55	65	75	85	95	...

In this, as the positions increase by 1, the terms increase by 10

What happens as the positions increase by 2?

Position	1	3	5	7	9	...
Term	5	25	45	65	85	...

What about the terms as the positions increase by 3 ?

On the other hand, what happens when the positions decrease?

- As the positions decrease by 1, the terms decrease by 10
- As the positions decrease by 2, the terms decrease by 20
- As the positions decrease by 3, the terms decrease by 30

Now let's look at an arithmetic sequence with decreasing terms:

Position	1	2	3	4	5	6	7	8	9	10	...
Term	50	45	40	35	30	25	20	15	10	5	...

What is the relation between the changes in positions and the changes in terms of this sequence?

- As the positions increase by 1, the terms decrease by 5
- As the positions increase by 2, the terms decrease by 10
- As the positions increase by 3, the terms decrease by 15

And when the positions decrease?

- As the positions decrease by 1, the terms increase by 5
- As the positions decrease by 2, the terms increase by 10
- As the positions decrease by 3, the terms increase by 15

In any other arithmetic sequence, the terms would be different, but the relation between the change in positions and the change in terms would be similar, right?

So what can we say in general ?

In any arithmetic sequence, the change in terms is the product of the change in position and a fixed number

Using the ideas discussed in the chapter **Proportion** of the Class 9 textbook, we can state this as follows:

In any arithmetic sequence, the change in terms is proportional to the change in position

Now look at this problem:

> The 3rd term of an arithmetic sequence is 12 and the 7th term is 32. What is its 15th term?

First let's write the position change and the term change from the 3rd to the 7th

Position change	7 – 3	=	4 increase
Term change	32 – 12	=	20 increase

From this, we can see that the terms increase by 20 ÷ 4 = 5 as the positions increase by 1

We want the 15th term. As the position change from 7th to the 15th, there's an increase of 15 – 7 = 8; and so the term increases by 8 × 5 = 40.

Thus the 15th term is 32 + 40 = 72

There's another way of doing this

(i) As the position increases by 4, the term increases by 20

(ii) When the position increases by $8 = 2 \times 4$, the term increases by $2 \times 20 = 40$

(iii) 15th term is 32 + 40 = 72

How do we compute the 5th term like this?

(i) 3rd term is 12

(ii) 5th term is 2 positions ahead of the 3rd term

(iii) As the position increases by 4, the term increases by 20

(iv) When the position increases by $2 = \frac{1}{2} \times 4$, the term increases by $\frac{1}{2} \times 20 = 10$

(v) 5th term is 12 + 10 = 22

Let's look at another problem:

> The 4th term of an arithmetic sequence is 81 and the 6th term is 71. What is its 20th term?

As in the first problem, let's first write down the changes in positions and terms:

Position change	6 – 4	=	2 increase
Term change	81 – 71	=	10 decrease

What we want is the 20th term; that is $20 - 6 = 14$ positions ahead of the 6th term. The calculations can be done like this:

(i) When the position increases by 2, the term decreases by 10

(ii) When the position increases by $14 = 7 \times 2$ the term decreases by $7 \times 10 = 70$

(iii) The 20th term is $71 - 70 = 1$

(1) The 3rd term of an arithmetic sequence is 15 and the 8th term is 35

   (i) What is its 13th term?

   (ii) What is its 23rd term?

(2) The 5th term of an arithmetic sequence is 21 and the 9th term is 41

   (i) What is its first term?

   (ii) What is its 3rd term?

(3) The 4th term of an arithmetic sequence is 61 and the 7th term is 31

   (i) What is its 10th term?

   (ii) What is its first term?

(4) The 5th term of an arithmetic sequence is 10 and the 10th term is 5

   (i) What is its 15th term?

   (ii) What is its 25th term?

We can use the fact that the term change in an arithmetic sequence is proportional to the position change, to check whether a number is a term of an arithmetic sequence.

For example, let's look at an earlier example of an arithmetic sequence:

$$19, 28, 37, \ldots$$

We have seen that the 10th term of this sequence is 100

Is 1000 a term of this sequence?

As we move from the first to the second term of this sequence, the term increases by $28 - 19 = 9$. So, the relation between position difference and term difference can be written like this:

Position change	1	2	3	...
Term change	9	2×9	3×9	...

Thus any term difference is a multiple of 9.

So, to check whether 1000 is a term of this sequence, we need only check whether 1000 – 19 = 981 is a multiple of 9.

Since we have

$$981 \div 9 = 109$$

We can see that 1000 is actually the 110th term of this sequence.

Now check whether 10000 is a term of this sequence, and if so find its position.

(1) Is 101 a term of the arithmetic sequence 13, 24, 35, ...? What about 1001?

(2) In the table below, some arithmetic sequences are given and two numbers against each. Check whether the numbers are terms of the respective sequences:

Sequence	**Numbers**	**Yes/No**
11, 22, 33, ...	123	
	132	
12, 23, 34, ...	100	
	1000	
21, 32, 43, ...	100	
	1000	
$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$	3	
	4	
$\frac{3}{4}, 1\frac{1}{2}, 2\frac{1}{4}, \ldots$	3	
	4	

(3) In the table above, find the position of the numbers that are terms of the respective sequences.

Term connections

We have noted that if we know the terms in two specified position of an arithmetic sequence, then we can calculate all its terms.

If we know just one term and its position, can we say anything about the sequence ?

For example, suppose we know that the first term of an arithmetic sequence is 5, can we say anything more about the sequence ?

The common difference can be any number; and so the sequence can be continued in whatever way we want.

What if we know that the second term is 5 ?

In this case also, we can take any number as the common difference and proceed.

Let's try few numbers as the common difference and the terms just before and after the 2nd term 5 (that is, the 1st and the 3rd terms)

Common difference 1 : 4, 5, 6

Common difference 2 : 3, 5, 7

Common difference $\frac{1}{2}$: $4\frac{1}{2}, 5, 5\frac{1}{2}$

Common difference $\sqrt{2}$: $5 - \sqrt{2}, 5, 5 + \sqrt{2}$

Common difference -1 : 6, 5, 4

Is there any relation between these numbers?

What is the sum of the first and last numbers in each case?

And the sum of all three?

Why does this happen?

Whatever be the common difference, the first term is this number subtracted from 5, and the third term is this number added to 5

So, adding these two numbers is in effect, adding two 5's, right?

Next, to add all three terms, the second term 5 must be added once more, which gives the sum of three 5's

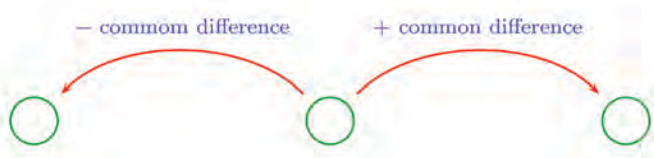
What if we take some number other than 5 as the second term?

In that case also, wouldn't the sum of the first and the third terms be twice that number?

And what about the sum of all three terms?

Now what if we take some other term instead of the second?

The term just before it is the common difference subtracted from it; and the term just after is the common difference added to it:



So, their sum is the twice number first chosen; that is the middle term

What about the sum of all three terms?

So, what can we say in general?

(i) In an arithmetic sequence, the sum of the terms just before and just after a term is twice this term.

(ii) In an arithmetic sequence, the sum of a term and the terms just before and just after it, is three times this term.

If we take a term of an arithmetic sequence and the terms just before and after that, they form three consecutive terms of the sequence, right?

So the second statement above can also put like this:

> In any arithmetic sequence, the sum of three consecutive terms is three times the middle term.

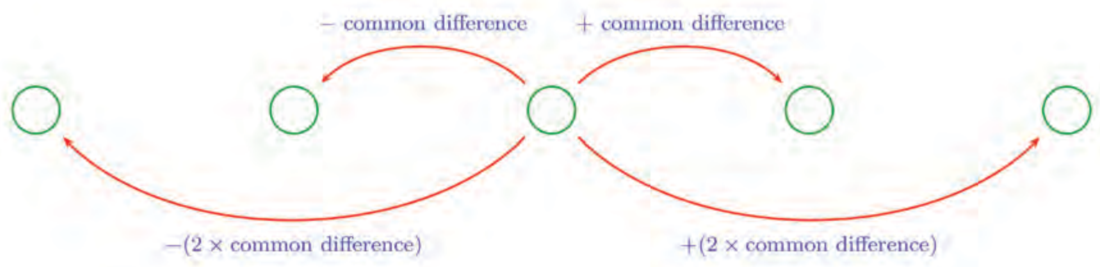
We can also state this in reverse:

> In any arithmetic sequence, the middle term of three consecutive terms is one-third of their sum

For example if the sum of the 5th, 6th and the 7th terms of an arithmetic sequence is 30, then the 6th term is $\frac{1}{3} \times 30 = 10$

Now in addition to the terms just before and after a term, what if we also take terms two positions behind and two positions ahead ?

Moving two positions behind means twice the common difference is subtracted from the term and moving two positions ahead means adding twice the common difference to the term:



We have already seen that the sum of the terms just before and just after a term is twice this term.

What about the sum of the terms two positions behind and two position ahead?

One of these is the twice common difference subtracted from the term first chosen and the other is twice the common difference added to this term. So their sum also is twice this term, isn't it?

What about the sum of all five terms?

The sum of the first and the fifth is twice the third number (which is the number in the middle) and so is the sum of the second and fourth. So the sum of all five is 2+2+1 = 5 times the third number.

Now what if we start from a term and take pairs of terms one, two and three positions behind and after?

What can we say in general?

(i) In an arithmetic sequence, the sum of the two terms, at the same distance behind and ahead a term, is twice this term

(ii) In an arithmetic sequence, the sum of a term and the consecutive terms at the same distance behind and ahead, is the product of this term and the number of terms

For example, if the 10th term of an arithmetic sequence is 25, what all sums can we compute using the first statement above ?

- The sum of the 9th and 11th terms is 2 × 25 = 50
- The sum of the 8th and 12th terms is 2 × 25 = 50

What other sums of terms can be calculated to be 50?

Using the second statement above, what all sums can we calculate?

- The sum of the terms in positions 9, 10, 11 is $3 \times 25 = 75$
- The sum of the terms in positions 8, 9, 10, 11, 12 is $5 \times 25 = 125$

What other sums can we compute ?

Did you notice another thing here? If we take a term and consecutive terms at the same distance before and ahead, we get an odd number of terms, right? And the term we started with, is the term in the middle of all these.

So, the second statement above can be stated in a different way:

The sum of an odd number of consecutive terms of an arithmetic sequence is the product of the middle term and the number of terms

(1) The 4th term of an arithmetic sequence is 8.

(i) Find the sum of the pairs of terms given below:

(a) 3rd and 5th

(b) 2nd and 6th

(c) 1st and 7th

(ii) What is the sum of the 3rd, 4th and the 5th terms?

(iii) What is the sum of the 5 terms from the 2nd to the 6th?

(iv) What is the sum of the 7 terms from the 1st to the 7th?

(2) The common difference of an arithmetic sequence is 2 and the sum of the 9th, 10th and 11th terms is 90. Calculate the first three terms of the sequence.

Now look at this problem:

> The first term of an arithmetic sequence is 10 and the sum of the first 5 terms is 250. Can you find the sequence?

Since the sum of the first 5 terms is 250, the 3rd term (the middle term) can be computed as 50 (How?).

Since the 1st term is 10 and the 3rd term is 50, the common difference can be found as 20
So, the sequence is 10, 30, 50, ...

(1) Write three arithmetic sequences with the sum of the first 7 terms as 70.

(2) The sum of the first 3 terms of an arithmetic sequence is 30 and the sum of the first 7 terms is 140.

(i) What is the 2nd term of the sequence?

(ii) What is the 4th term of the sequence?

(iii) What are the first three terms of the sequence?

(3) The sum of the first five terms of an arithmetic sequence is 150, and the sum of the first ten terms is 550

(i) What is the third term of the sequence?

(ii) What is the eighth term of the sequence?

(iii) Write the first three terms of the sequence

(4) The sum of the 11th and 21st terms of an arithmetic sequence is 80. What is the 16th term?

(5) The angles of a pentagon are in arithmetic sequence

(i) If the angles are written according to their magnitude, what would be the third angle?

(ii) If the smallest angle is 40°, what are the other angles?

(iii) Can the smallest angle be 36°?

We have seen that just by knowing the term at a specific position, we can compute the sum of many terms.

Let's look at a different kind of problem:

> If the sum of the 2nd and 5th terms of an arithmetic sequence is known to be 35, can we say anything more about the sequence?

We can take any number as the 2nd term. For example, let's take the 2nd term as 10.

Then the 5th term must be $35 - 10 = 25$

Now we can compute the common difference as $(25 - 10) \div 3 = 5$ and then write the first few terms of the sequence as

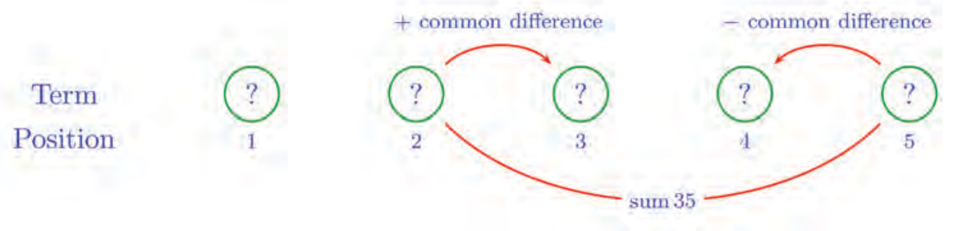
$$5, 10, 15, 20, 25, 30, \ldots$$

Did you notice something about the sequence?

sum of the 3rd and 4th terms = 15+ 20 = 35

sum of the 1st and 6th terms = 5 + 30 = 35

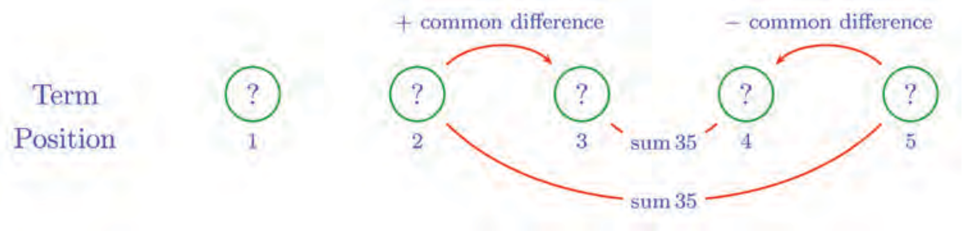
Take some other number as the 2nd term and calculate these sums. Why are they always 35? Let's draw a picture showing this:



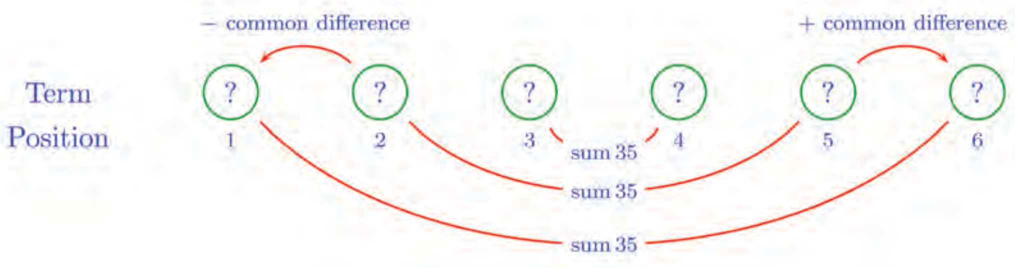
To go from the 2nd term to the 3rd term, we must add the common difference

To go from the 5th term to the 4th term, we must subtract the common difference

So, the sum of the 3rd and 4th terms is also 35, isn't it?



What about the sum of the 1st and 6th terms?



Similarly, if the sum of the 4th and 8th terms is known to be 45, what other sums can we calculate?

- Moving one position forward from the 4th and one position backward from the 8th, the sum of the 5th and the 7th terms is 45
- Moving one position backward from the 4th and one position forward from the 8th, the sum of the 3rd and the 9th terms is 45

What other terms have the same sum 45?

Positions	4, 8	5, 7	3, 9	...	...
Sum of terms	45	45	45	...	...

Similarly, if the sum of the 2nd and 6th terms is said to be 30, the sum of the terms at what other pairs of positions can we say is 30 ?

What can we say in general about this ?

In an arithmetic sequence, if one position is increased and another position is decreased by the same amount, the sum of the terms at these positions do not change

The sum of two numbers doesn't change, if one number is increased and the other decreased by the same amount, right? On the other hand, if the sum of a pair of numbers is equal to the sum of another pair, then one of the numbers has increased and the other decreased by the same amount.

So, the above fact can be stated like this also:

In an arithmetic sequence, if the sum of two positions is equal to the sum of other two positions, then the sum of the terms at each pair is the same

So, if we know that the sum of the first four terms of an arithmetic sequence is 100, what other things can we say about the sequence?

The first four terms can be split into two pairs with the same sum of positions:

- 1st and 4th
- 2nd and 3rd

Since the sum of all four terms is 100, the sum of the terms at each pair above must be 50. That is

- The sum of the 1st and 4th terms is 50
- The sum of the 2nd and 3rd terms is 50

Now suppose that we also know that the 1st term is 10

Then we can proceed like this

- The 4th term is 50 – 10 = 40
- The 1st term is 10 and the 4th term is 40; so, 3 times the common difference is 40 – 10 = 30
- The common difference is 30 ÷ 3 = 10
- The first four terms of the sequence are 10, 20, 30, 40

(1) Write four arithmetic sequence with sum of the first four terms 100

(2) The 1st term of an arithmetic sequence is 5 and the sum of the first 6 terms is 105. Calculate the first six terms of the sequence.

(3) The sum of the 7th and the 8th terms of an arithmetic sequence is 50. Calculate the sum of the first 14 terms.

(4) Write the first three terms of each of the arithmetic sequences given below:

(i) The 1st term is 30 and the sum of the first three terms is 300

(ii) The 1st term is 30 and the sum of the first four terms is 300

(iii) The 1st term is 30 and the sum of the first five terms is 300

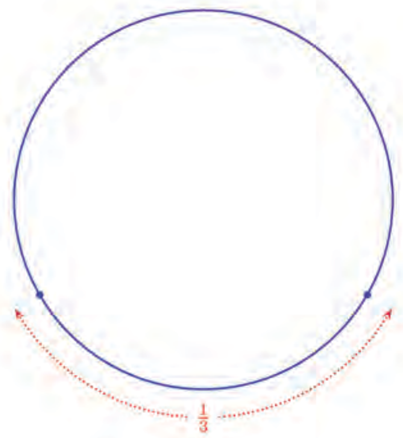
(iv) The 1st term is 30 and the sum of the first six terms is 300

2

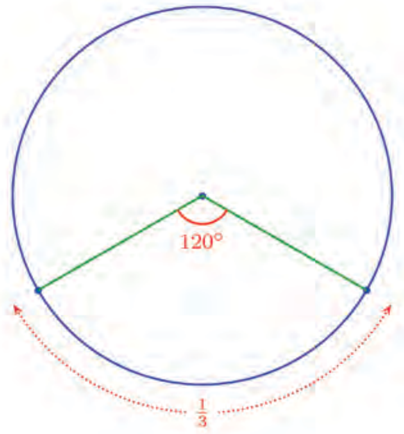
CIRCLES AND ANGLES

Parts of a circle

How do we cut out $\frac{1}{3}$ of a circular ring ?

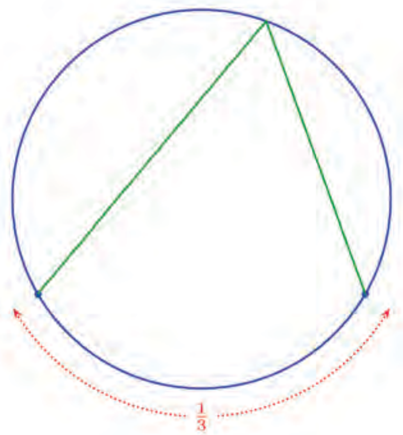


As we saw in the section **Length and angle** of the chapter **Parts of Circles** in the Class 9 textbook , the central angle of this arc is $\frac{1}{3} \times 360° = 120°$.

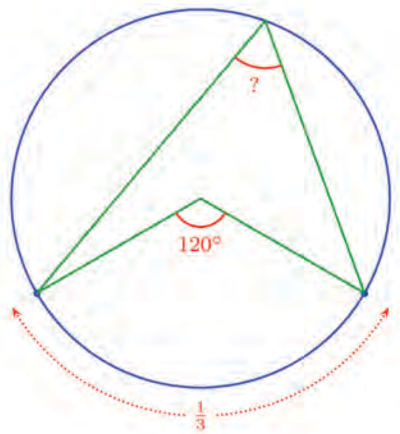


So, one method is to first locate the centre of the circle using any of the techniques discussed in the section **Chords** of the chapter **Circles** in the Class 8 textbook, and then draw an angle of 120° at the centre to mark an arc which is $\frac{1}{3}$ of the circle.

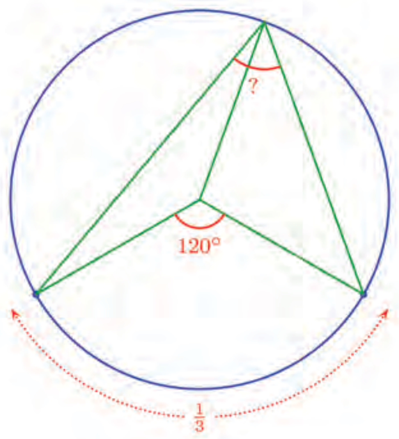
Can we mark such an arc without locating the centre ? In other words, is there a way to mark such an arc by drawing two lines from some point of the circle itself ?



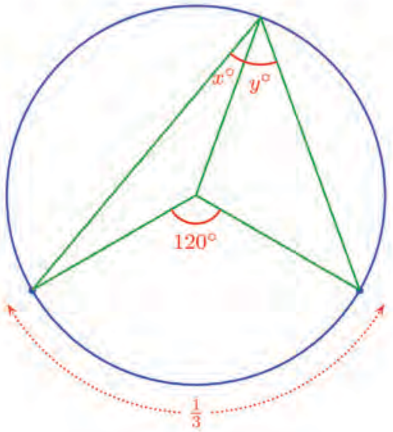
Let's make this question more precise : what angle should we choose in the picture above, so that the angle at the centre is 120° ?

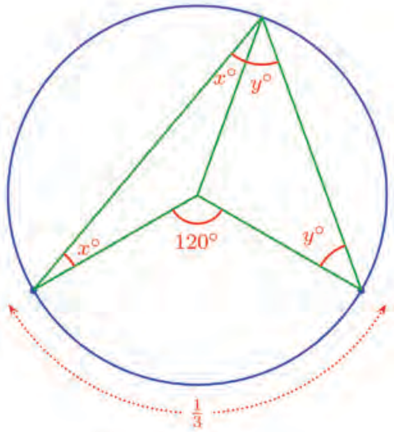


To compute this angle , we join the centre of the circle to the corner of this angle and split it into two parts :

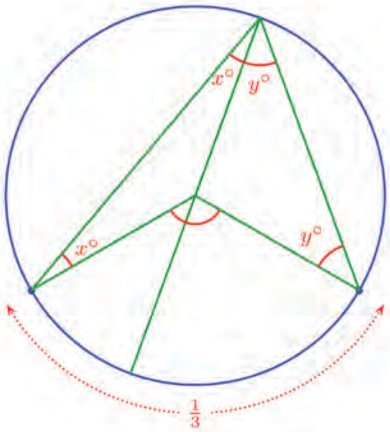


Now there are two triangles in the picture. In both, two of the sides are radii of the circle, and so are of equal length. In other words, both are isosceles triangles. So, if we take the parts of the top angle as $x°$ and $y°$, then they are also the bottom angles of the triangles :

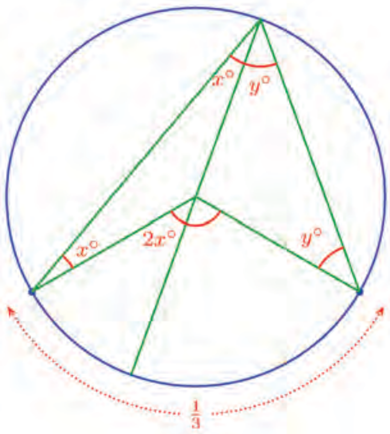


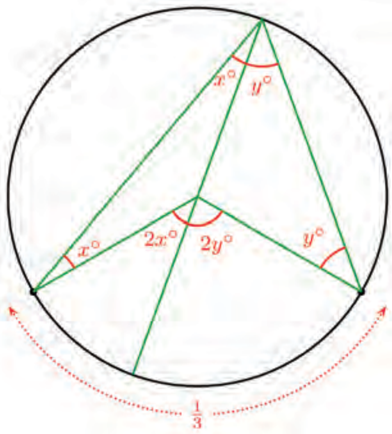


Now if we extend the common side of these triangles, then the central angle also is split into two :



These parts of the central angles are the outer angles at the third vertices of the triangles on the left and right ; and so they are equal to the sum of the inner angles at the other two vertices (the section **Outer angles** of the chapter **Polygons**, in the Class 8 textbook).





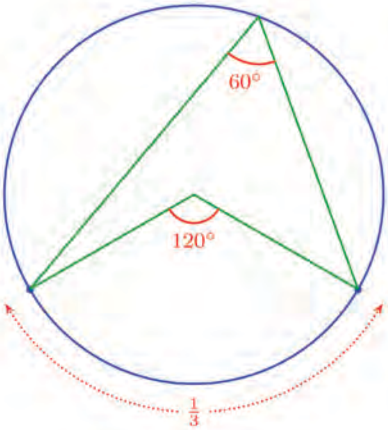
Since the central angle is 120°, we have

$$2x + 2y = 120$$

and from this, we get

$$x + y = 60$$

This means , the angle made by the ends of the arc, at the point on the circle is 60°.



Make an angle slider a. Draw a circle centred at a point A and mark a point B on it. Select the Angle with Given Size tool and click on B and then on A. In the dialogue window, give a as Angle. We get a point B' on the circle. Join AB and AB', and mark the central angle. Make sure that the central angle is less than 180°. Mark the point C on the larger arc and join CB and CB', and mark the angle at C. What is the relation between the angles at A and C ? Change the position of C. What if we change the value of a, using the slider ?

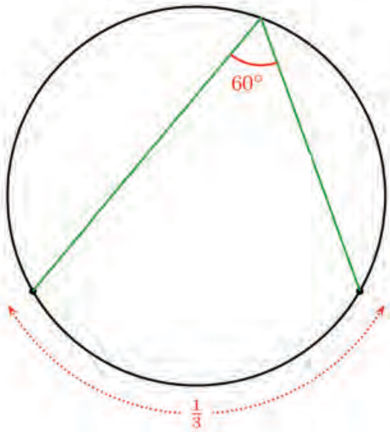
What do we see here ?

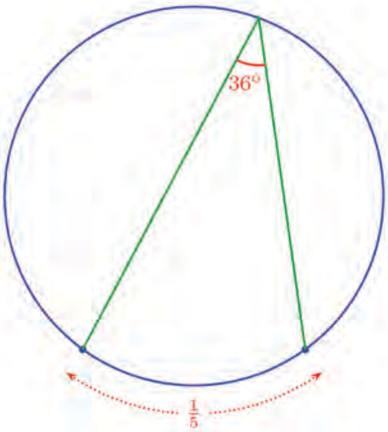
To mark $\frac{1}{3}$ of a circle , we need only make a 60° angle at a point on the circle, instead of a 120° angle at the centre.

Compute what angle we should mark at a point on the circle, to mark off $\frac{1}{5}$ of the circle.

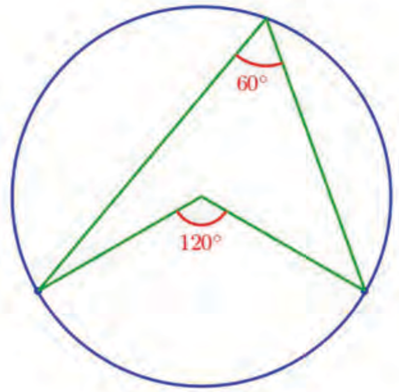
Arcs and angles

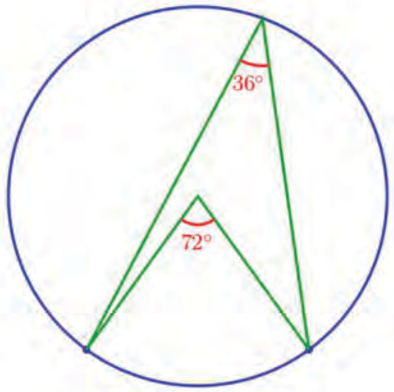
We have seen that if we mark a 60° angle at a point on a circle, we get an arc which is $\frac{1}{3}$ of the circle; and if we mark a 36° angle at a point on a circle, we get an arc which is $\frac{1}{5}$ of the circle





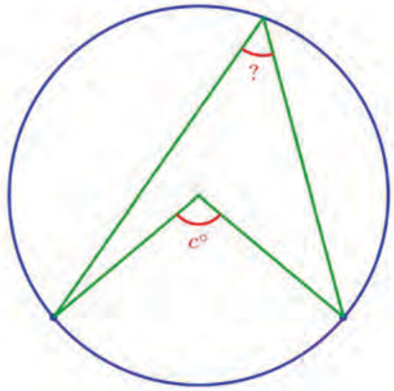
We can state this using central angles of arcs, instead of parts of a circle :





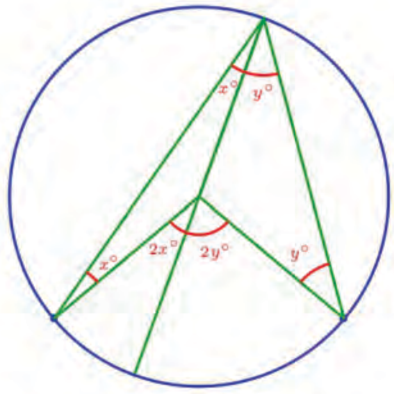
Now here's a question : if we join the ends of an arc to a point on the circle, would we get half the central angle ?

Let's take *c*° as the central angle of an arc :



As before , let's join the centre of the circle and the point on the circle and take the parts into which this cuts the top angle as *x*° and *y*°.

We then extend this line to cut the central angle also. We can then mark the angles as below:

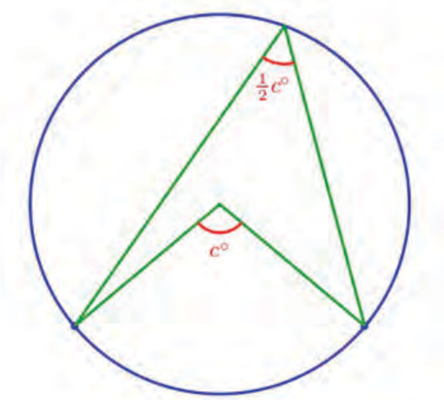


We took the central angle as $c°$. So, we have

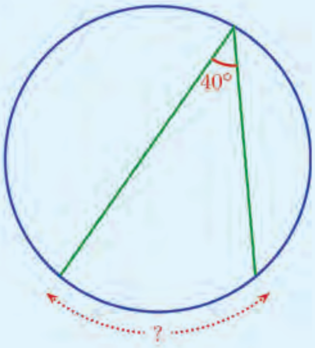
$$2x + 2y = c$$

and from this we get

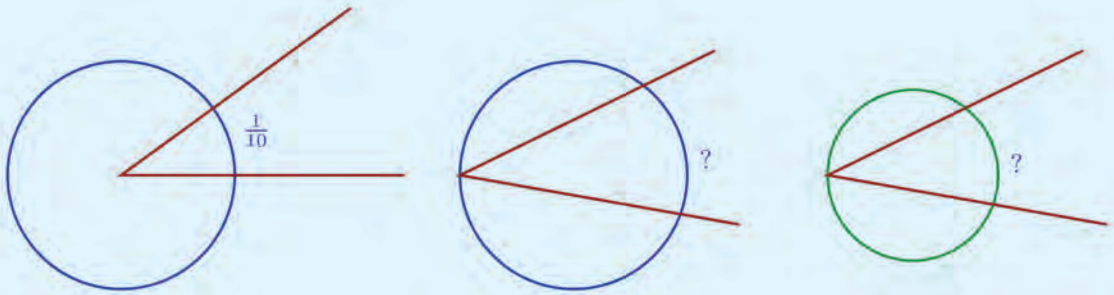
$$x + y = \frac{1}{2}c$$



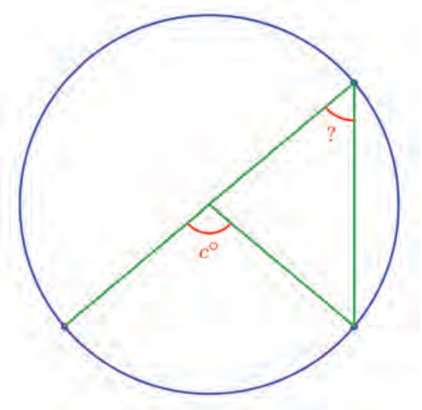
(1) What fraction of the circle is the arc marked in the picture below ?



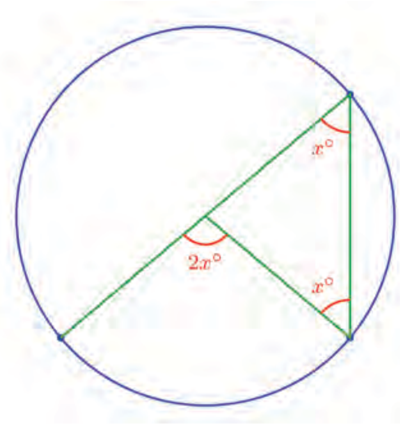
(2) When the corner of a bent wire was placed at the centre of a circle, $\frac{1}{10}$ of the circle was contained within it. If the corner of this wire is placed on a point of this circle as in the second picture, what fraction of the circle would it contain ? What if it is placed at a point on another circle as in the third picture ?

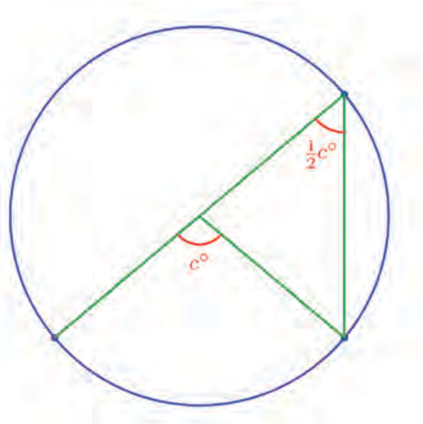


But then , there's another question : would we get half the central angle , if the ends of an arc are joined to any point on the circle ? What if it's like this ?

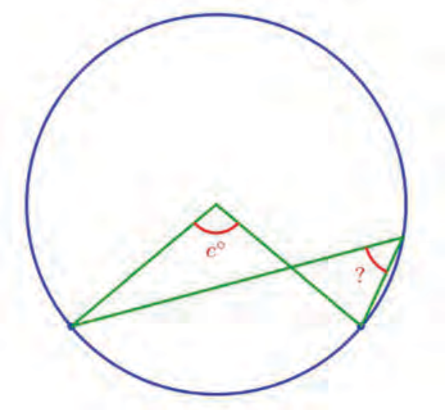


Here the point on the circle is the other end of a diameter through one end of the arc. If we take the angle which the arc makes at this point as $x°$, then we can compute the other angles of the figure:

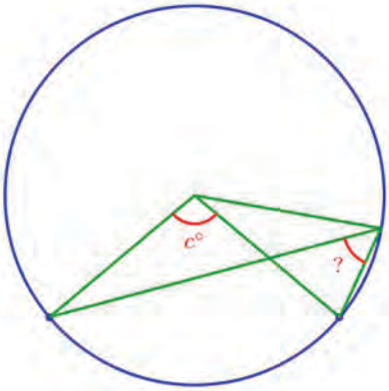




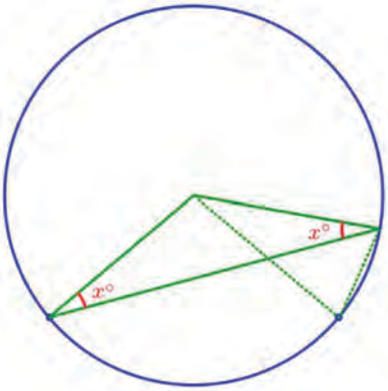
What if we choose a point still lower on the circle as in the picture below ?

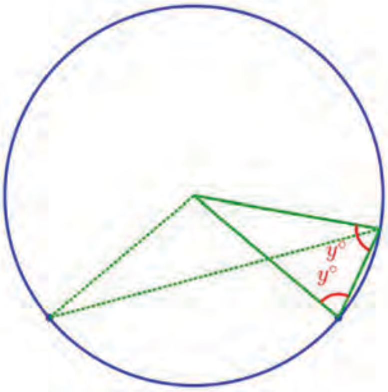


As before , we join this point to the centre of the circle:

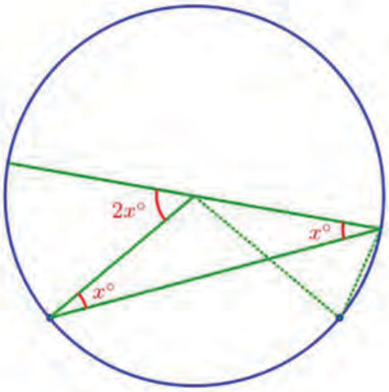


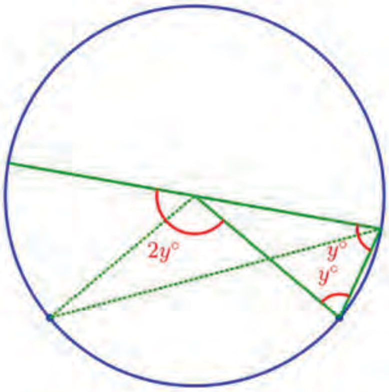
And take the equal angles in one isosceles triangle as *x*° and in the other as *y*°:



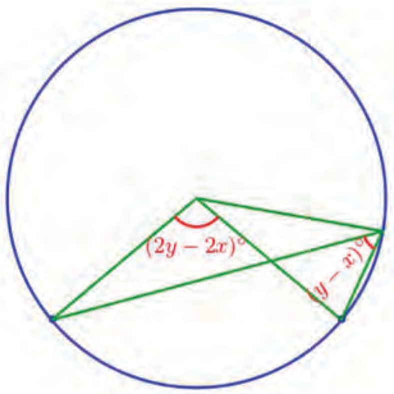


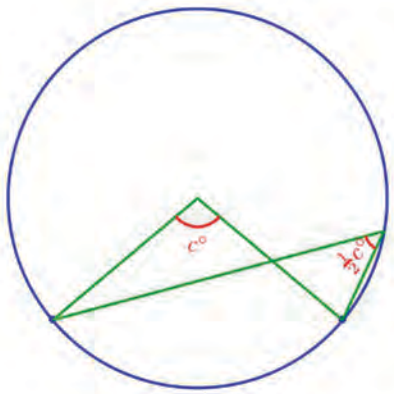
Again we extend the common side of these triangles , and mark the outer angles:



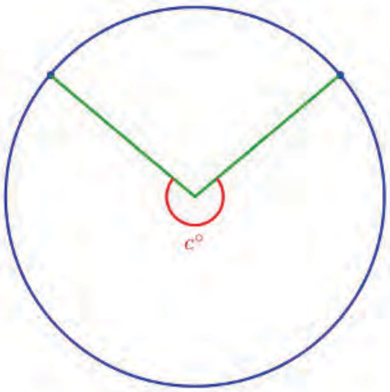


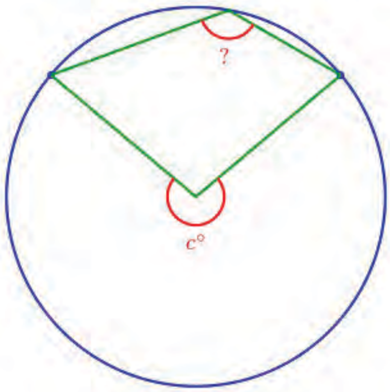
Now we can compute the central angle of the arc and the angle made by its ends at the point on the circle:



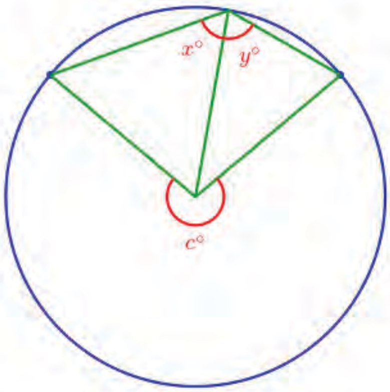


In all the discussion so far, we considered only arcs with central angle less than 180°. What if the central angle is greater than 180° ?

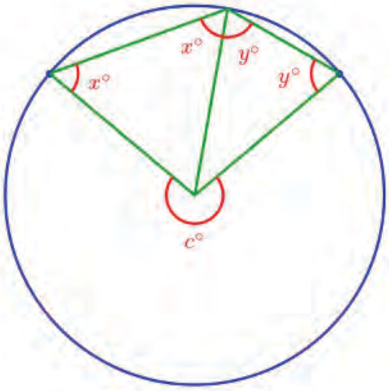




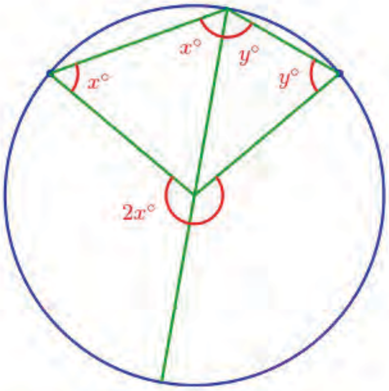
As before, let's join the centre of the circle and the point on the circle ; and take as $x°$ and $y°$ the parts of the angles into which this line splits the angle made at the point on the circle :

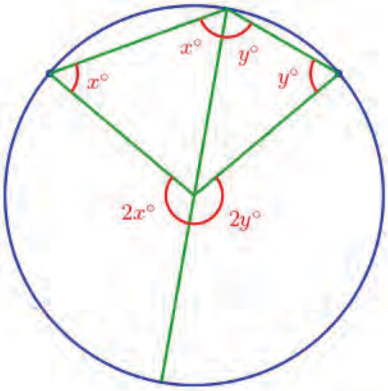


Since the two triangles in the circle are isosceles triangles, the left and right angles are also *x*° and *y*°:

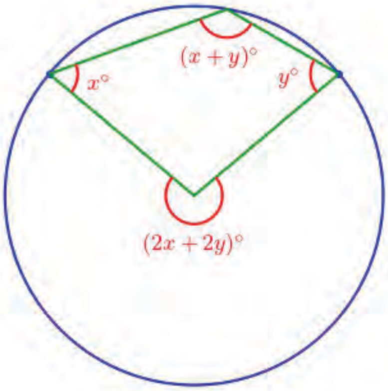


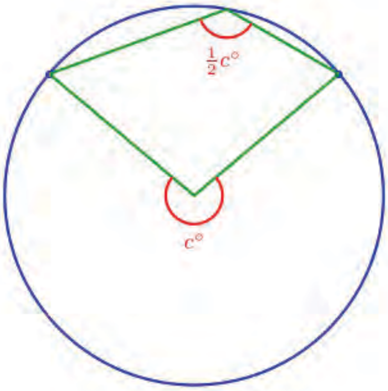
Again, as before, let's extend this line and split the central angle into outer angles of the triangles :



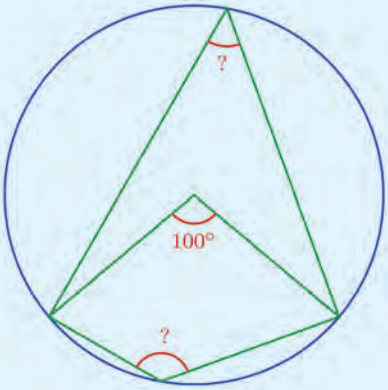


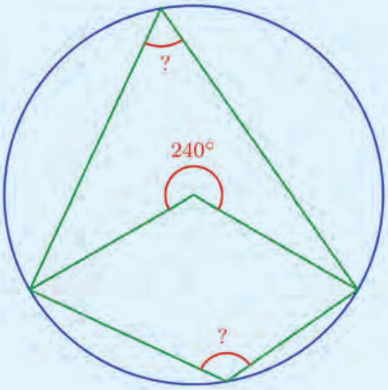
From this we can calculate the central angle of the arc and the angle made by its endpoints at the point on the circle :





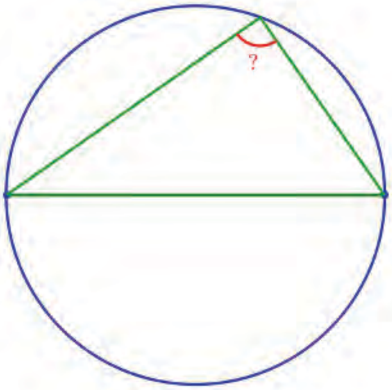
In each picture below, the central angle of an arc of a circle is shown :



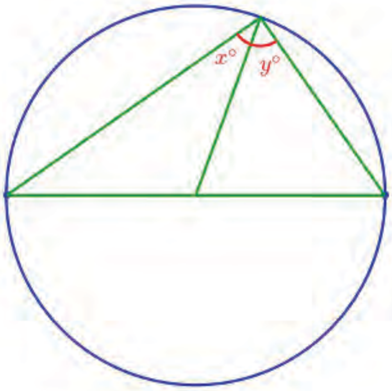


In each, calculate the angles which the arc makes at the other two points

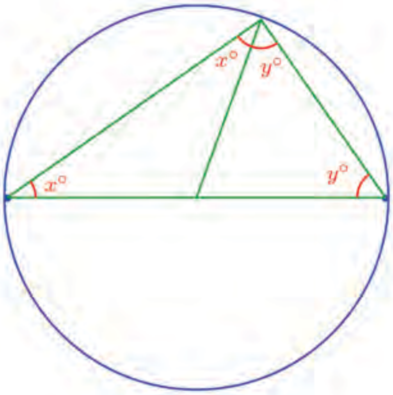
Now what if the central angle is 180° ? That is if the arc is a semicircle ?

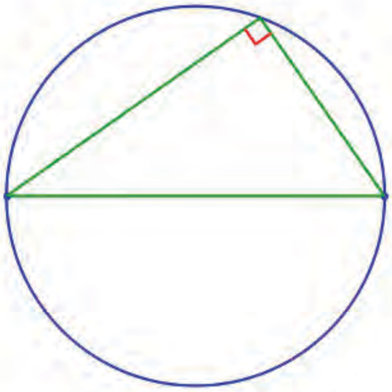


Let's again join the centre of the circle to the point on it, and take as $x°$ and $y°$, the angles into which this line cuts the angle at the point on the circle :



The bottom angles of the isosceles triangles are also $x°$ and $y°$





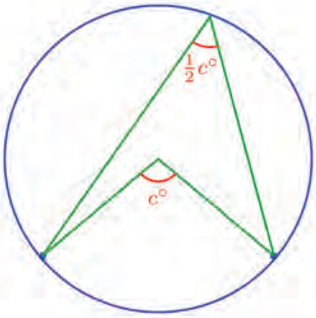
Since the sum of the angles of the large triangle in the circle is 180°, we have

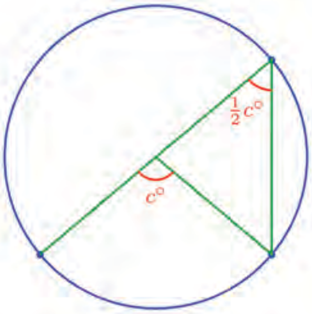
$$(x + y) + x + y = 180$$

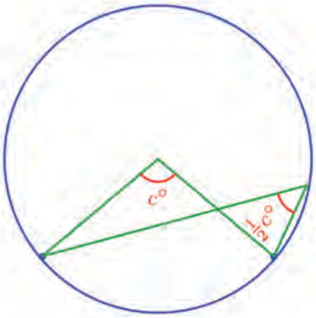
and from this we get $x + y = 90$

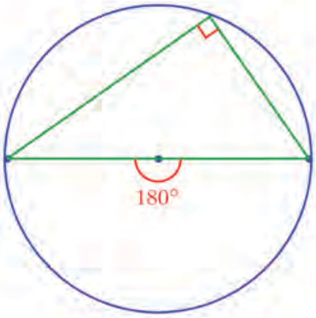
What did we do in all these discussions ?

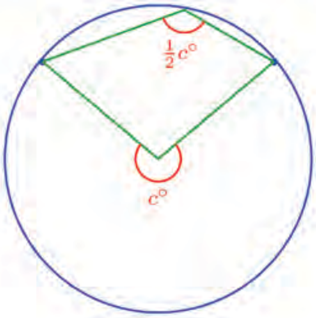
We found out the relation between the central angle of an arc of a circle and the angle which the ends of the arc make with a point on the circle :





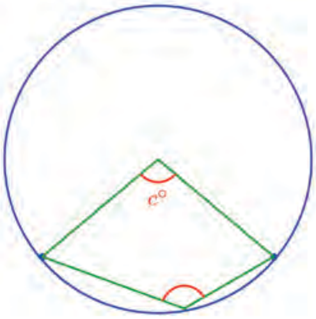






Note that in all these, we join the ends of the arc to a point on the circle, *outside* the arc.

We can readily see that if we join the ends of an arc to a point *within* the arc, then we won't get half the central angle :



So all that we have seen so far can be combined to a general principle :

If the ends of an arc of a circle are joined to a point on the circle, which is not a point on the arc itself, then the angle so made is half the central angle of the arc

The case of a semicircle is worth special mention :

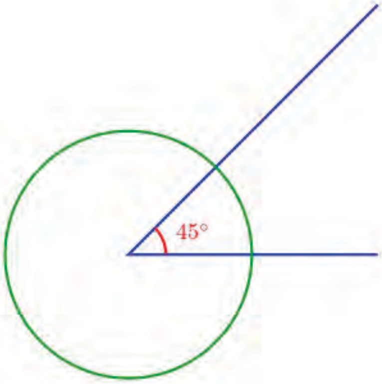
If the ends of a semicircle are joined to another point on the circle, the angle made is a right angle

This can be shortened like this:

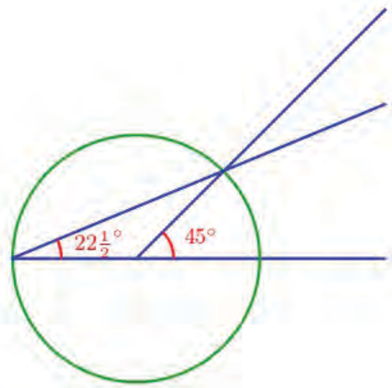
The angle in a semicircle is a right angle

Let's look at some applications of these results

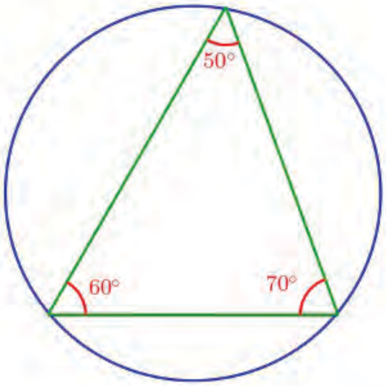
We can use the first result to draw half a given angle . See this picture :



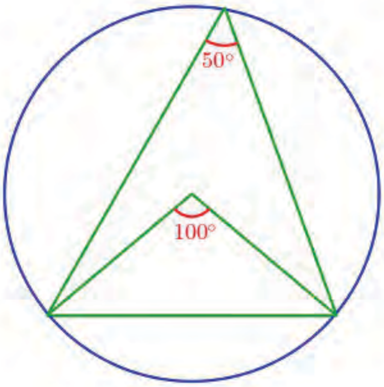
The circle is drawn with centre at the vertex of the angle . Now if we extend the bottom line of the angle to meet the circle, and join this point to the point where the top line of the angle intersects the circle , we have half the angle

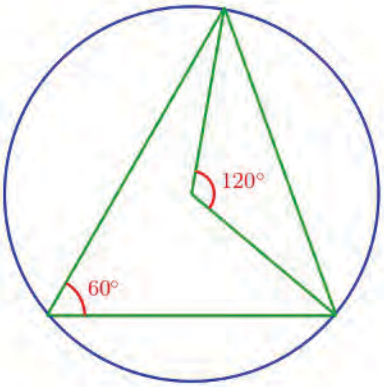


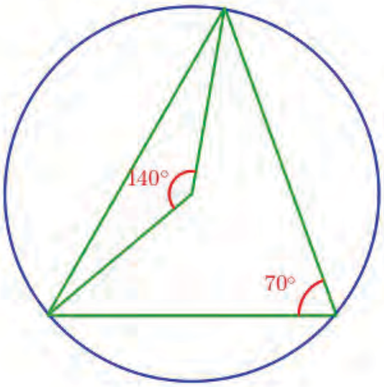
We can also use the result to draw a triangle with specified angles within a circle. For example, let's see how we can draw a triangle of angles 50°, 60° and 70° within a circle.



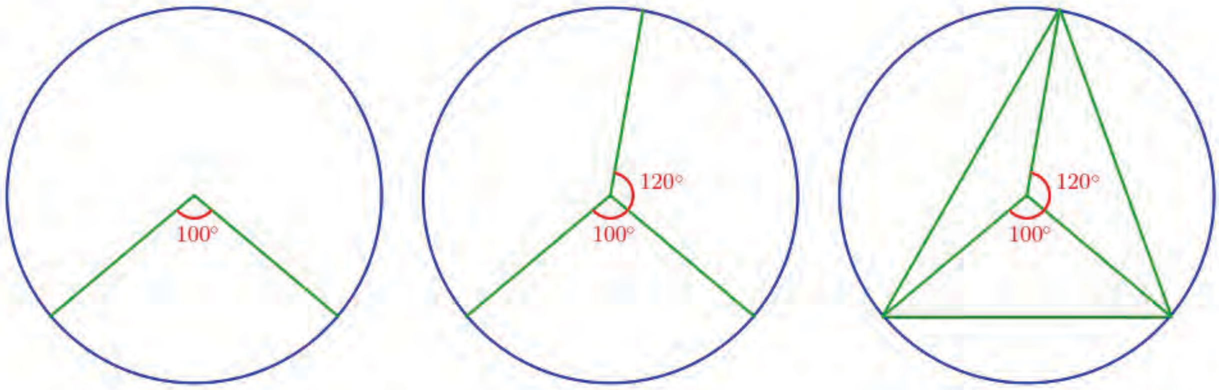
The vertices of the triangle split the circle into three arcs. What are their central angles ?





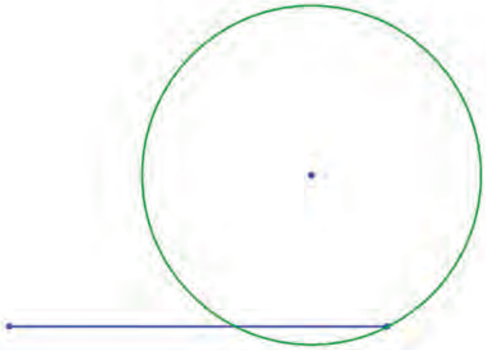


So to draw the triangle , we need only draw two of these angles at the centre :

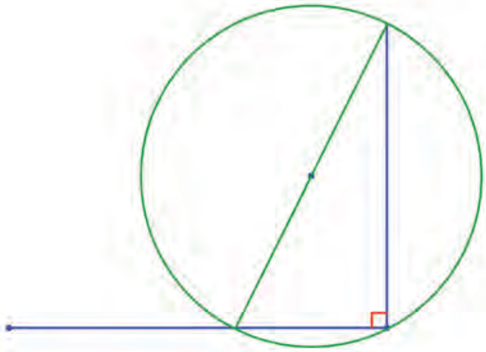


Next we use the result that the angle in a semicircle is a right angle, to draw a perpendicular to a line. Suppose we want to draw the perpendicular to the line below through its right endpoint :

For that , we first draw a circle which passes through this point and intersects the line at another point:



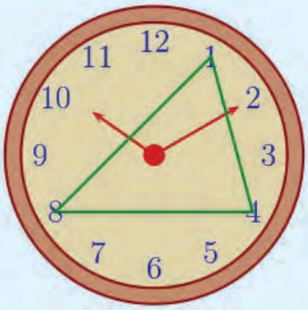
Next draw the diameter through this point and join its other end to the end of the first line:



Since the angle in a semicircle is a right angle , this line is perpendicular to the first line

Remember another method of drawing a perpendicular to a line using ruler and compass, seen in Class 8 ? (The section **Bisectors of a line** in the lesson **Bisectors**).

(1) A triangle is drawn joining the numbers 1, 4 and 8 on a clock face:



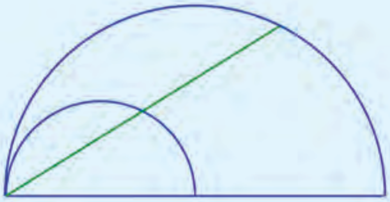
(i) Calculate the angles of this triangle.

(ii) How many equilateral triangles can we make by joining the numbers on a clock face ?

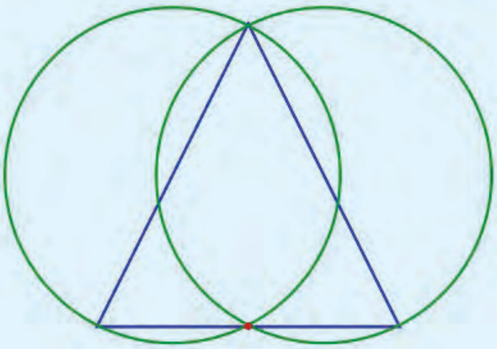
(2) Draw an equilateral triangle with circumradius 3.5 centimetres.

(3) Draw a triangle with circumradius 3 centimetres and two of the angles $32\frac{1}{2}^{\circ}$ and $37\frac{1}{2}^{\circ}$.

(4) In the picture , a semicircle is drawn with a line as diameter and a smaller semicircle with half this line as diameter . Prove that a line joining the point where the semicircles meet, to any point on the larger semicircle is bisected by the smaller semicircle:

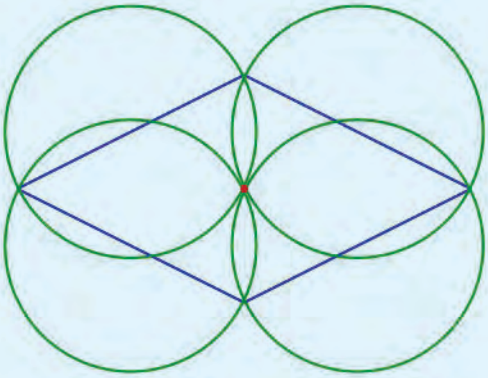


(5) Prove that circle drawn on the equal sides of an isosceles triangle as diameters pass through the midpoint of the third side:



(Hint : Consider the circles one by one).

(6) Prove that all the circles drawn on the four sides of a rhombus as diameters pass through a common point:

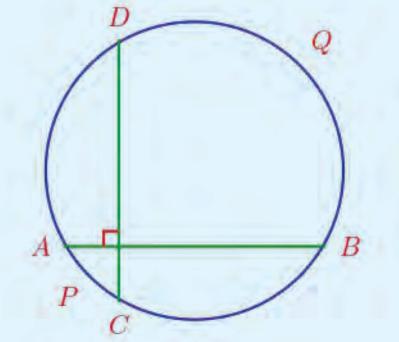


(7) In the picture below, a triangle is drawn joining the ends of the diameter of a circle and another point on the semicircle; and then semicircles on the other sides of the triangle as diameters:

Prove that the sum of the areas of the blue and red crescents in the second picture is equal to the area of the triangle

(Hint : See the last problem of the lesson, **Parts of Circles** of Class 9 textbook).

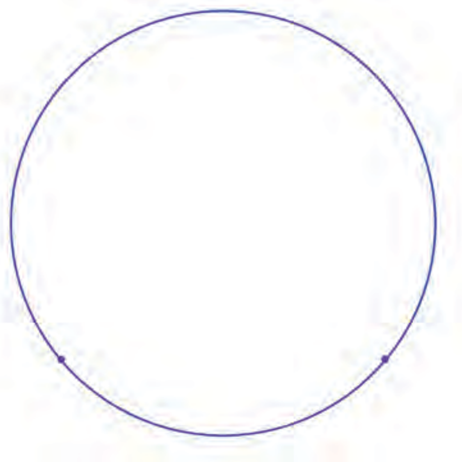
(8) In the picture, AB and CD are perpendicular chords of the circle:

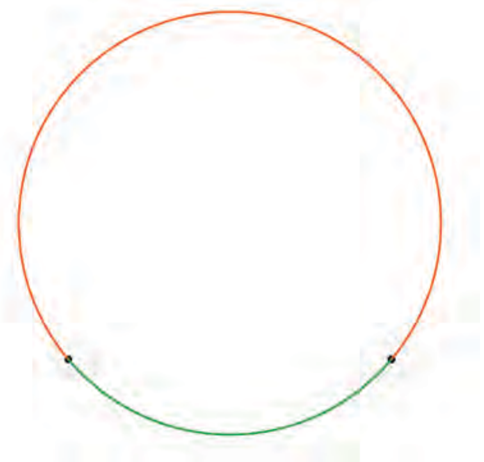


Prove that the arcs APC and BQD joined together make a semicircle.

Segments of a circle

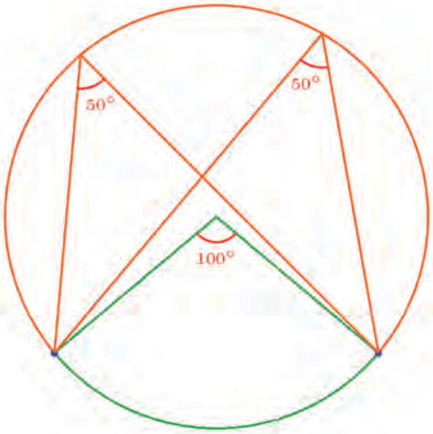
Any two points on a circle splits the circle into two arcs :

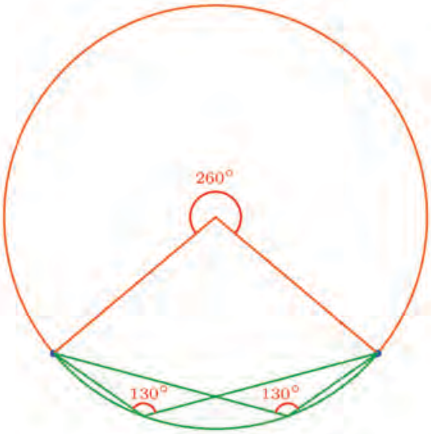




Each of these arcs may be called the **alternate arc** of the other . Any point on the circle is either on one of these arcs or its alternate arc . So , the relation between the central angle of an arc and the angle made by joining the ends of the arc to a point on the circle can be stated like this

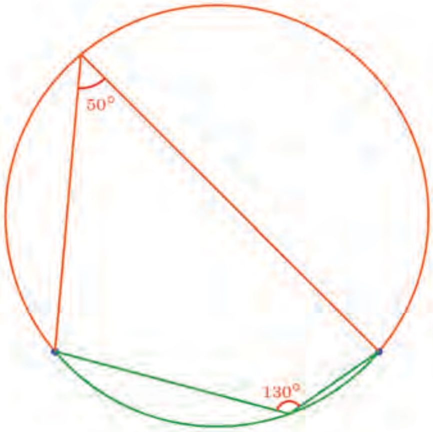
The angle made by joining the ends of an arc of a circle to any point on the alternate arc is half the central angle of the arc



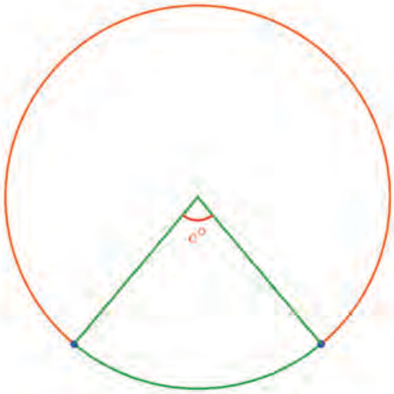


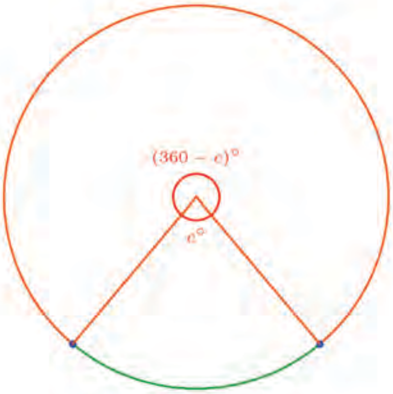
In other words , two points on a circle divide the circle into two arcs. The angles made by joining these points to any point on one of the arcs are all equal.

What is the relation between the angle made by joining these points to points on one arc and its alternate arc ?

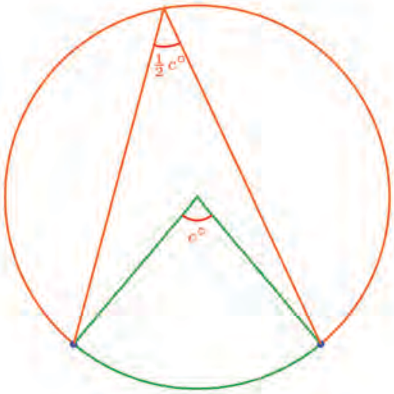


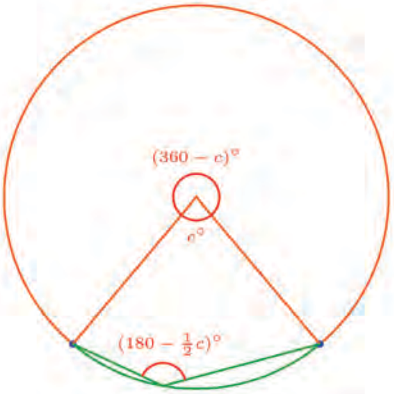
The sum of the central angles of an arc and its alternate arc is 360°, isn't it ?





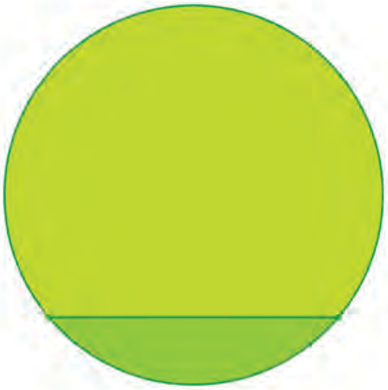
And the angles made at points on each arc is half its central angle:





So what can we say ?

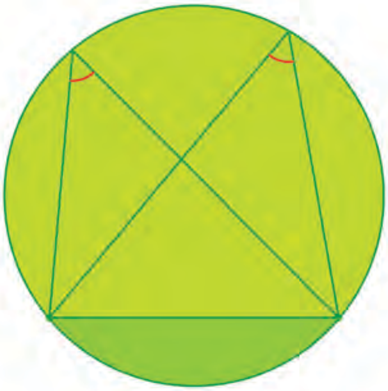
> The sum of the angles made by two points on a circle, at points on one of the arcs and the alternate arc, is 180°

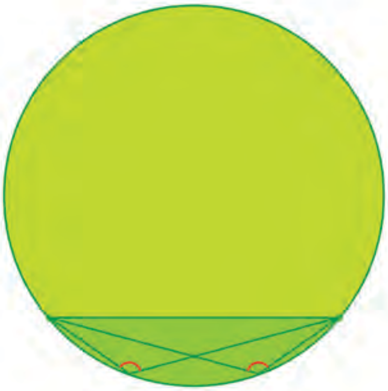


This can put in another way. A chord joining two points on a circle splits the entire circular region into two parts.

Such parts of a circle are called **segments** of the circle. Of the two segments into which a chord cuts a circle, each may be called the **alternate segment** of the other.

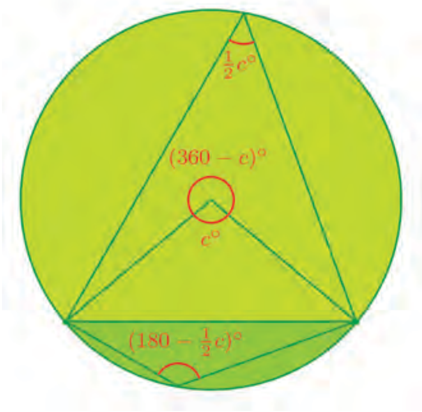
So the result about the angles in a circle can be stated in terms of segments also, instead of arcs.



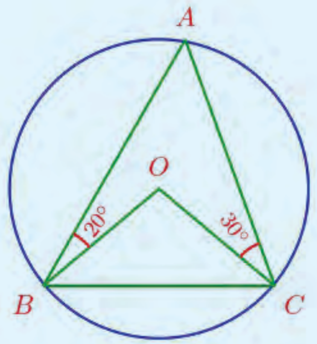


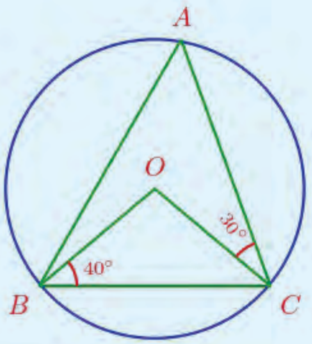
In a circle , angles in the same segment are equal; the sum of the angles in alternate segments is 180°

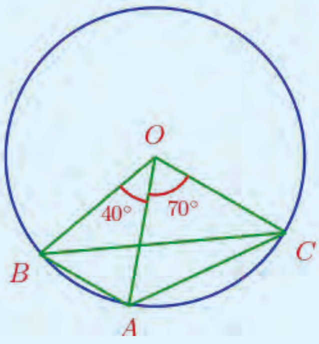
The relations between the central angles of the two arcs into which two points divide a circle, and the angles in the segments into which the chord joining these points divide the circular region, are shown in the figure below:



(1) In all three pictures below, O is the centre of the circle and A, B, C are points on the circle. In each , calculate all the angles of triangles ABC and OBC:







(2) In each of the problem below , a circle and a chord is to be drawn to split the circle into two parts. The parts must be as specified:

(i) All angles in one part must be 80°

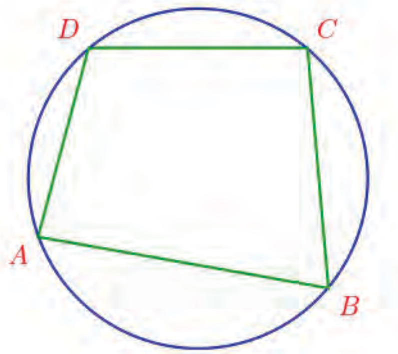
(ii) All angles in one part must be 110°

(iii) All angles in one part must be half the angles in the other part

(iv) All angles in one part must be one and a half times the angles in the other part

Circle and quadrilateral

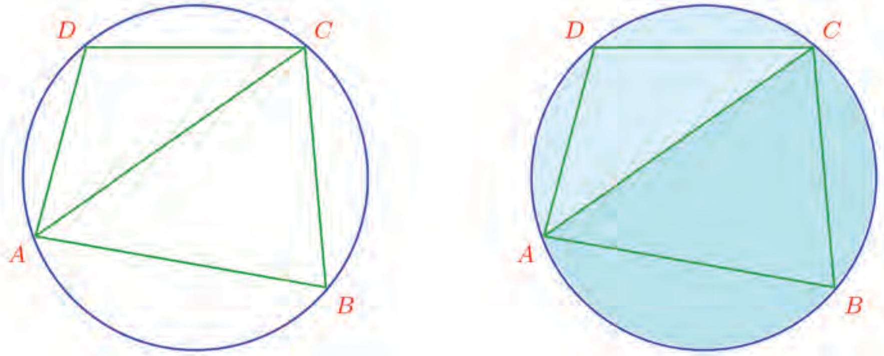
See this picture:



Draw a circle in GeoGebra and draw a quadrilateral with vertices on the circle, using the Polygon tool. Mark all its angles (Just select the Angle tool and click inside the quadrilateral) Check the relation between the opposite angles. Move the vertices on the circle and see what happens

A quadrilateral is drawn joining four points on a circle. Is there any relation between the opposite angles in this quadrilateral ?

If you don't see it, join *AC* and look again:



Now the angles at *B* and *D* are the angles in alternate segments into which the chord *AC* splits the circle. So their sum is 180°

If we join *BD* instead of *AC*, we can similarly see that the sum of the angles at *A* and *C* is also 180°

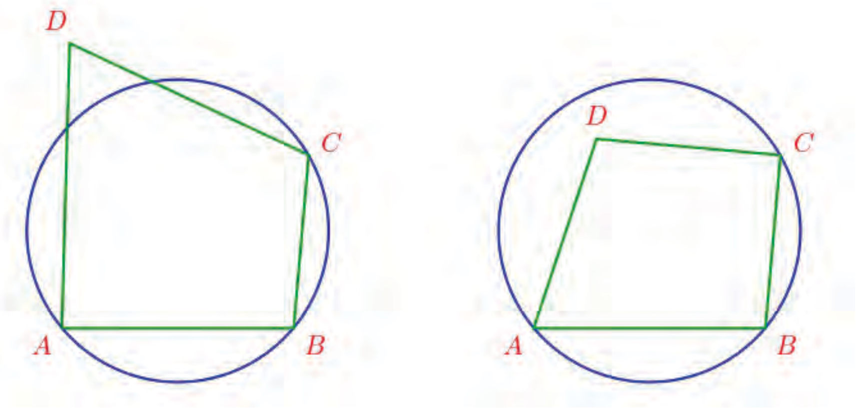
So what can we say in general ?

If all vertices of a quadrilateral are on a circle, then the sum of its opposite angles is 180°

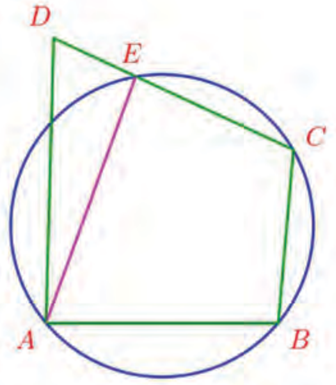
Is the reverse statement true ? That is, if the sum of opposite angles of a quadrilateral is 180°, can we draw a circle through all its vertices ?

We think like this. We can draw a circle through three vertices of any quadrilateral (The section **Lines and circles** of the chapter **Circles** in the Class 8 textbook).

The fourth vertex may be on the circle; or it may be inside or outside the circle. We have seen that if it is on the circle , then the sum of the angles at this vertex and the opposite one is 180°



Look at the first picture, If we join the point where the circle intersects *CD*, and the point *A*, we get a quadrilateral with all four vertices on the circle:



Since *A*, *B*, *C*, *E* are on the circle,

$$\angle B + \angle AEC = 180° \qquad (1)$$

Here $\angle AEC$ is the outer angle at *E* of triangle *AED*. So,

$$\angle AEC = \angle D + \angle EAD$$

This shows that

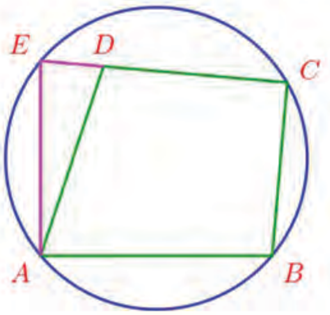
$$\angle D < \angle AEC \qquad (2)$$

Now if we think about the meaning of the algebraic statements marked (1) and (2), we can see that

$$\angle B + \angle D < 180°$$

Thus if the fourth vertex is outside the circle , the sum of the angles at this vertex and the opposite one is less than 180°

Next in the second picture, extend the line *CD* and join the point where it meets the circle, and the point *A*:



Using the Polygon tool, draw a quadrilateral ABCD. Using the Circle through 3 points tool, Draw a circle passing through points A, B, C. Mark the angles of the quadrilateral. Observe ∠D, ∠B. Change the position of D. What is the relationship between these angles when the point D is outside the circle ? When inside the circle ? What if the point D is on the circle ?

In this we have

$$\angle B + \angle E = 180° \qquad (3)$$

Also from triangle *AED*, we can see that

$$\angle ADC = \angle E + \angle EAD$$

so that

$$\angle ADC > \angle E \qquad (4)$$

From the relations (3) and (4) we get

$$\angle B + \angle ADC > 180°$$

So if the fourth vertex is inside the circle, the sum of the angles at this vertex and the opposite one is greater than 180°

> If a circle is drawn through three vertices of a quadrilateral and the fourth vertex is outside this circle, then the sum of the angles at this vertex and the opposite vertex is less than 180°; if the fourth vertex is inside the circle, the sum of these angles is greater than 180°

Now suppose that in a quadrilateral *ABCD*, we have $\angle B + \angle D = 180°$ and we draw the circle through *A*, *B* and *C*.

Can *D* be outside the circle ? If it is, then the sum of the angles at *B* and *D* would be less than 180°. So, *D* is not outside the circle.

Can *D* be inside the circle ? If it is , then the sum of the angles at *B* and *D* would be greater than 180°. So, *D* is not inside the circle.

Since it is neither outside nor inside the circle, *D* must be on the circle.

Thus we have this result

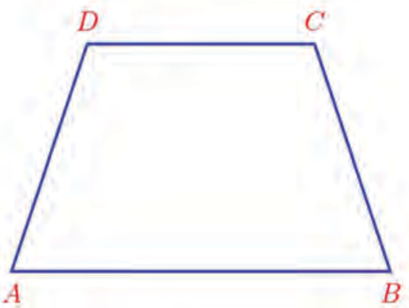
If in a quadrilateral, the sum of opposite angles is 180°, then a circle can be drawn passing through all four of its vertices

A quadrilateral with the property that a circle could be drawn through all four of its vertices, is called a **cyclic quadrilateral.**

What we have seen just now is that a cyclic quadrilateral is a quadrilateral with the sum of opposite angles 180°.

All rectangles are cyclic quadrilaterals, right ?

We can also show that any isosceles trapezium is cyclic. See this picture:



$ABCD$ is an isosceles trapezium. So,

$$\angle A = \angle B$$

Also, since AB and CD are parallel,

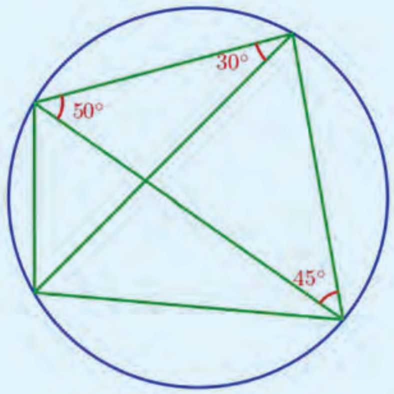
$$\angle A + \angle D = 180°$$

From these two equations, we get

$$\angle B + \angle D = 180°$$

So $ABCD$ is a cyclic quadrilateral

(1) Calculate the angles of the quadrilateral shown below and also the angles between its diagonals

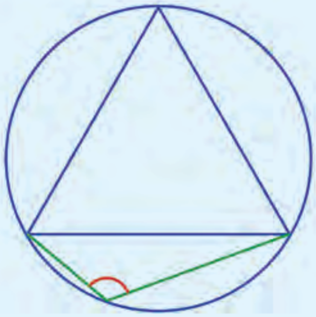


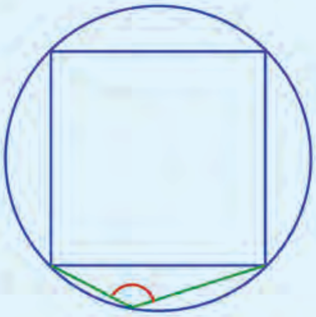
(2) Prove that in a cyclic quadrilateral, the outer angle at any vertex is equal to the inner angle at the opposite vertex

(3) Prove that any parallelogram, which is not a rectangle, is not cyclic.

(4) Prove that a non - isosceles trapezium is not cyclic.

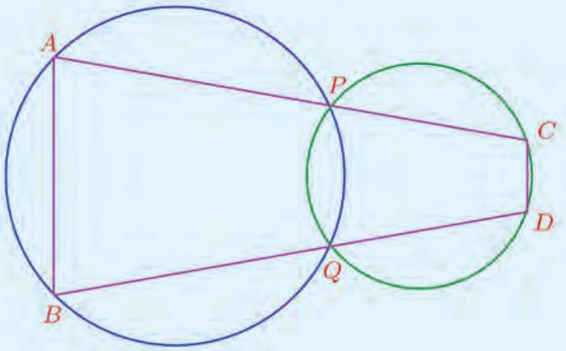
(5) In the first picture below , an equilateral triangle is drawn with vertices on a circle and two of its vertices are joined to a point on the circle. In the second picture, a square is drawn with vertices on a circle and two of its vertices are joined to a point on the circle :



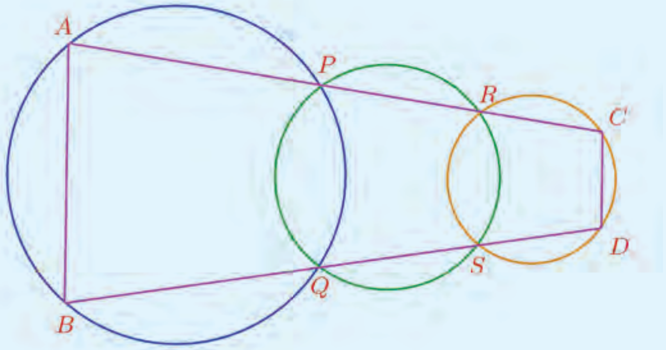


In each picture, calculate the angle marked.

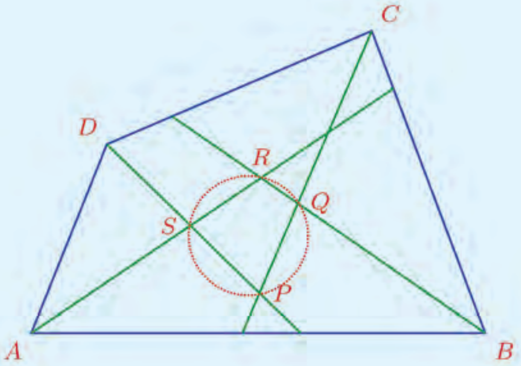
(6) (i) In the picture below , two circles intersect at *P* and *Q*. Lines through these points meet the circles at *A*, *B*, *C*, *D*. The lines *AC* and *BD* are not parallel. Prove that if these lines are of equal length, then *ABDC* is a cyclic quadrilateral.



(ii) In the picture , the circles on the left and right intersect the middle circle at *P*, *Q*, *R*, *S*. Lines joining these meet the left and right circles at *A*, *B*, *C*, *D*. Prove that *ABDC* is a cyclic quadrilateral.



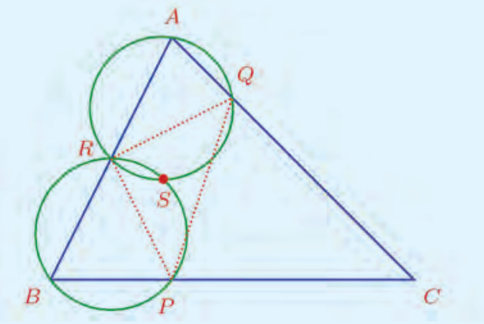
(7) In the picture, the bisectors of the angles of the quadrilateral *ABCD* intersect at *P*, *Q*, *R*, *S*.

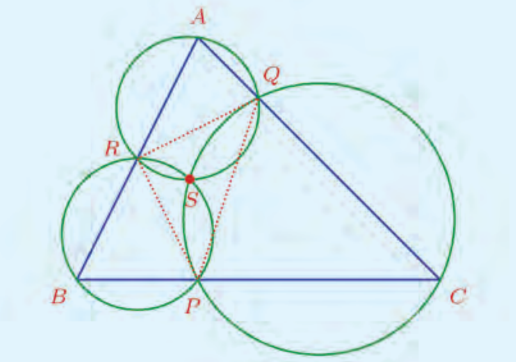


Prove that *PQRS* is a cyclic quadrilateral.

(Hint : Look at the sum of the angles of triangles *PCD* and *RAB*).

(8) In the first picture below, points P, Q, R are marked on the sides BC, CA, AB of triangle ABC and circumcircles of triangles AQR and BRP are drawn. They intersect at the point S inside the triangle :





Prove that the circumcircle of triangle CPQ also passes through S, as in the second picture.

(Hint : In the first figure, join PS, QS and RS. Then find the relations of the angles formed at the point S with $\angle A$, $\angle B$ and $\angle C$).

3

ARITHMETIC SEQUENCES AND ALGEBRA

Algebraic form

We have noted that a number sequence is an arrangement of numbers according to a definite rule. How do we state this rule?

For example, how do we describe the sequence 2, 4, 6, ... of even numbers?

The term at every position is twice the number giving the position:

Position	1	2	3	...
Term	$2 \times 1 = 2$	$2 \times 2 = 4$	$2 \times 3 = 6$	...

What about the sequence 1, 3, 5, ... of odd numbers?

The term at every position is one less than twice the number giving the position:

Position	1	2	3	...
Term	$(2 \times 1) - 1 = 1$	$(2 \times 2) - 1 = 3$	$(2 \times 3) - 1 = 5$	...

We can use algebra to concisely state these.

Thus we can describe the first sequence like this:

The number in the n^{th} position is $2n$

And the second one?

The number in the n^{th} position is $2n - 1$

In writing the terms of a general sequence as letters, it is not possible to use a different letter for each term. So, we use a single letter together with the position number to denote each term of a sequence. For example, we can write x_1 for the first term, x_2 for the second term, x_3 for the third term and so on. We can use different letters like x or y to denote different sequences

Using this notation, the two sequences above can be written like this:

The sequence 2, 4, 6, ... of even numbers: $x_n = 2n$

The sequence 1, 3, 5, ... of odd numbers: $y_n = 2n - 1$

This representation of the relation between the positions and terms of a sequence can be called the algebraic form of the sequence.

Now let's see how we can write the algebraic form of any arithmetic sequence

For example, consider the sequence

12, 23, 34, ...

What is the next term of this sequence? And the term after that?

Instead of computing term by term like this, we can directly find the term at any position. How do we compute the 10th term of this sequence?

(i) The 1st term is 12

(ii) The common difference is 11

(iii) The change in position from the 1st term to the 10th term is $10 - 1 = 9$

(iv) The 10th term is $12 + (9 \times 11) = 111$

We can compute the term at any position like this.

In general how do we write the n^{th} term?

(i) The change in position from the 1st term to the n^{th} term is $n - 1$

(ii) The n^{th} term is $12 + ((n - 1) \times 11) = 12 + 11(n - 1)$

General forms

We have noted that the algebraic form of the sequence 1, 3, 5, ... of odd numbers is $x_n = 2n - 1$. Here the number n indicates the position of each term. Thus when n is taken as 1, 2, 3, ... we get the first terms as 1, second term as 3 and so on for all odd numbers.

In the lesson Algebra of the Class 7 textbook, the general form of an odd number is given as $2n + 1$ ($n = 0, 1, 2, ...$). In this n denotes the quotient on dividing the odd number by 2. So when we take $n = 0, 1, 2, ...$ we get the numbers with quotient 0, 1, 2, ... and remainder 1 on division by 2; that is the odd numbers 1, 3, 5, ...

In this, we can rewrite $11(n - 1)$ as

$$11(n - 1) = 11n - 11$$

This gives

$$12 + 11(n - 1) = 12 + (11n - 11) = 11n + 1$$

Thus the algebraic form of this sequence is

$$x_n = 11n + 1$$

In this if we take n as 1, 2, 3, ... we get the terms in these positions as $x_1, x_2, x_3, ...$

$$x_1 = (11 \times 1) + 1 = 12$$

$$x_2 = (11 \times 2) + 1 = 23$$

$$x_3 = (11 \times 3) + 1 = 34$$

....

Let's look at another example: the arithmetic sequence starting with $\frac{1}{4}$ and proceeding by successively adding $\frac{1}{2}$ is

$$\frac{1}{4}, \frac{3}{4}, 1\frac{1}{4}, 1\frac{3}{4}, \ldots$$

How do we compute its nth term?

(i) 1st term is $\frac{1}{4}$

(ii) Common difference is $\frac{1}{2}$

(iii) The position change from the 1st term to the n^{th} term is $n - 1$

(iv) n^{th} term is $\frac{1}{4} + (n-1) \times \frac{1}{2}$

$$\frac{1}{4} + (n-1) \times \frac{1}{2} = \frac{1}{4} + \left(n \times \frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}n - \frac{1}{4}$$

The algebraic form of the sequence is

$$x_n = \frac{1}{2}n - \frac{1}{4}$$

In the first example, the n^{th} term is got by multiplying n by 11 and adding 1

In the second example, the n^{th} term is got by multiplying n by $\frac{1}{2}$ and subtracting $\frac{1}{4}$, which means adding $-\frac{1}{4}$

Is the n^{th} term of every arithmetic sequence got by multiplying n by a fixed number and adding a fixed number ?

Let's take the first term of a general arithmetic sequence as f and the common difference as d. Then the n^{th} term of the sequence is got by adding $(n - 1)$ times the common difference d to the first term f.

Thus, the n^{th} term is

$$f + (n - 1)d = dn + (f - d)$$

This means the n^{th} term is got by multiplying d by n and adding $(f - d)$

So, what can we say in general?

In any arithmetic sequence, the term in any position is got by multiplying the position number by a fixed number and then adding a fixed number. In other words, the algebraic form of any arithmetic sequence is

$$x_n = an + b$$

The algebraic form of any arithmetic sequence is

$$x_n = an + b$$

where ***a*** and ***b*** are specific numbers.

In this, if we take as n, the natural numbers 1, 2, 3, ... we get the terms of the sequence as

$$a + b, 2a + b, 3a + b, \ldots$$

In other words,

> The terms of any arithmetic sequence are got by multiplying the natural numbers, 1, 2, 3, ... by a fixed number and adding a fixed number.

In such an arithmetic sequence, the common difference is the fixed number used to multiply the natural numbers.

Using this idea, the algebraic form of an arithmetic sequence can be easily found. For example, consider the arithmetic sequence got by starting with $\frac{1}{2}$ and successively adding $\frac{1}{3}$:

$$\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \ldots$$

Since the common difference is $\frac{1}{3}$, the terms of this sequence are got by multiplying the natural numbers by $\frac{1}{3}$ and adding another fixed number.

What is the number added?

The first term is this number added to $\frac{1}{3}$ itself

The first term is $\frac{1}{2}$. What number is to be added to $\frac{1}{3}$ to get $\frac{1}{2}$?

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Thus the terms of this arithmetic sequence are got by multiplying the natural numbers by $\frac{1}{3}$ and then adding $\frac{1}{6}$.

That is,

$$x_n = \frac{1}{3}n + \frac{1}{6}$$

From the algebraic form of an arithmetic sequence, we can gain much information about the sequence. For example, first note that the algebraic form of the sequence in this example can be written

$$x_n = \frac{n}{3} + \frac{1}{6} = \frac{2n + 1}{6}$$

Now $2n + 1$ is an odd number, whatever natural number we take as n; and the denominator 6 is an even number. So, in none of the fractions $\frac{2n + 1}{6}$ is the numerator a multiple of the denominator. Thus none of the terms of this sequence are natural numbers

The language of the law

We have noted that to find the terms of a sequence, the law of formation should be specified. And we have seen some examples of how such rules can be algebraically specified.

But not all sequences can be algebraically described. For example, no algebraic formula to find the n^{th} prime number has been discovered; that is, no formula to directly compute the number at a specified position in the sequence 2, 3, 5, 7, 11, 13, ... of primes is known.

Again, there is no algebraic formula to compute the digit at a specified location in the sequence 3, 1, 4, 1, 5, 9, ... got from the decimal representation of π. In such cases, we can only specify the rule of forming the terms in ordinary language.

Next look at the sequence got by starting with $\frac{1}{2}$ and repeatedly adding $\frac{1}{4}$.

The algebraic form of this sequence can be found as

$$x_n = \frac{n}{4} + \frac{1}{4}$$

This we can write as

$$x_n = \frac{n+1}{4}$$

In this, if we take n as 3, 7, 11, ... we get

$$x_3 = 1$$

$$x_7 = 2$$

$$x_{11} = 3$$

....

In other words, all natural numbers are terms of this sequence

(1) Find the algebraic form of the arithmetic sequences given below:

(i) 1, 6, 11, 16, ... (ii) 2, 7, 12, 17, ... (iii) 21, 32, 43, 54, ...

(iv) 19, 28, 37, ... (v) $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$ (vi) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \ldots$

(2) The terms of some arithmetic sequences in two specified positions are given below. Find the algebraic form of each:

(i) 1st term 5, 10th term 23

(ii) 1st term 5, 7th term 23

(iii) 5th term 10, 10th term 5

(iv) 8th term 2, 12th term 8

(3) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{1}{6}$ contains all natural numbers.

(4) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{2}{3}$ contains all odd numbers, but no even numbers.

(5) Prove that in the arithmetic sequence 4, 7, 10, ... the squares of all terms are also terms of the sequence.

(6) Prove that the arithmetic sequence 5, 8, 11, ... does not contain any perfect square.

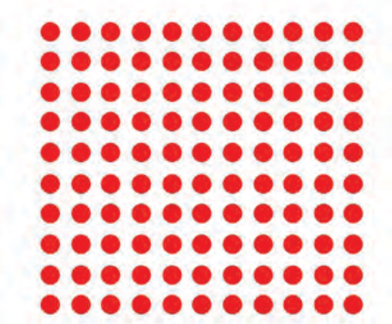
In the sequence of all even numbers, the square of each term is also a term of the sequence. How about other powers?

What about the sequence of odd numbers?

Are there other arithmetic sequences in which the powers of every term is again a term of the sequence?

Sums

See this picture:



How many dots are there?

No need to count one by one. There are 10 rows of dots with 11 dots in each, giving a total of

$$10 \times 11 = 110$$

How many dots in this triangle?

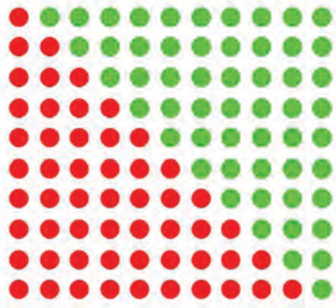
We can count one by one:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

Is there any easier way for this also?

For that, we make another triangle like this:

Turn it upside down and join with the first triangle:



This rectangle has 10 × 11 = 110 dots, as seen earlier.

So how many in each triangle?

Half of 110, which is 55

What we did with picture can be done with numbers also

We first write

$$s = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Adding in the other direction,

$$s = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Let's add the numbers in the same position in the two equations:

$$2s = 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 = 110$$

So,

$$s = 110 \div 2 = 55$$

In the same way, we can find the sum of natural numbers from 1 to 100:

$$\begin{aligned} s &= 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\ s &= 100 + 99 + 98 + \cdots + 3 + 2 + 1 \end{aligned}$$

Adding numbers in the same position,

$$\begin{aligned} 2s &= \overbrace{101 + 101 + 101 + \cdots + 101 + 101 + 101 + 101}^{100 \text{ times}} \\ &= 100 \times 101 \end{aligned}$$

A math tale

Haven't you heard about the mathematician, Gauss? It is said that he showed extraordinary mathematical ability right from an early age. There is a tale about it.

This happened when Gauss was just ten and in school. His teacher asked the children to add all numbers from 1 to 100, just to keep them quiet. Young Gauss did it in a flash, explaining his answer like this: 1 and 100 make 101, so does 2 and 99 and so on. 50 pairs of numbers, each with sum 101 make 50 × 101 = 5050

From this we get

$$s = \frac{1}{2} \times 100 \times 101 = 5050$$

We can find the sum up to any number instead of 100 in the same way. Thus

The sum of any number of consecutive natural numbers starting from one is the half the product of the last number and the next

In the language of algebra,

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$$

We can use this to find the sum of other arithmetic sequences. For example, consider the sequence 2, 4, 6, ..., 100 of even numbers.

We can write

$$2 + 4 + 6 + \cdots + 100 = 2(1 + 2 + 3 + \ldots +50)$$

In this,

$$1 + 2 + 3 + \cdots + 50 = \frac{1}{2} \times 50 \times 51$$

as seen before

So,

$$2 + 4 + 6 + \cdots + 100 = 2 \times \frac{1}{2} \times 50 \times 51 = 2550$$

In general, the first n even numbers are

$$2, 4, 6, \ldots, 2n$$

and their sum is

$$2 + 4 + 6 + \cdots + 2n = 2 \times \frac{1}{2} \times n(n + 1) = n(n + 1)$$

Now the even numbers 2, 4, 6, ... are multiples of 2; what if we take multiples of 3 instead?

The first n terms of this sequence are

$$3, 6, 9, \ldots, 3n$$

and their sum is

$$\begin{aligned} 3 + 6 + 9 + \cdots + 3n &= 3(1 + 2 + 3 + \cdots + n) \\ &= 3 \times \frac{1}{2}n(n+1) \\ &= \frac{3}{2}n(n + 1) \end{aligned}$$

Can't you find the sums of multiples of 4 and multiples of 5 like this? Try:

Let's now look at the sum of the first n odd numbers

First let's write the sequence of odd numbers in algebraic form

This is the arithmetic sequence which starts at 1 and proceeds by repeatedly adding 2.

So if we write x_n for the n^{th} odd number,

$$x_n = 1 + ((n - 1) \times 2) = 2n - 1$$

We get the sequence of odd numbers by taking 1, 2, 3, ... as n.

So, we can write this sequence as

$$\begin{aligned} x_1 &= (2 \times 1) - 1 \\ x_2 &= (2 \times 2) - 1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= (2 \times n) - 1 \end{aligned}$$

If we add these from top to bottom, we get

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= ((2 \times 1) + (2 \times 2) + \cdots + (2 \times n)) - \overbrace{(1 + 1 + \cdots + 1)}^{n \text{ times}} \\ &= 2(1 + 2 + \cdots + n) - n \end{aligned}$$

In this, we have

$$1 + 2 + \cdots + n = \frac{1}{2}\, n(n + 1)$$

Using this, we get

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 2 \times \frac{1}{2} n(n + 1) - n \\ &= n(n + 1) - n \\ &= n^2 + n - n = n^2 \end{aligned}$$

Thus the sum of any number of consecutive odd numbers starting with 1 is the square of that number.

We can compute the sum of consecutive terms of any arithmetic sequence like this

Any arithmetic sequence we can write as

$$x_n = an + b$$

To compute the sum of the first n terms of these, we take $n = 1, 2, 3, \ldots$ in this and add

$$\begin{aligned} x_1 &= a + b \\ x_2 &= 2a + b \\ &\ldots\ldots\ldots\ldots \\ x_n &= na + b \end{aligned}$$

Adding as before,

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= (a + 2a + \cdots + na) + \overbrace{(b + b + \cdots + b)}^{n \text{ times}} \\ &= a(1 + 2 + \cdots + n) + nb \\ &= \tfrac{1}{2}an(n + 1) + nb \end{aligned}$$

For the arithmetic sequence given by

$$x_n = an + b$$

the sum of the first n terms is,

$$x_1 + x_2 + \cdots + x_n = \tfrac{1}{2}an(n + 1) + nb$$

As an example, let's compute the sum of the first 100 terms of the arithmetic sequence

$$1, 4, 7, \ldots$$

The algebraic form of this sequence is

$$x_n = 3n - 2$$

So, the sum of the first 100 terms is

$$3 \times \tfrac{1}{2} \times 100 \times 101 - (2 \times 100) = 14950$$

In general, the sum of the first n terms of this sequence is

In GeoGebra, we can use the sum command to compute the sum of a sequence.

The command

```
L=Sequence(n^2,n,1,10)
```

in the input bar gives the list of first ten perfect squares. The command `sum (L)` gives the sum of these ten numbers. We can directly type `sum(n^2,n,1,10)`. To get the sum of the first 20 terms of the arithmetic sequence 5, 8, 11, ... type

```
sum(3n+2,n,1,20)
```

To get the sum of the 10th to the 20th terms of this sequence, type

```
sum(3n + 2,n,10,20).
```

$$3 \times \tfrac{1}{2}n(n + 1) - 2n = 3 \times \tfrac{1}{2}n^2 + \tfrac{3}{2}n - 2n = \tfrac{1}{2}(3n^2 - n)$$

The sum of an arithmetic sequence can be computed in another manner. For this, we first rewrite the algebraic form of the sum like this:

$$\begin{aligned} \tfrac{1}{2}an(n + 1) + bn &= \tfrac{1}{2}n(a(n + 1) + 2b) \\ &= \tfrac{1}{2}n(an + a + 2b) \\ &= \tfrac{1}{2}n\,((an + b) + (a + b)) \end{aligned}$$

In this, $an + b$ is the n^{th} term of the sequence and $a + b$ is the first term

The sum of the first n terms of the arithmetic sequence $x_1, x_2, x_3, \ldots$ is

$$x_1 + x_2 + \ldots + x_n = \frac{1}{2}n(x_1 + x_n)$$

This can be put in ordinary language

The sum of consecutive terms of an arithmetic sequence is half the product of the sum of the first and last terms by the number of terms

To compute the sum of the first 100 terms of the arithmetic sequence 1, 4, 7, ... using this, first find the 100th term as

$$1 + (99 \times 3) = 298$$

Then we can compute the sum of the first 100 terms as

$$\frac{1}{2} \times 100 \times (1 + 298) = 14950$$

The sum of n terms of all arithmetic sequences have a common algebraic form. To see this, we write the sum as

$$\frac{1}{2}an(n + 1) + bn = \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$$

In this $\frac{1}{2}a$ and $\frac{1}{2}a + b$ are definite numbers associated with the sequence. So, the sum is the sum of the products of n^2 and n by fixed numbers

The algebraic form of the sum of the first n terms of an arithmetic sequence is

$$pn^2 + qn$$

where p and q are specific numbers

For example, it is not difficult to see that the algebraic form of the sequence

$$3, 10, 17, \ldots.$$

is

$$7n - 4$$

What is the sum of the first n terms?

$$\frac{1}{2} \times 7 \times n(n + 1) - 4n = \frac{1}{2}(7n^2 + 7n - 8n) = \frac{1}{2}(7n^2 - n)$$

In detail, if we put $n = 1$ in the algebraic expression $\frac{1}{2}(7n^2 - n)$, we get 3, which is the first term of the sequence.

Putting $n = 2$, we get 13, which is the sum of the first two terms

Putting $n = 3$, we get 30, which is the sum of the first three terms and so on

Now a question in the other direction:

> The sum of the first n terms of an arithmetic sequence is
>
> $$3n^2 + n$$
>
> What is the sequence?

How do we do this?

We can think like this:

(i) We get the first term by taking $n = 1$ in the algebraic form of the sum

(ii) $(3 \times 1^2) + 1 = 4$

(iii) We get the sum of the first two terms by taking $n = 2$ in the algebraic form of the sum

(iv) The sum of the first two terms is $(3 \times 2^2) + 2 = 14$

(v) Subtracting the first term from this gives the second term

(vi) Second term is $14 - 4 = 10$

(vii) The sequence is 4, 10, 16, ...

(viii) The algebraic form of the sequence is $x_n = 6n - 2$

If we know the algebraic form of a sequence, we can use the sum function of the Python language to find the sum of any number of consecutive terms. For example, to get the sum of the first 100 perfect squares, type.

```
sum(x**2 for x in range(1,101))
```

at the python prompt.

Sum of squares

We have seen the identity $(x + 1)^2 = x^2 + 2x + 1$. Another such identity is

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$$

From this, we can see that for any number

$$(x + 1)^3 - x^3 = 3x^2 + 3x + 1$$

If we take $x = 1, 2, 3, \ldots, n$ in this we get,

$$\begin{array}{ll} x = 1, & 2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1 \\ x = 2, & 3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1 \\ \cdots\cdots & \cdots\cdots\cdots \\ x = n - 1, & n^3 - (n - 1)^3 = 3(n - 1)^2 + 3(n - 1) + 1 \\ x = n, & (n + 1)^3 - n^3 = 3n^2 + 3n + 1 \end{array}$$

If we add all these, we get

$$2^3 - 1^3 + 3^3 - 2^3 + \ldots + n^3 - (n - 1)^3 + (n + 1)^3 - n^3$$
$$= 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + n$$

$$(n + 1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + n$$

That is,

$$n^3 + 3n^2 + 3n$$
$$= 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + \frac{3}{2}n(n + 1) + n$$

So, we get

$$1^2 + 2^2 + 3^2 + \ldots + n^2$$

$$= \frac{1}{3}\left(n^3 + 3n^2 + 3n - \frac{3}{2}n(n + 1) - n\right)$$

Simplifying the right side of this equation, we get

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

We can put it like this. The algebraic form of the arithmetic sequence

$$4, 10, 16, \dots$$

is

$$x_n = 6n - 2$$

If we calculate the first term, the sum of the first two terms, the sum of the first three terms and so on, we get another sequence

$$4, 14, 30, \dots$$

And the algebraic form of this sequence

$$y_n = 3n^2 + n$$

(1) Calculate in head the sum of the arithmetic sequences below:

(i) $51 + 52 + 53 + \dots + 70$

(ii) $1\frac{1}{2} + 2\frac{1}{2} + \dots + 12\frac{1}{2}$

(iii) $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + \dots + 12\frac{1}{2}$

(iv) $\frac{1}{101} + \frac{3}{101} + \frac{5}{101} + \dots + \frac{201}{101}$

(2) Calculate the sum of the first 25 terms of each of the arithmetic sequences below:

(i) 11, 22, 33, ... (ii) 12, 23, 34, ... (iii) 21, 32, 43, ... (iv) 19, 28, 37, ...

(3) Find the sum of all multiples of 9 among three-digit numbers.

(4) The n^{th} term of some arithmetic sequences are given below. Find the sum of the first n terms of each:

(i) $2n + 3$ (ii) $3n + 2$ (iii) $2n - 3$ (iv) $3n - 2$

(5) The sum of the first n terms of some arithmetic sequences are given below. Find the n^{th} term of each:

(i) $n^2 + 2n$ (ii) $2n^2 + n$ (iii) $n^2 - 2n$ (iv) $2n^2 - n$ (v) $n^2 - n$

(6) (i) Calculate the sum of the first 20 natural numbers

(ii) Calculate the sum of the first 20 numbers got by multiplying the natural numbers by 5 and adding 1. Calculate also the sum of the first *n* terms.

(7) How much more is the sum of the first 25 terms of the arithmetic sequence 15, 21, 27, ... than the sum of the first 25 terms of the arithmetic sequence 7, 13, 19, ...?

(8) The 10th term of an arithmetic sequence is 50 and the 21st term is 75. Calculate the sum of the first 30 terms of this sequence.

We know that the sum of any number of consecutive odd numbers starting with 1 is a perfect square

Calculate the sum of the first few consecutive terms of the sequence 4, 12, 20, ...

Are there other arithmetic sequences with the sum of any number of consecutive terms from the first is a perfect square?

4

MATHEMATICS OF CHANCE

Chances as numbers

There are ten beads in a box, nine black and one white. If we pick one from this, without looking, it's most likely to be black, can be white though.

There is another box, with eight black and two white beads. If we pick one from it, again it's more likely to be black than white.

A third box contains five black and five white beads. What if we pick one from this? Could be white or black; can't say anything more, can we?

These can be put in slightly different words: from the first and second, drawing a black is more probable; from the third, drawing a black and drawing a white are equally probable.

Think of a game with boxes and beads. A box contains five black beads and five white; another contains six black and four white. We have to choose a box and then pick a bead from it. If it's a black, we win. Which box should we choose?

The second box contains more black beads; so getting a black from it is more probable, right?

What if we take a black bead from the second box and put it in the first?

This is how the boxes and beads are now:

First box : 6 black 5 white

Second box : 5 black 4 white

Now if we play the game, which box is the better choice?

Now the first box has more black beads. Is the probability of getting a black from it also more?

Let's think in terms of totals.

The first box has 11 beads in all, 6 of them black. That is, $\frac{6}{11}$ of the total is black

What about the second box? $\frac{5}{9}$ of the total number of beads is black

Which is greater, $\frac{6}{11}$ or $\frac{5}{9}$?

$\frac{5}{9}$ is larger than $\frac{6}{11}$, right ?

(The section **Large and Small** of the lesson **Arithmetic of Parts** in the Class 6 textbook)

So, the second box is still the better choice, isn't it?

In other words, getting a black bead from the second box is more probable.

In fact we can say that the probability of a black bead from the first box is $\frac{6}{11}$, and the probability of a black bead from the second box is $\frac{5}{9}$.

What about the probability of getting a white bead?

$\frac{5}{11}$ from the first and $\frac{4}{9}$ from the second; which is greater?

So, if the winning draw is white, which box is the better choice?

We can tabulate the various probabilities in this problem like this:

		First box		Second box	
		Black	White	Black	White
Beginning	Number	5	5	6	4
	Probability	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Later	Number	6	5	5	4
	Probability	$\frac{6}{11}$	$\frac{5}{11}$	$\frac{5}{9}$	$\frac{4}{9}$

Another question: we have seen that both at the beginning and also after transferring a bead, the probability of getting a black from the second box is higher; is the probability less or more after the transfer?

(1) A box contains 6 black and 4 white balls. If a ball is picked from it, what is the probability that it is black? And the probability that it is white?

(2) A bag contains 3 red balls and 7 green balls. Another contains 8 red and 7 green

(i) If a ball is drawn from the first bag, what is the probability that it is red?

(ii) From the second bag?

(iii) The balls in both bags are put together in a single bag. If a ball is drawn from this, what is the probability that it is red?

(3) A bag contains 3 red beads and 7 green beads. Another bag contains one more of each. The probability of getting a red from which bag is greater?

Number probability

Let's look at another problem:

Each of the numbers from 1 to 25 is written in a slip and all these are put in a box. One slip is taken out. What is the probability that the number drawn is an even number ?

Among the 25 numbers, 13 are odd and 12 are even, right ? So, the probability of getting an even number is $\frac{12}{25}$.

What about the probability of drawing an odd number ?

What is the probability of getting a multiple of 3 from this box? And a multiple of 6?

Now look at this problem:

If a number is drawn from the numbers 1, 2, 3, ..., 1000, what is the probability that it is a factor of 1000?

To calculate it, we must first compute the number of factors of 1000 among the numbers from 1 to 1000.

All the factors of 1000 are among these, right ? So, we need only find the number of factors of 1000.

Remember doing such computations in Class 7 ? (The section **Number of factors** in the lesson **Number Relations**).

Writing 1000 as the product of powers of distinct primes, we get

$$1000 = 2^3 \times 5^3$$

So, the number of factors is

$$(3 + 1) \times (3 + 1) = 4 \times 4 = 16$$

Thus 16 of the numbers among the numbers 1, 2, 3, ..., 1000 are factors of 1000.

So, the probability of a number drawn from these numbers to be a factor of 1000 is

$$\frac{16}{1000} = \frac{2 \times 8}{125 \times 8} = \frac{2}{125}$$

Probabilities are sometimes given in decimal form or as a percent.

The probability in this problem can be put in decimal form. Since

$$\frac{16}{1000} = 0.016$$

we can say that the probability is 0.016

Or we can convert the fraction into a percent. Since

$$\frac{16}{1000} \times 100 = \frac{16}{10} = 1.6$$

we can say that the probability is 1.6%

(1) In each of the problems below, compute the probability as a fraction and then write it in decimal form and as a percent

   (i) If one number from 1, 2, 3, ..., 10 is chosen, what is the probability that it is a prime number?

   (ii) If one number from 1, 2, 3, ..., 100 is chosen, what is the probability that it is a two-digit number?

   (iii) Every three digit number is written in a slip of paper, and all the slips are put in a box. If one slip is drawn, what is the probability that it is a palindrome ? (See Problem (4) at the end of the section **Growing numbers** of the lesson **Number World** in the Class 5 textbook).

(2) A person is asked to say a two-digit number. What is the probability that it is a perfect square?

(3) A person is asked to say a three-digit number.

   (i) What is the probability that all three digits of this number are the same?

   (ii) What is the probability that the digit in one's place of this number is zero?

   (iii) What is the probability that this number is a multiple of 3?

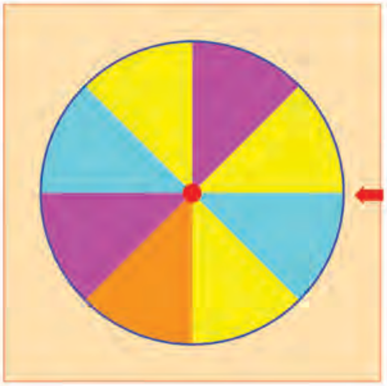
(4) Four cards with numbers 1, 2, 3, 4 on them, are joined to make a four-digit number.

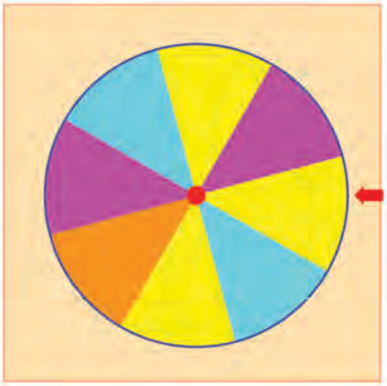
   (i) What is the probability that the number is greater than four thousand?

   (ii) What is the probability that the number is less than four thousand?

Geometrical probability

A multicoloured disc is mounted on a board, so that it can freely rotate. What is the probability of getting yellow against the arrow when it stops rotating?





When the disc stops rotating, any of the eight sectors may be against the arrow mark. And three of these are yellow. So,the probability of getting yellow against the mark is $\frac{3}{8}$.

Now compute the probabilities of getting the other colours.

Let's look at another problem

Cut out a rectangular piece of cardboard and draw a triangle joining the midpoint of one side to the endpoints of the opposite side:

If we mark a point within the rectangle, with eyes closed, what is the probability that it would be within the triangle?

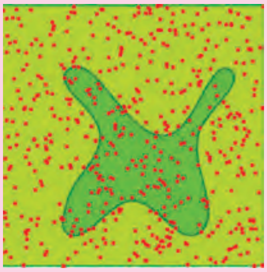
The triangle has the same base and height as the rectangle. So, it is half the rectangle.

Draw a circle in GeoGebra and mark three points on it. Draw the triangle with these points as vertices (Use the Polygon tool). If we close our eyes and touch within the circle, what is the probability that it is within the triangle? If the circle is named c and the triangle t1, then to calculate this probability, type Area(t1)/Area(c) in the input bar.

Move the vertices of the triangle and see when this probability is maximum and when it is minimum. To get more decimal places in the probability value, choose Options → Rounding and increase number of decimal places.

Probability and area

We can use probability to estimate the area of complicated figures. The figure is drawn within a square and then a large number of dots are marked within the square, without any order or scheme.

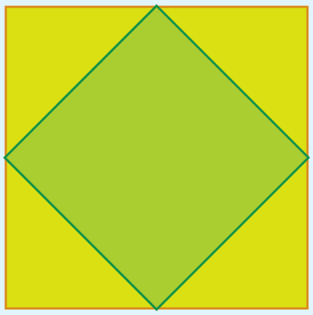


The number of dots falling within our figure, divided by the total number of dots gives an approximation of the area of the figure divided by the area of the square. And this approximation gets better, as we increase the number of dots. Both the geometric operation of marking the dots and the arithmetic operation of division can be done very fast, using computers. This is called the Monte Carlo Method

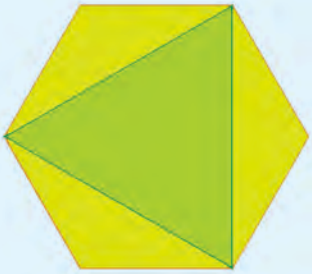
That is, the area of the triangle is half the area of the rectangle. So, the probability of the point being within the triangle is also $\frac{1}{2}$.

In each picture below, the description of the green region inside the yellow one is given. In each, find the probability that a dot put in the picture, without looking, falls within the green region:

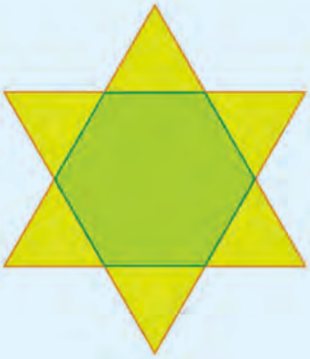
(1) The square got by joining the mid-points of a larger square



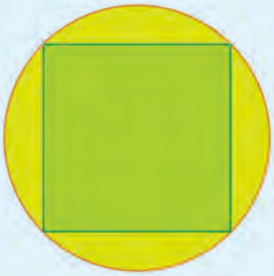
(2) The triangle got by joining alternate vertices of a regular hexagon



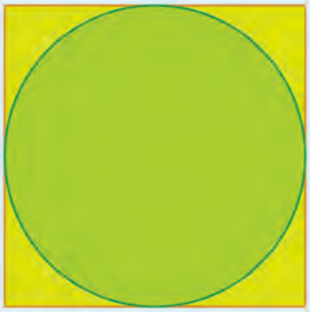
(3) The regular hexagon formed between two equal equilateral triangle



(4) A square drawn with vertices on a circle:



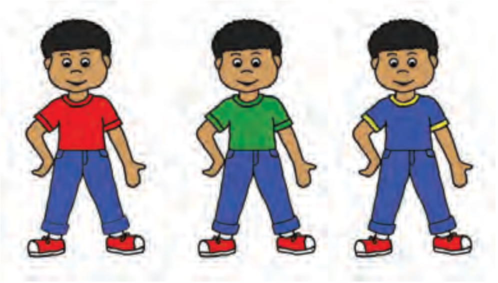
(5) The circle that just fits within a square



Draw a circle in GeoGebra and make an integer slider n. Draw a regular polygon of n sides as follows. Choose Angle with Given Size and click on a point B on the circle and the centre A of the circle in this order. In the dialogue window that opens, give the Angle as (360/n)°. We get a point B'. Choose Regular Polygon and click on B and B' and give Vertices as n. If the name of the circle is c and the name of the polygon is poly1, type Area (poly1)/ Area (c). This gives the probability of a dot put within the circle, without looking, actually falls within the polygon. When is this probability the least? What happens to the probability as the number of sides is increased?

Pairs

Searching for a clean dress, Johnny found a pair of blue pants and three shirts, red, green and blue. "in what all ways can I dress?", thought Johnny.



Searching again, He got a pair of green pants also. Now there are three more ways, wearing this with each of the three shirts, Johnny calculated.

Thus Johnny can dress in six different ways. In how many of these are the pants and shirt of the same colour?

$$\frac{2}{6} = \frac{1}{3}$$

A problem

The famous scientist Galileo writes about a problem asked by a gambler friend. He had computed that when three dice are rolled together, 9 or 10 can occur as the sum in 6 different ways:

	9	10
1.	1 + 2 + 6	1 + 3 + 6
2.	1 + 3 + 5	1 + 4 + 5
3.	1 + 4 + 4	2 + 2 + 6
4.	2 + 2 + 5	2 + 3 + 5
5.	2 + 3 + 4	2 + 4 + 4
6.	3 + 3 + 3	3 + 3+ 4

But then in actual experience, he found 10 occurring more often than 9 as the sum. He wanted an explanation of this.

In the list above 1,2,6, for example, stands for 1 coming up in some die, 2 in some other die and 6 in yet another die. Galileo argued instead of this, he must denote by the triple (1,2,6), the occurrence of 1 in the first die, 2 in the second die, 6 in the third die; by the triple (1,6,2), the occurrence of 1 in the first die, 6 in the second die and 2 in the third die and so on. This gives six different triples, (1,2,6), (1,6,2), (2,1,6), (2,6,1),(6,1,2), (6,2,1) all denoting the occurrence of the same numbers 1, 2, 6 in the three dice. Expanding other triples also like this, Galileo shows that the sum 9 can occur in 25 different ways, while 10 can occur in 27 different ways (Try this).

Let's look at another problem:

> A box contains four slips with numbers 1, 2, 3, 4 and another contains two slips with numbers 1 and 2. One slip is drawn from each box. What are the possible pairs of numbers that could be got?

Suppose 1 is drawn from the first box; the number got from the second may be 1 or 2, so that there are two possible pairs. Let's write these as (1,1) and (1,2).

Similarly, let's consider all number pairs, considering each number from the first box in turn and combining it with the numbers from the second:

(1, 1)	(1, 2)
(2, 1)	(2, 2)
(3, 1)	(3, 2)
(4, 1)	(4, 2)

8 pairs in all.

In how many of them are both the numbers odd?

Only two pairs, (1,1) and (3,1), right?

So, when one slip is drawn from each box, the probability of getting both the numbers odd is

$$\frac{2}{8} = \frac{1}{4}$$

Can you find like this, the probability of getting both numbers even and the probability of getting one even and the other odd? What about the probability of both number being the same ?

(1) Rajani has three necklaces and three earrings of green, blue and red stones. In how many different ways can she wear them ? What is the probability of her wearing a necklace and earrings of the same colour ? Of different colours?

(2) A box contains four slips numbered 1, 2, 3, 4 and another box contains two slips numbered 1 and 2. If one slip is drawn from each box, what is the probability of the sum of the numbers being odd? What is the probability of the sum being even?

(3) A box contains four slips numbered 1, 2, 3, 4 and another box contains three slips numbered 1, 2, 3. If one slip is drawn from each box, what is the probability of the product of the numbers being odd? What is the probability of the product being even?

(4) From all two-digit numbers using only the digits 1, 2 and 3, one number is chosen

  (i) What is the probability of both digits being the same?

  (ii) What is the probability of the sum of the digits being 4?

(5) A game for two players. Before starting, each player has to decide whether he wants an odd number or even number. Then both raise some fingers of one hand at the same time. If the sum of numbers of fingers is odd, the one who chose odd number at the beginning wins; if even, the one who chose even number wins. Which is the better choice at the beginning, odd or even?

More pairs

Again two boxes, one now containing ten slips numbered 1 to 10, and the other containing five slips numbered 1 to 5. One slip is drawn from each box, as usual. What is the probability of both being odd ?

The method of solution is simple. Compute all the possible pairs of numbers and then the number of pairs in which both the numbers are odd, as we want. Dividing the second number by the first gives the probability.

Easier said than done. It is tedious to list all pairs and count.

Let's think about this. The number drawn from the first box can be any of the ten in it. The number from the second can be any of the five in it. So, how many possible pairs are there with the first number 1? And how many with first number 2 ?

In short, once we fix the first number, we can make 5 different pairs, changing the second number. And the first number can be fixed in 10 different ways.

Probability and frequency

When a coin is tossed a large number of times, the number of heads and tails got are almost equal, and so we can take the probability of each coming up as $\frac{1}{2}$. But due to some reason, such as a manufacturing defect in the coin, it may happen that the probability of head coming up is higher. How do we recognize this?

We suspect such a case, if in a large number of tosses, one face comes up very much more than the other. Then we toss the coin more and more times and tabulate the number of times each face comes up. For example, see this table:

Tosses	Heads	Tails
10	6	4
100	58	42
1000	576	424
10000	5865	4135

This shows that, instead of taking the probability of each face as 0.5, it is better to take the probability of head as 0.6 and the probability of tail as 0.4.

There are mathematical methods for making such computations more accurate, which we will see in the further study of the branch of mathematics called *Probability Theory*.

So, we can imagine all the number pairs arranged like this:

(1,1)	(1,2)	. . .	(1,5)
(2,1)	(2,2)	. . .	(2,5)
. . .	. . .	. . .	. . .
(10,1)	(10,2)	. . .	(10,5)

- Pairs with first number 1 in the 1st row, pairs with first number 2 in the 2nd row and so on till the first number is 10, giving 10 rows
- 5 pairs in each row, with the second number 1 to 5
- Altogether, $5 \times 10 = 50$ pairs

In how many of these pairs are both the numbers odd?

For that, the first number must be one among the 5 numbers 1, 3, 5, 7, 9 and the second number one among the 3 numbers 1, 3, 5

$5 \times 3 = 15$ pairs in all

(1,1)	(1,3)	(1,5)
(3,1)	(3,3)	(3,5)
(5,1)	(5,3)	(5,5)
(7,1)	(7,3)	(7,5)
(9,1)	(9,3)	(9,5)

So the probability of getting odd numbers from both the boxes is

$$\frac{15}{50} = \frac{3}{10}$$

Can you compute like this, the probability of getting both the numbers even and the probability of getting one even and other odd ?

Measuring uncertainty

Have you noticed that calendars show the time of sun-rise and sun-set for each day? It is possible to compute these, since the earth and sun move according to definite mathematical laws.

Because of this, we can also predict the months of rain and shine; but we may not be able to predict a sudden shower during summer. It is the largeness of the number of factors affecting rainfall and the complexity of their inter-relations that makes such predictions difficult.

But in such instances also, we can analyze the context mathematically and compute probabilities. This is the reason why weather predictions are often given as possibilities. And unexpected changes in the circumstances sometimes make these predictions wrong.

If we look at such situations rationally, we can see that such probability predictions are more reliable than predictions sounding exact, but made without any scientific basis.

One more problem:

> A basket contains 50 mangoes and 20 of them are not ripe. Another contains 40 mangoes and 15 of them are not ripe. If one mango is picked from each basket, what is the probability of both being ripe?

In how many different ways can we choose a pair of mangoes, one from each basket? (If necessary, imagine the mangoes in the first basket numbered 1 to 50 and the mangoes in the second basket numbered 1 to 40).

So, there are $40 \times 50 = 2000$ ways of choosing two mangoes, one from each basket.

In how many of these pairs are both mangoes ripe?

In the first basket, there are $50 - 20 = 30$ ripe mangoes, and in the second basket, $40 - 15 = 25$ are ripe

If we pair each ripe mango in the first with each ripe mango in the second, we would have $25 \times 30 = 750$ pairs.

So, the probability of both being ripe is $\frac{750}{2000} = \frac{3}{8}$

Can't you compute similarly, the probability of both being unripe?

What is the probability of at least one of the mangoes being ripe?

At least one ripe means, one ripe and the other unripe, or both ripe

One ripe and the other unripe can occur in two ways:

(i) ripe from the first, unripe from the second

(ii) unripe from the first, ripe from the second

How many pairs are there of each possibility?

So the total number of pairs with only one ripe is

$$(15 \times 30) + (25 \times 20) = 450 + 500$$
$$= 950$$

We have already seen there are 750 possible pairs with both ripe. Together with this, the total number of pairs with at least one ripe is

$$950 + 750 = 1700$$

So, the probability of at least one ripe is $\frac{1700}{2000} = \frac{17}{20}$

We can also think like this: at least one ripe means both cannot be unripe, and among all 2000 possible pairs, only in 20 × 15 = 300 pairs are both unripe; so in 2000 – 300 = 1700 pairs, at least one should be ripe.

This again gives the probability of at least one being ripe as $\frac{1700}{2000} = \frac{17}{20}$.

(1) There are 30 boys and 20 girls in Class 10 A and 15 boys and 25 girls in Class 10 B. One student is to be chosen from each Class.

   (i) What is the probability of both being girls ?

   (ii) What is the probability of both being boys ?

   (iii) What is the probability of one being a boy and one being a girl ?

   (iv) What is the probability of at least one being a boy ?

(2) One is asked to say a two-digit number

   (i) What is the probability of both digits being the same ?

   (ii) What is the probability of the first digit being greater than the second ?

   (iii) What is the probability of the first digit being less than the second ?

(3) Two dice with faces numbered from 1 to 6 are rolled together. What are the possible sums that can be got ? Which sum has the maximum possibility ?

(4) One box contains 10 slips numbered 1 to 10 and another contains slips numbered with multiples of 5, up to 25. One slip is drawn from each box.

   (i) What is the probability of both numbers being odd ?

   (ii) What is the probability of the product of the numbers being odd ?

   (iii) What is the probability of the sum of the numbers being odd ?

5

SECOND DEGREE EQUATIONS

Square problems

Let's start with a problem:

> When a square was enlarged, extending each side by 1 metre, its perimeter became 36 metres. What was the length of a side of the original square?

Easy, isn't it?

The length of a side of the new square is 36 ÷ 4 = 9 metres and so, the length of a side of the original square was 9 – 1 = 8 metres.

What if we change the question to this:

> When a square was enlarged, extending each side by 1 metre, its area became 36 square metres. What was the length of a side of the original square?

What is the length of a side of the new square?

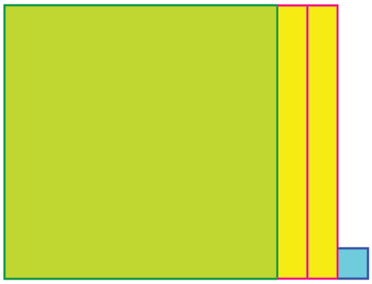
$\sqrt{36} = 6$ metres, right?

So, the length of a side of the original square was 6 – 1 = 5 metres

(1) When each side of a square was reduced by 2 metres to make a smaller square, its area became 49 square metres. What was the length of a side of the original square?

(2) There is a 2 metre wide path around a square ground. The area of the ground and path together is 1225 square metres. What is the area of the ground alone?

See this picture:

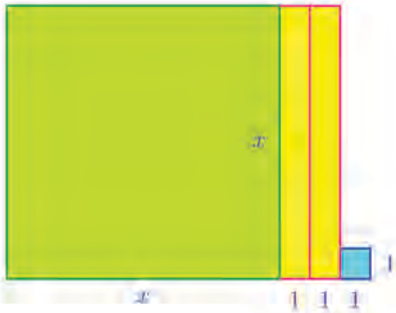


A green square, two yellow rectangles of the same height and a small blue square joined together. The width of each yellow rectangle and a side of the blue square are both 1 metre. And the area of the whole figure is 100 square metres.

We want to calculate the length of a side of the green square.

Difficult to do directly, isn't it?

Let's try algebra, taking the length of a side of the green square as x:



Then the total area of the figure is

$$x^2 + x + x + 1 = x^2 + 2x + 1$$

The total area is given to be 100 square metres. So, we can translate the problem to algebra like this:

If $x^2 + 2x + 1 = 100$, then what is x ?

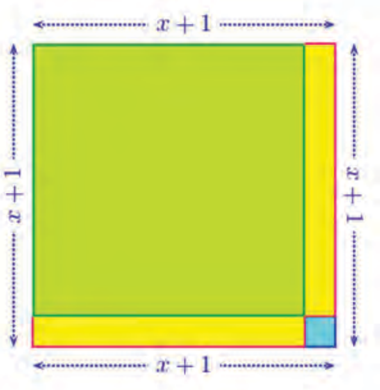
Does the expression $x^2 + 2x + 1$ look familiar?

Remember the equation

$$(x + 1)^2 = x^2 + 2x + 1$$

seen in the lesson **Square Identities** of the Class 8 textbook?

This can also be seen by rearranging the pieces of our figure:



So we can rephrase our problem like this:

If $(x + 1)^2 = 100$, then what is x?

Now it is easy to see that $x + 1 = 10$ and so $x = 9$

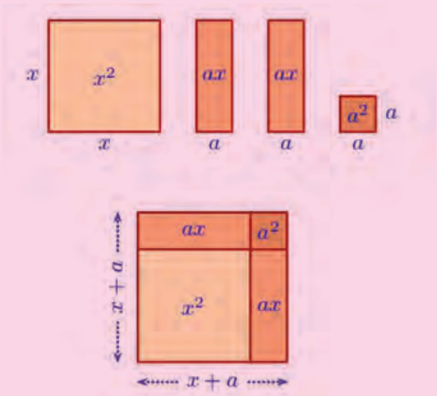
That is, the lengths of a side of of the green square is 9 metres.

Squares - algebra and geometry

For any two numbers x and a, we have

$$x^2 + 2ax + a^2 = (x + a)^2$$

If x and a are positive numbers, we can translate this algebraic identity to a geometrical picture:



(1) 1 added to the product of two consecutive even numbers gave 289. What are the numbers?

(2) 9 added to the product of two consecutive multiples of 6 gave 729. What are the numbers?

(3) The first few terms of the arithmetic sequence 9, 11, 13, ... were added and then 16 added to the sum, to get 256. How many terms were added?

Square completion

Look at this problem:

> One side of a rectangle is 2 metres longer than the other and its area is 224 square metres. What are the lengths of the sides?

Let's first translate this to algebra. If we take the length of the shorter side as x metres, then the length of the longer side is $x + 2$ metres and the area is $x\ (x + 2) = x^2 + 2x$ square metres.

Now we have the algebraic problem:

If $x^2 + 2x = 224$, then what is x?

What next?

Remember how we rewrote $x^2 + 2x + 1$ as $(x + 1)^2$ and went ahead? In this problem, we have only $x^2 + 2x$

Why not just add a 1?

So, we can continue like this:

(i) Since $x^2 + 2x = 224$, we have $x^2 + 2x + 1 = 224 + 1 = 225$

(ii) That is, $(x + 1)^2 = 225$

(iii) Since $(x + 1)^2 = 225$ we have $x + 1 = \sqrt{225} = 15$

(iv) Since $x + 1 = 15$, we have $x = 14$

Thus we get the shorter side of the rectangle as 14 metres.

So, the longer side is 16 metres

Suppose we change this problem slightly like this:

> One side of a rectangle is 20 metres longer than the other and its area is 224 square metres. What are the lengths of the sides?

The algebraic form of the problem changes like this:

If $x^2 + 20x = 224$ then what is x ?

Here also, if we add 1, the number on the right side of the equation becomes $225 = 15^2$, but the expression on the left of the equation becomes $x^2 + 20x + 1$

Can we put it in the form $(x + a)^2$?

Whatever number we take as a,

$$(x + a)^2 = x^2 + 2ax + a^2$$

In our equation, we have $x^2 + 20x$; that is $20x$ in the place of $2ax$ in the general equation.

So, what if we take a as 10?

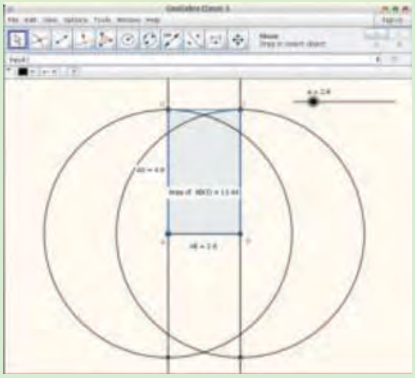
$$(x + 10)^2 = x^2 + 20x + 100$$

In our problem, we have $x^2 + 20x = 224$. How about adding 100, following what we saw just now?

$$\begin{aligned} x^2 + 20x &= 224 \\ x^2 + 20x + 100 &= 324 \\ (x + 10)^2 &= 324 \\ x + 10 &= \sqrt{324} = 18 \\ x &= 8 \end{aligned}$$

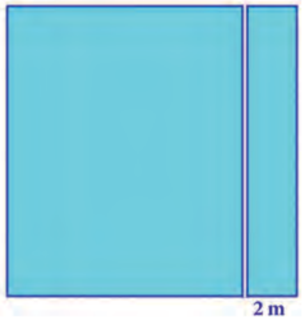
Thus we can see that the lengths of the sides of the rectangle are 8 metres and 28 metres

We can draw rectangles with one side 2 longer than the other. For that, first create a slider a with minimum value 0. Draw a line AB of length a, and draw perpendiculars through its endpoints. Draw circles of radius a + 2, centres at A and B. Mark the points C and D, where the perpendiculars meet the circle. Draw the rectangle joining these four points. Now we can hide the perpendiculars and the circles. Mark the area of the rectangle. At what value of a does the area becomes 224?



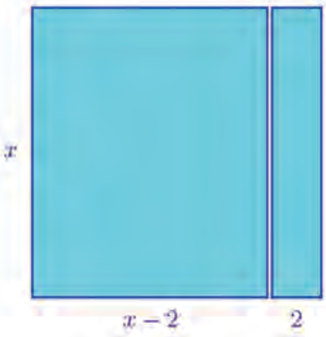
Here is another problem on rectangles:

From a square, a rectangle of width 2 metres is cut off.



The area of the remaining rectangle is 99 square metres. What was the length of a side of the original square?

Taking the length of a side of the square as x metres, the lengths of the sides of the remaining rectangle are x metres and $x - 2$ metres;



And the area of the remaining rectangle is $x(x - 2) = x^2 - 2x$ square metres.

So, the algebraic form of the problem is this:

If $x^2 - 2x = 99$, then what is x ?

Can you change $x^2 - 2x$ to a squared expression, as we did with $x^2 + 2x$?

Recall another identity from Class 8:

$$x^2 - 2x + 1 = (x - 1)^2$$

Now can't we find the x in our problem?

$$\begin{aligned} x^2 - 2x &= 99 \\ x^2 - 2x + 1 &= 100 \\ (x - 1)^2 &= 100 \\ x - 1 &= 10 \\ x &= 11 \end{aligned}$$

The length of the side of the original square is 11 metres.

Another problem:

> One of the perpendicular sides of a right triangle is 5 centimetres more than the other. The area of the triangle is 12 square centimetres. What are the lengths of these sides?

Taking the length of the shorter of the perpendicular sides as x, the length of the other side is $x + 5$. What about the area?

So, translating the given information into algebra, we get

$$\frac{1}{2}x(x + 5) = 12$$

This we can write as

$$x^2 + 5x = 24$$

What number added to $x^2 + 5x$ changes it to the form $x^2 + 2ax + a^2$?

Write

$$x^2 + 5x = x^2 + \left(2 \times \frac{5}{2}\right)x$$

and think what number is to be added to change it to a squared expression

$$x^2 + \left(2 \times \frac{5}{2}\right)x + \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2$$

So, let's add $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ to both sides of our equation:

$$x^2 + 5x + \frac{25}{4} = 24 + \frac{25}{4}$$

That is,

$$\left(x + \frac{5}{2}\right)^2 = \frac{121}{4}$$

From this, we get

$$x + \frac{5}{2} = \sqrt{\frac{121}{4}} = \frac{11}{2}$$

and then

$$x = \frac{11}{2} - \frac{5}{2} = 3$$

Thus the lengths of the perpendicular sides of the triangle are 3 centimetres and 8 centimetres.

Now look at this problem:

> A rectangle is to be constructed with perimeter 100 metres and area 525 square metres. What should be the lengths of its sides?

The sum of the length and breadth of the rectangle is 50 centimetres. So, taking the length of one side as x metres, the length of the other side is $50 - x$ metres.

And the area is $x(50 - x) = 50x - x^2$ square metres

So, we can rewrite our problem like this:

If $50x - x^2 = 525$, then what is x ?

We cannot write $50x - x^2$ as a squared expression. So, let's rewrite the equation in another form. The number x^2 subtracted from the number $50x$ gives 525. So, the reverse subtraction gives the negative, –525. Thus the problems can be written like this:

If $x^2 - 50x = -525$, then what is x ?

Now we have to add a number to $x^2 - 50x$ to make it a squared expression. What number should we add?

$$x^2 - 50x + 625 = (x - 25)^2$$

So, we can solve our problem like this:

$$\begin{aligned} x^2 - 50x &= -525 \\ x^2 - 50x + 625 &= -525 + 625 = 100 \\ (x - 25)^2 &= 100 \\ x - 25 &= 10 \\ x &= 35 \end{aligned}$$

Thus the lengths of the sides of the rectangle are 35 metres and 15 metres.

This problem can be done in a different way

The length of the side of a square of perimeter 100 metres is 25 metres and its area is 625 square metres. But the area of the rectangle in our problem is 525 square metres. So, it is not a square. So the length of all the sides is not 25 metres.

If both sides of this rectangle are longer than 25 metres, the perimeter would be greater than 100 metres; and if both are shorter than 25 metres, then the perimeter would be less than 100 metres. So, one side must be longer than 25 metres and the other less than 25 metres.

Since the sum of these lengths must be 50 (why?), the increase in one length and the decrease in the other must be the same.

If that is taken as x metres, then the lengths of the sides are $25 + x$ metres and $25 - x$ metres.

And the area $(25 + x)(25 - x)$ square metres

So, the problem in algebra is this:

If $(25 + x)(25 - x) = 525$, then what is x?

We can write $(25 + x)(25 - x)$ as

$$(25 + x)(25 - x) = 625 - x^2$$

(The section **Difference of squares** of the lesson **Square Identities** in the Class 8 textbook).

Using this, the problem can be changed to this

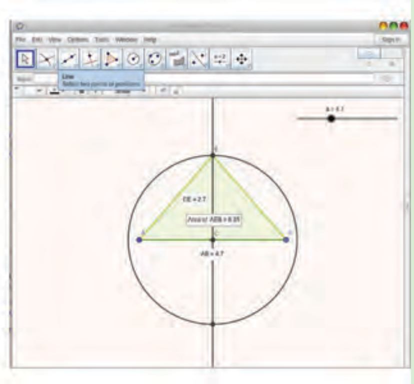
$625 - x^2 = 525$, then what is x ?

From this, we can see that $x^2 = 625 - 525 = 100$ and so $x = 10$.

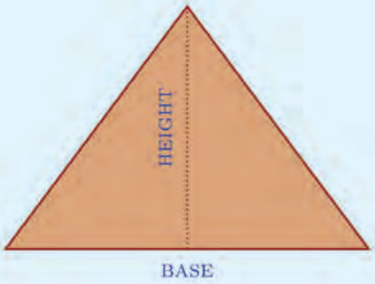
This means the lengths of the sides are

$25 + 10 = 35$ metres and $25 - 10 = 15$ metres

In GeoGebra, make a slider a with Min:2 and Max:10 and draw a line of length a. Mark its midpoint and perpendicular bisector. With the midpoint as centre, draw the circle of radius a−2. Mark the point where the bisector meets the circle and complete the triangle. Mark the area of the triangle. Now change the value of a using the slider. When is the area 12?



(1) An isosceles triangle like the one shown below must be made:



The height must be 2 metres less than the base and the area should be 12 square metres. What should be the lengths of the sides of the triangle?

(2) One side of a right triangle is one centimetre shorter than twice the side perpendicular to it and the hypotenuse is one centimetre longer than twice the length of this side. What are the lengths of the sides?

(3) A pole 2.6 metres long leans against a wall. The foot of the pole is 1 metre away from the wall. When the foot of the pole was pushed a little away, the top end slid down by the same length. How much was the foot moved?

(4) The product of two consecutive odd numbers is 195. What are the numbers?

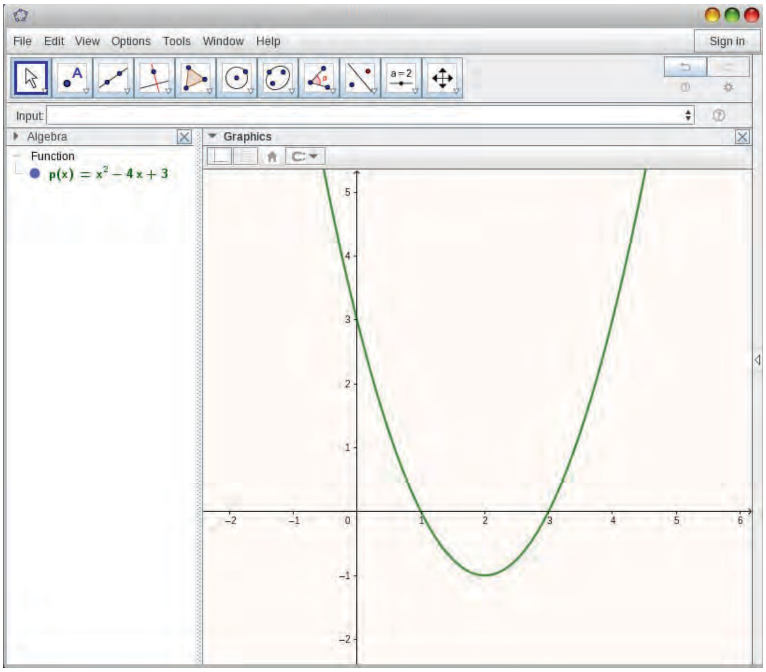
(5) How many terms of the arithmetic sequence 5, 7, 9, ... must be added to get 140 as the sum ?

Two solutions

We have seen graphs of second degree polynomials in the lesson **Polynomial Pictures** of the Class 9 textbook (The section **Second degree polynomials**).

For example, using GeoGebra, we get this as the graph of the polynomial

$$p(x) = x^2 - 4x + 3$$



The graph cuts the horizontal line at the points marked 1 and 3, right?

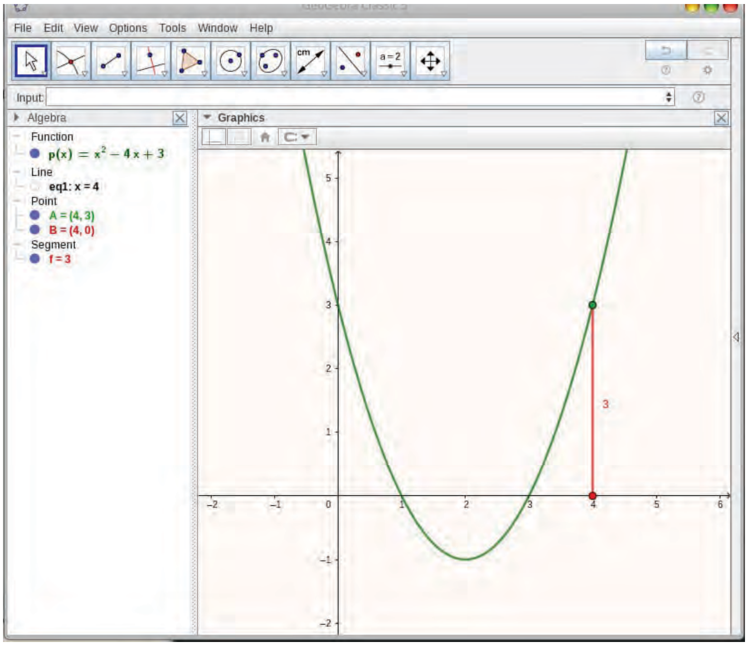
So, what are the numbers $p(1)$ and $p(3)$?

In general, for any number x on the horizontal line, the number $p(x)$ is the height from this point to the graph of $p(x)$.

For example

$$p(4) = 4^2 - (4 \times 4) + 3 = 3$$

and we can see from the picture that the height from the point 4 on the horizontal line to the graph of the polynomial is 3



This height from the points 1 or 3 on the horizontal line is zero. Are $p(1)$ and $p(3)$ equal to zero?

Let's check:

$$p(1) = 1^2 - (4 \times 1) + 3 = 1 - 4 + 3 = 0$$

$$p(3) = 3^2 - (4 \times 3) + 3 = 9 - 12 + 3 = 0$$

Can we compute the numbers x for which $p(x) = 0$ algebraically, without drawing its graph?

The question is this:

If $x^2 - 4x + 3 = 0$, then what is x ?

Can we solve this problem like we did the earlier ones?

Can we add a number to $x^2 - 4x + 3$ to make it a squared expression?

We know that

$$x^2 - 4x + 4 = (x - 2)^2$$

This we can write like this

$$(x^2 - 4x + 3) + 1 = (x - 2)^2$$

Now we can find the x we want:

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ x^2 - 4x + 3 + 1 &= 0 + 1 \\ x^2 - 4x + 4 &= 1 \\ (x - 2)^2 &= 1 \\ x - 2 &= 1 \\ x &= 3 \end{aligned}$$

That is $p(3) = 0$, as seen earlier

But we have seen earlier that $p(1) = 0$ also. In other words, $x = 1$ is also a solution to the equation $x^2 - 4x + 3 = 0$.

Why didn't we get this answer now?

Let's look again at the steps in finding the solution through algebra.

At one stage in this process, we got

$$(x - 2)^2 = 1$$

and from that we found $x - 2 = 1$

If the square of a number is 1, can we say that the number itself is 1?

What is the square of -1?

$$(-1)^2 = (-1) \times (-1) = 1$$

(The section **Oscillation** of the lesson **Negative Numbers** in the Class 9 textbook).

So in our problem, from the equation

$$(x - 2)^2 = 1$$

We can only say that

$$x - 2 = 1 \text{ or } x - 2 = -1$$

In this

$$\text{If we take } x - 2 = 1 \text{ we get, } x = 1 + 2 = 3$$
$$\text{If we take } x - 2 = -1 \text{ we get, } x = -1 + 2 = 1$$

Thus using the algebraic method also, we get both the numbers we can take as x to get $p(x) = 0$

This raises a question: if in the problems done so far, would we have got another solution, had we taken the negative root?

For example, let's look at a problem about rectangles done earlier: finding the sides of a rectangle with one side 2 metres longer than the other and area 224 square metres.

To solve this, we took x metres as the length of the shorter side and got the equation $(x + 1)^2 = 225$; then we took $x + 1 = 15$ and found the length of the shorter side as 14 metres.

If we just look at the algebra only, we should also consider $x + 1 = -15$, so that $x = -16$

But here, x is the length of the side of a rectangle and so is a positive number. Thus the solution $x = -16$ is not possible in our problem on rectangles.

Let's look at another rectangle problem done earlier: rectangle of perimeter 100 metres and area 525 square metres.

In this, taking the length of one side as x metres, we got the equation $(x - 25)^2 = 100$; then took $x - 25 = 10$ to get the length of one side as $(25 + 10) = 35$ metres and the length of the other side as $50 - 35 = 15$ metres

Suppose we take the negative square root?

We get $x - 25 = -10$ and from this $x = 15$; that is, the length of one side as 15 metres and the length of the other as $50 - 15 = 35$ metres.

Thus in this problem, we get the same rectangle, whether we take the positive or negative square root.

In general, when we translate a practical problem to an algebraic problem and think only about the mathematics, we may get more than one solution. Some of these or sometimes even all of them may not be suitable answers to the practical problem.

So what we do is to find all the solutions algebraically and then choose only those solutions suitable to the context.

Now look at this problem:

> How many terms of the arithmetic sequence 99, 97, 95, ... starting from the first must we add to get 900 as the sum?

We can find the n^{th} term of this sequence as $101 - 2n$ and the sum of n terms as

$$101n - n(n + 1) = 100n - n^2$$

So, the problem can be translated to algebra as

If $100n - n^2 = 900$, what is n ?

We can proceed like this:

$$\begin{aligned} 100n - n^2 &= 900 \\ n^2 - 100n &= -900 \\ n^2 - 100n + 50^2 &= -900 + 2500 \\ (n - 50)^2 &= 1600 \end{aligned}$$

What do we get from this ?

1600 is the square of both 40 and –40

So from the above equation we get

$$n - 50 = 40 \text{ or } n - 50 = -40$$

and this gives two solutions

$$n = 90 \text{ or } n = 10$$

Thus whether we add the first 10 terms of the sequence or the first 90 terms of the sequence, we get the same sum 900.

(1) The product of a number and 2 added to it is 168. What are the numbers?

(2) Find two numbers with sum 4 and product 2

(3) Consider the arithmetic sequence 55, 45, 35, ...

  (i) How many terms of this, starting from the first, must be added to get 175 as the sum?

  (ii) How many terms of this, starting from the first, must be added to get 180 as the sum?

  (iii) How many terms of this, starting from the first, must be added to get 0 as the sum?

6

TRIGONOMETRY

Angles and sides

See this triangle:

What are the lengths of its other two sides?

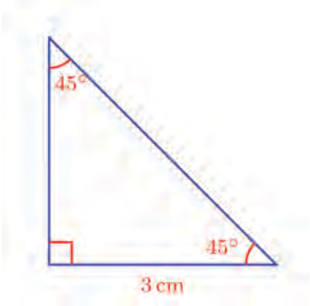
Since two of the angles are 90° and 45°, the third angle is also 45°

Now since two angles are equal, the sides opposite them are also equal (The section **Isosceles triangles** of the lesson **Equal Triangles** in the Class 8 textbook).

What about the third side?

We have seen that its length is $\sqrt{2}$ centimetres in Class 9 (the section **Lengths and numbers** in the lesson **New Numbers**).

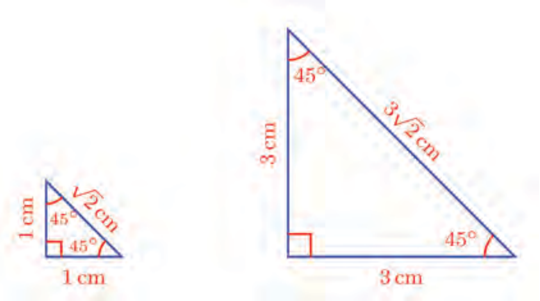
Now can you calculate the lengths of the other two sides of this triangle ?



This triangle has the same angles as the first one, right ?

We have seen, in the lesson **Similar Triangle** of the Class 9 textbook, that in triangles with the same angles, the sides are scaled by the same factor (that is, multiplied by the same number). Here, the side opposite the top 45° angle in the large triangle is 3 times the same angle in the small triangle.

So, the other sides of the large triangle must also be 3 times the other sides of the small triangle



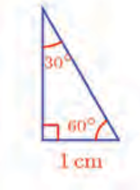
Like this, if we know one side of a triangle with angles 45°, 45°, 90°, then we can calculate the lengths of the other two sides.

For example, if one of the perpendicular sides of such a triangle is $\frac{1}{2}$ metre, then the lengths of the other two sides are $\frac{1}{2}$ metre and $\frac{1}{2}\sqrt{2}$ metres.

So, what can we say in general?

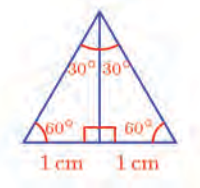
In any triangle with angles 45°, 45°, 90° both the shorter sides have the same length and the length of the longest side is $\sqrt{2}$ times this length

Now look at this triangle:



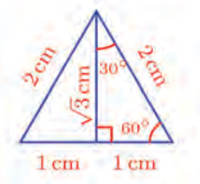
How do we calculate the lengths of its other two sides?

We can make an equilateral triangle by joining two of these like this:

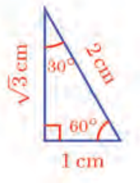


Since the bottom side of this equilateral triangle is 2 centimetres, the length of each of the other two sides is also 2 centimetres.

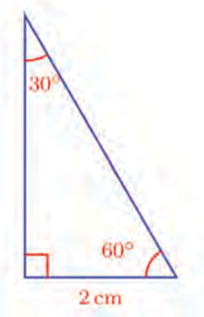
We have also seen that its height is $\sqrt{3}$ centimetres (the section **Lengths and numbers** of the lesson **New Numbers** in the Class 9 textbook).



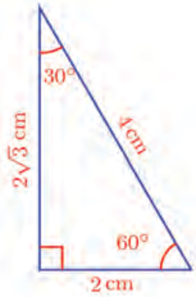
So, we have got the lengths of the sides of the first triangle:



Now look at another triangle with the same angles 30°, 60°, 90°:



In this, the length of the side opposite the 30° angle is double the length of the 30° angle of the small triangle. So, the lengths of the sides opposite other angles must also be double:



So, what can we say in general about the relation between sides of such a triangle?

In any triangle with angles 30°, 60°, 90°, the length of the longest side is 2 times the length of the shortest side; and the length of the side of medium size is $\sqrt{3}$ times the length of the shortest

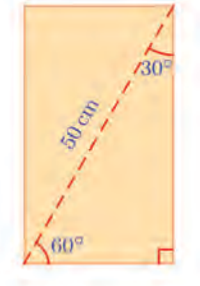
Let's do a problem using this:

An equilateral triangle is to be made by cutting a rectangular board and rearranging the pieces as shown:

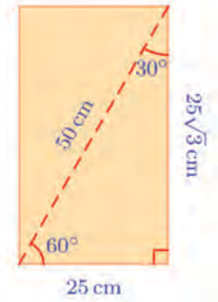
The sides of the triangle must be 50 centimetres each. What should be the lengths of the sides of the rectangle?

One side of the triangle is a diagonal of the rectangle. So, the diagonal of the rectangle must be 50 centimetres:

Since we must get an equilateral triangle on joining the triangles got by cutting the rectangle, the angles of these triangles must be 30°, 60°, 90°:



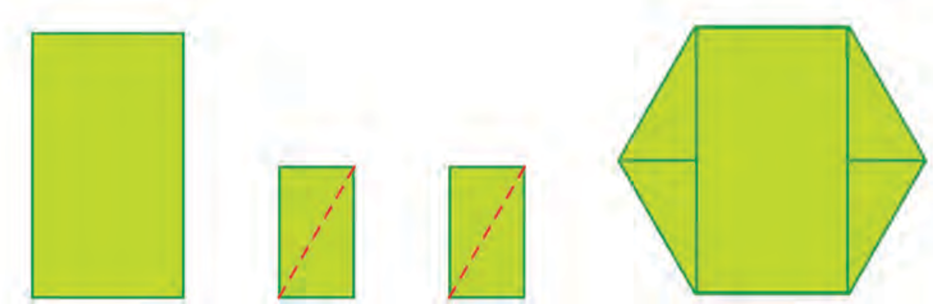
In such a triangle, the longest side is twice the shortest side. Here, the longest side is 50 centimetres. So, the shortest side must be 25 centimetres. And the third side?



Now we have the length of the sides of the rectangle, right?

Next try this problem:

> Two rectangles are cut along their diagonals and joined to a third rectangle, to make a regular hexagon, as shown below:



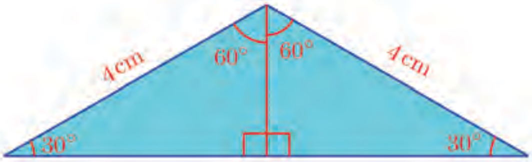
The length of a side of the hexagon is to be 30 centimetres. What should be the lengths of the side of the rectangles?

We can use these facts to compute some areas also. For example look at this problem:

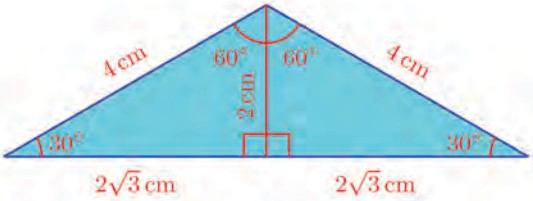
What is the area of the triangle below?

Two sides of this triangle are of the same length and so the angles opposite them must also be the same. Since the top angle is 120°, the sum of the other two angles is 60° and since they are equal, each must be 30°

The perpendicular from the top vertex to the bottom side bisects the top angle (the section **Isosceles triangles** of the lesson, **Equal Triangles** in the Class 8 textbook):



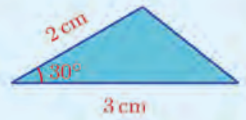
Now can't we compute the length of this perpendicular and the length of the bottom side?

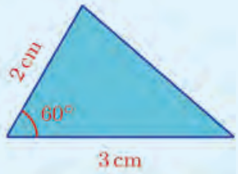


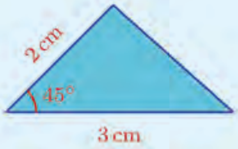
So, the area of the triangle is $2 \times 2\sqrt{3} = 4\sqrt{3}$ square centimetres

(1) Calculate the area of the parallelograms shown below:

(2) Calculate the area of the triangles shown below:

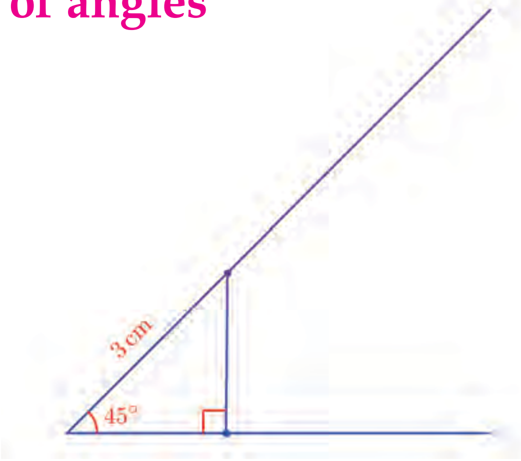




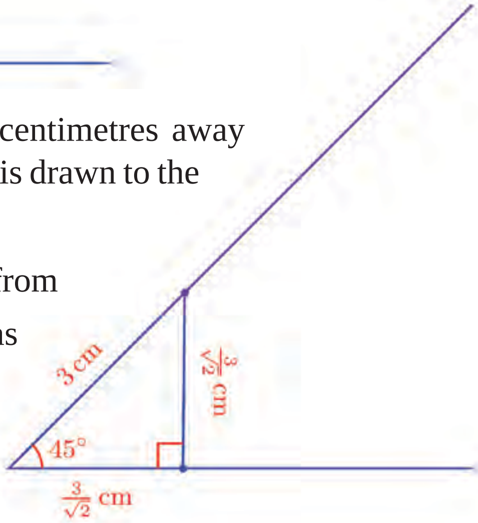


New measures of angles

See this picture:

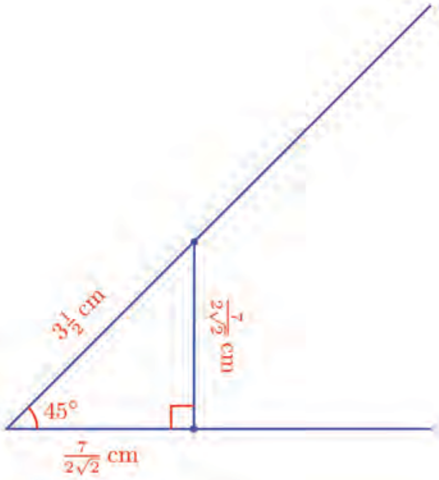


A point is chosen on one side of a 45° angle, 3 centimetres away from the vertex and a perpendicular from this point is drawn to the other side of the angle.



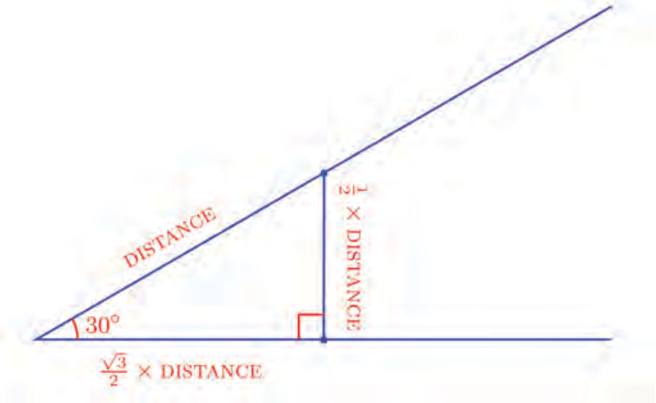
The height of this perpendicular and the distance from the vertex to its foot are both $\frac{3}{\sqrt{2}}$ centimetres, as seen in the last section.

What if the first point chosen is $3\frac{1}{2}$ centimetres away from the vertex?

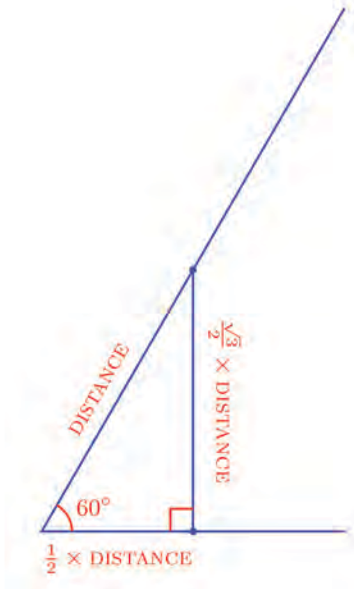


In general, at whatever distance from the vertex we take the first point, the height of the perpendicular to the other side and the distance to its foot from the vertex are both $\frac{1}{\sqrt{2}}$ distance.

What if we take the angle as 30° instead of 45°?

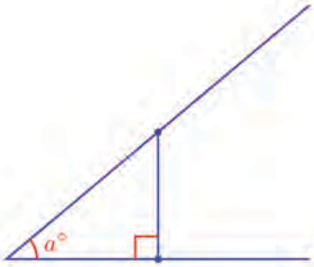


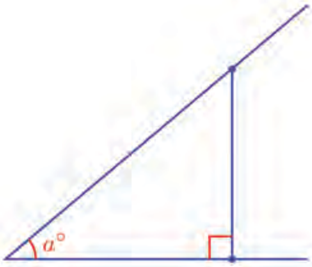
And if it is 60°?



For other angles also, would the height of the perpendicular and the distance to its foot from the vertex be fixed multiples of the distance of the first point from the vertex?

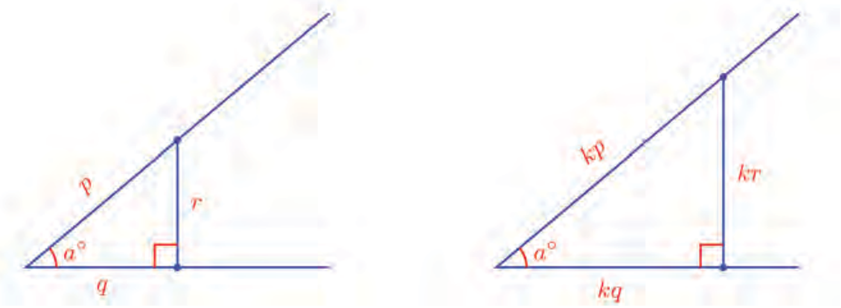
See these pictures:





On one side of an angle, two points are chosen at different distances from the vertex and perpendicular drawn to the other side

The two triangles got from these have equal angles and so their sides are scaled by the same factor. So, if the lengths of the sides of the triangle on the left are taken as p, q, r in decreasing order, then the lengths of the sides of the triangle on the right, in this order are p, q, r multiplied by the same number. Taking this number as k, the sides of the triangle on the right are kp, kq, kr:



In the triangle on the left, the height of the perpendicular is $\frac{r}{p}$ of the distance of the chosen point from the vertex and the distance of the foot of the perpendicular from the vertex is $\frac{q}{p}$ of this distance

And in the triangle on the right?

These fractions are $\frac{kr}{kp} = \frac{r}{p}$ and $\frac{kq}{kp} = \frac{q}{p}$

Thus these fractions do not change

So, what can we say in general?

> If on one side of an angle, points at different distances from the vertex are chosen and perpendiculars drawn to the other side, the heights of these perpendiculars and the distance of their feet from the vertex change; but these as fractions of the distances of the chosen points from the vertex do not change.

We have calculated these fractions for 45°, 60° and 30°. It is not easy to calculate these for other angles. But mathematicians have devised methods to compute these and tabulated them long ago.

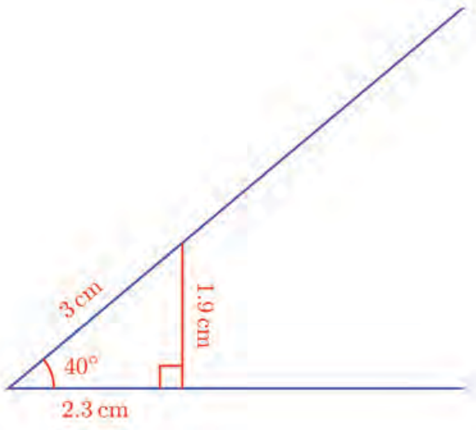
For example, we can see from this table that in a 40° angle, the height of the perpendicular from a point on one side to the other side is about 0.6428 times the distance of the point from the vertex and the foot of this perpendicular is at a distance of about 0.7660 times the distance of the point from the vertex.

Thus if in a 40° angle, we mark a point on one side, 3 centimetres from the vertex and draw a perpendicular to the other side, then we can calculate the height of the perpendicular as approximately

$$3 \times 0.6428 \approx 1.9 \text{ cm}$$

and the distance of the foot of the perpendicular from vertex as approximately

$$3 \times 0.7660 \approx 2.3 \text{ cm}$$



If the distance of the point from the vertex is 3 metres, we can compute the height of the perpendicular as approximately 1 metre, 928 millimetres and the distance of the foot of the perpendicular from the vertex as approximately 2 metres, 298 millimetres.

Numbers computed like these have special names. In the problem just seen, the number 0.6428... shows what fraction of the distance between a point on one side of a 40° angle and the vertex of the angle, is the height of the perpendicular from this point to the other side. This number is called the **sine** of 40° and is written sin 40°. Thus

$$\sin 40° \approx 0.6428$$

The second number 0.7660... shows what fraction of the distance between a point on one side of a 40° angle and the vertex of the angle, is the distance between the foot of the perpendicular from this point to the other side and the vertex of the angle. This number is called the **cosine** of 40° and is written cos 40°. Thus

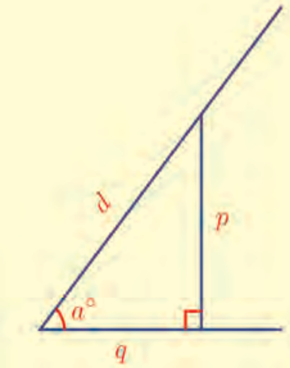
$$\cos 40° \approx 0.7660$$

In general,

Consider a point on one side of an angle of $a°$, at distance d from the vertex, and the perpendicular from this point to the other side; if the height of the perpendicular is p and the distance of the foot of the perpendicular from the vertex is q, then.

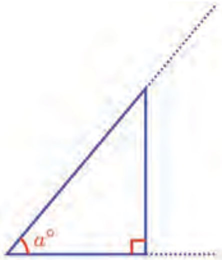
$$\sin a° = \frac{p}{d}$$

$$\cos a° = \frac{q}{d}$$



It must be noted that as we take different points on one side of the angle, the number d, p, q all change. But $\sin a°$ and $\cos a°$ do not change; only when the size $a°$ of the angle changes, do these change. In other words, they are numbers indicating the size of an angle.

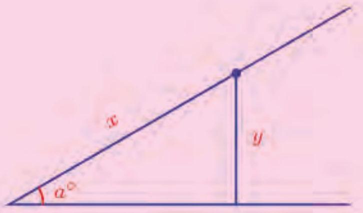
We can describe sin and cos in a slightly different way. When we draw a perpendicular from a point on one side of an angle to the other side, we get a right triangle:



The line joining the vertex of the angle and the point chosen is the hypotenuse of this triangle. The perpendicular from the point to the other side is the **opposite side** of the a° angle in the triangle. The line joining the vertex of the angle and the foot of the perpendicular is called the **adjacent side** of the $a°$ angle. So, $\sin a°$ and $\cos a°$ can be described like this:

Proportionality

We have seen in Class 9 that if we take all points on one side of an angle, the height of a point from the other side changes proportionally with respect to the distance of the point from the vertex of the angle.



(The section **Proportional changes** of the lesson, **Proportion**).

What is the proportionality constant in this change?

$\frac{y}{x}$ is what we call $\sin a°$, isn't it?

Thus $\sin a°$ is the proportionality constant in the change of height of the perpendicular with respect to the distance of the point from the vertex.

Can you describe $\cos a°$ also as a proportionality constant?

In a right triangle with one angle $a°$

$$\sin a° = \frac{\text{Opposite side}}{\text{Hypotenuse}} \qquad \cos a° = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

The relations between the sides of right triangles with one angle 45°, 60° or 30°, seen in the first section can be written in terms of sin and cos:

Angle	**Picture**	**sin**	**cos**
30°		$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60°		$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

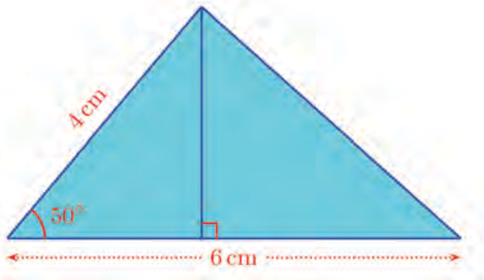
(The table at the end of the lesson gives the sine and cosine of all angles from 1° to 89° at intervals of 1°, correct to four decimal places).

We can use this to compute the areas of triangles, as done in the first section.

For example, let's find the area of the triangle shown below:

Draw a line AB of length 1 in GeoGebra and create an angle slider a. Select Angle with Given Size and click on B and then on A. In the dialogue window, give a as Angle. We get a new point B'. Draw the perpendicular to AB' through B and mark the point C where it meets AB'. Draw triangle ABC. Then AB' can be hidden. We can see that as we change the angle using the slider, the lengths of the sides of the triangle also change. In this the length of BC is equal to the sine of the angle and the length of AC is the cosine of the angle (why?). Make a table of sine and cosine of angles using this. What is the maximum value of sine and cosine?

To find the area, we draw the perpendicular from the top vertex to the bottom side:



We can directly find the sine and cosine of angles using GeoGebra. Typing sin(30°) in the input bar gives sin 30° (To type 30°, first type 30, then click the α symbol at the right of the input bar and select the ° symbol).

The area of the triangle is half the product of the lengths of the bottom side and the height of this perpendicular, isn't it?

How do we calculate the height of the perpendicular?

This perpendicular is the side opposite the 50° angle in the right triangle on the left. So, its height divided by the hypotenuse gives the sine of 50°. This means the height of the perpendicular is the hypotenuse multiplied by sin50°; that is $4 \times \sin 50°$

Now from the table we get

$$\sin 50° \approx 0.7660$$

From this we get the height as approximately

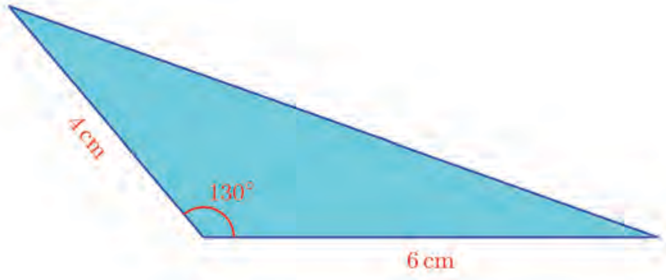
$$4 \times 0.7660 = 3.064$$

Now we can compute the area

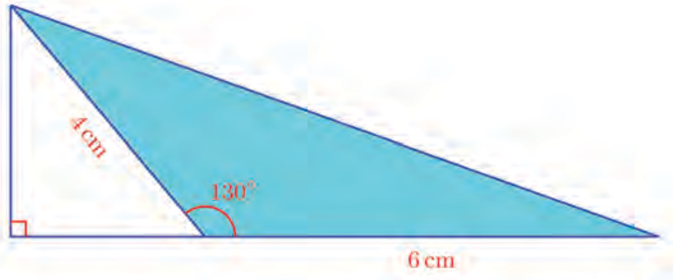
$$\frac{1}{2} \times 6 \times 3.064 \approx 9.19$$

Thus the area is about 9.19 square centimetres.

Suppose we change the angle to 130° instead of 50°



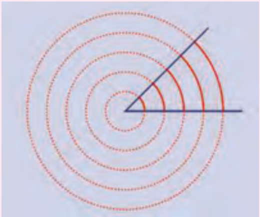
In this, the perpendicular from the top vertex is outside the triangle:



Degree measure of angles

What does it mean, when we say that an angle is 45°?

We can draw several circles centered at the vertex of such an angle. And the length of the arc of such circles within the sides of this angle are all different.

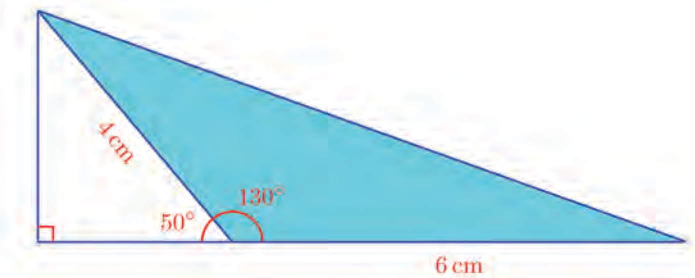


But each of these arcs is $\frac{1}{8}$ of the corresponding full circle. And 45 is the number got by multiplying this fraction $\frac{1}{8}$ by 360. What if the measure of the angle is 60°? For any circle centered at the vertex of an angle of this size, the length of the arc within the sides of the angle would be $\frac{1}{6}$ of the entire circle. And 60 is the number got by multiplying this $\frac{1}{6}$ by 360.

Generally speaking, the degree measure of any angle is the number got by first drawing a circle centered at its vertex, dividing the length of the arc within its sides by the circumference, and then multiplying this number by 360.

How do we calculate the height of this perpendicular?

One of the angles of right triangle outside our triangle is 50°; and the perpendicular we want is the side opposite the 50° angle in that triangle:



So, as in the first problem, the height of the perpendicular is again $4 \times \sin 50° \approx 4 \times 0.766 = 3.064$ centimetres. And the area is also approximately

$$\frac{1}{2} \times 6 \times 3.064 \approx 9.19 \text{ square centimetres}$$

(1) The lengths of two sides of a triangle are 8 centimetres and 10 centimetres and the angle between them 40°

  (i) What is the area of this triangle?

  (ii) What is the area of the triangle with lengths of two sides the same, but the angle between them 140°?

(2) The length of sides of a rhombus is 5 centimetres and one of its angles is 100°. Calculate its area.

(3) The lengths of the diagonals of a parallelogram are 8 centimetres and 12 centimetres and the angle between them 50°. Calculate its area.

(4) A triangle is to be drawn with one side 8 centimetres long and one of the angles on it 40°. What is the minimum length of the side opposite this angle?

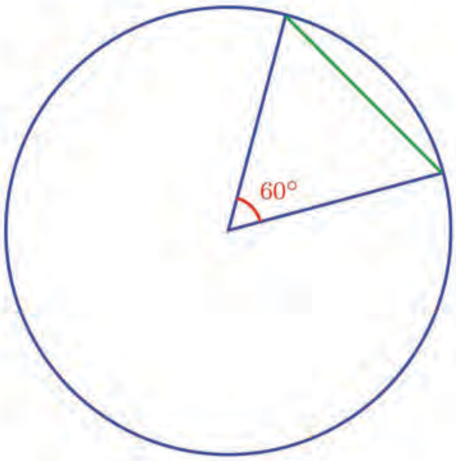
(5) The length of the sides of a rhombus is 5 centimetres and one of its angles is 70°. Calculate the length of its diagonals.

Triangles and circles

We have seen in Class 9 textbook that the length of an arc of a circle can be calculated using its central angle (the section **Length and angle** of the lesson, **Parts of Circles**).

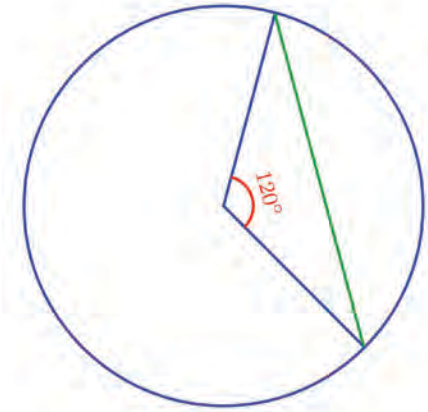
How do we compute the length of a chord?

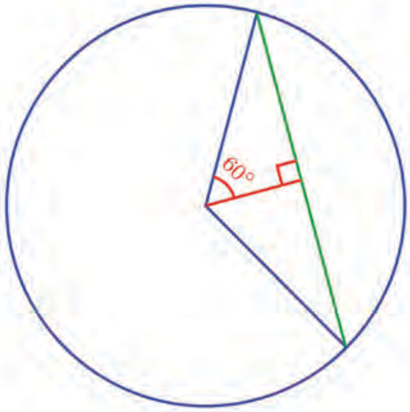
For example, we can easily see that the length of a chord of central angle 60° is equal to the radius of the circle (how?).



What about the length of a chord of central angle 120° ?

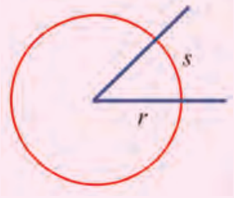
To calculate it, we draw the perpendicular from the centre of the circle to the chord. It bisects both the chord and the central angle (why?).





Another measure of an angle

We have seen what the degree measure of an angle means:



We noted that for circles of different radii, r and s in the picture above will change, but $\frac{s}{2\pi r}$ will not change and the degree measure of the angle is this fixed number multiplied by 360. In other words,

degree measure of the angle = $\frac{s}{2\pi r} \times 360$

In this, we can change r and s, but the numbers 2π and 360 don't. So, isn't it enough if we take $\frac{s}{r}$ as a measure of the angle?

That's right. This gives another measure of the angle, called its *radian measure*.

Thus

radian measure of the angle = $\frac{s}{r}$

We use the symbol ° to denote the degree measure, right? The radian measure is written rad.

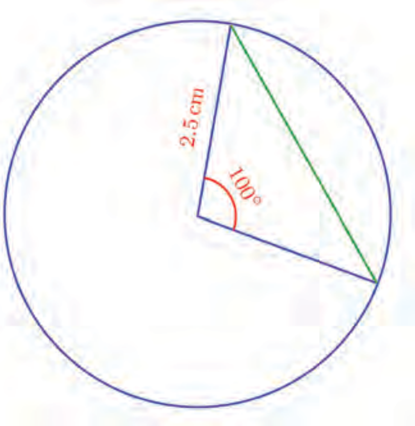
This idea was first proposed by the English mathematician, Roger Cotes in the eighteenth century. The name radian was first used by the English physicist, James Thomson in the nineteenth century

In the right triangle formed by the radius and half the chord, the hypotenuse is the radius; and the side opposite the 60° is half the chord.

Thus we can see that half the chord is $\frac{\sqrt{3}}{2}$ of the radius.

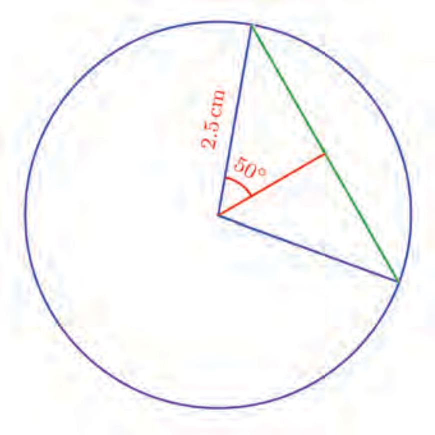
So, the length of the chord is $\sqrt{3}$ times the radius.

We have seen in Class 9 that as the central angle is doubled, the length of the arc is also doubled; but note that the chord is not doubled, as seen in this example. In other words, the length of a chord is not scaled by the same factor as the central angle.



Now how do we compute the length of the chord in this picture?

As before, let's draw the perpendicular from the centre to bisect the chord and the central angle



The hypotenuse of the right triangle thus got is 2.5 centimetres. So, the length of the side opposite the 50° angle is 2.5 × sin 50°. This is half the chord. So, the length of the chord is

$$5 \times \sin 50° \approx 5 \times 0.7660 \approx 3.8 \text{ cm}$$

We can use this method to find the length of any chord.

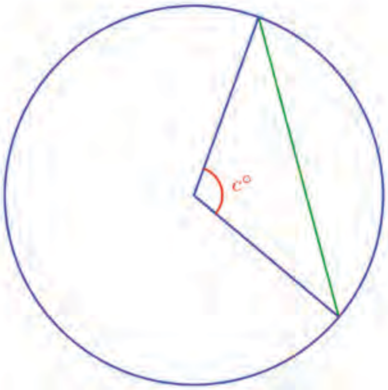
What all things did we do to compute the length of a chord?

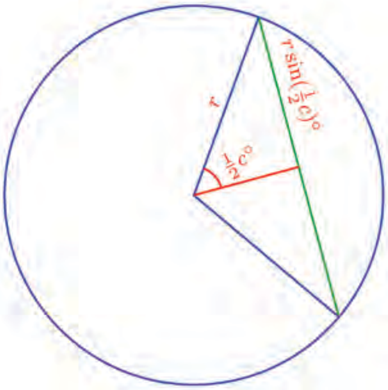
Straightening

In measuring an angle, we actually measure the length of an arc of a circle, whether we use degrees or radians. Instead of this, the Greek astronomer Hipparchus started using lengths of chords, in the second century BCE.

Later mathematicians often refer to a table of chords of various central angles computed by Hipparchus, but it has not been found. However, such a table of chords done by the Egyptian astronomer Claudius Ptolemy in the second century CE, has been found. He has computed accurately lengths of chords of central angles up to 180° in a circle of radius 60 units, at $\frac{1}{2}^{\circ}$ intervals.

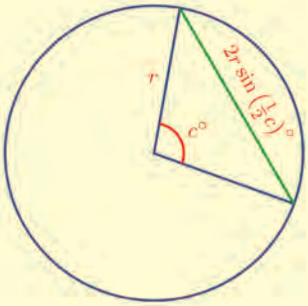
Multiplying the radius by the sine of half the central angle gave half the chord; doubling it gave the length of the chord





So what can we say in general?

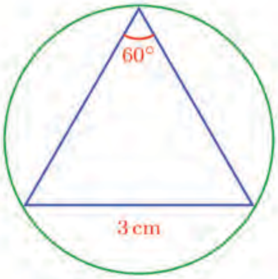
The length of any chord of a circle is twice the product of the radius and the sine of half the central angle



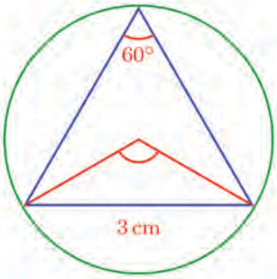
For example, look at this problem:

What is the radius of the circumcircle of an equilateral triangle of sides 3 centimetres

See this picture:



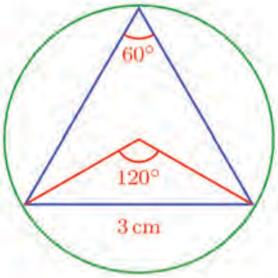
Each side of the triangle is a chord of the circle. What is its central angle?



The central angle of the chord, which is the bottom side of the triangle, is also the central angle of the arc joining the bottom two vertices of the triangle.

And the top vertex of the triangle is a point on the circle, which is not on this arc.

So, the central angle of this arc (which is also the central angle of the chord) is 2 × 60° = 120°, as seen in the section, **Arcs and angles** of the lesson **Circles and Angles**



As stated above, the length of this side is the product of the circumradius and 2 sin 60°

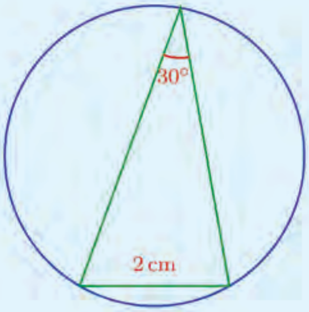
$$2\sin 60^\circ = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

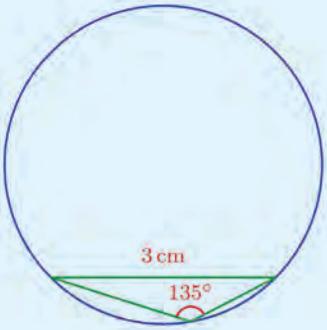
So the circumradius can be computed as

$$3 \div \sqrt{3} = \sqrt{3}$$

Thus the circumradius is $\sqrt{3}$ centimetres

(1) The pictures below show two triangles and their circumcircles:

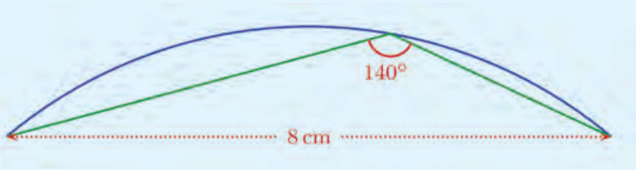




Calculate the radius of each circle

(2) A circle is to be drawn, passing through the ends of a line 5 centimetres long; and the angle in one of the segments made by the line should be 80°. What should be the radius of the circle?

(3) The picture shows part of a circle:



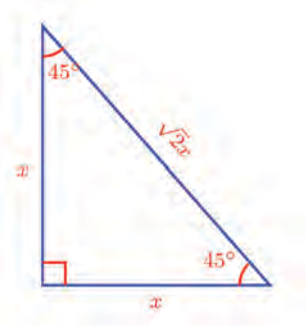
What is the radius of the circle?

Ratio of sides

We have seen in Class 9 textbook that in triangles with the same three angles, the sides are in the same ratio (the section **Scale and proportion** of the lesson **Proportion**). In other words, the angles of a triangle determine the ratio of the sides.

For example, we have seen at the beginning of this lesson that in any triangle with angles 45°, 45°, 90° both the shorter sides are of the same length and the longest side is $\sqrt{2}$ times this length.

That is, in any such triangle, the lengths of the sides are the numbers 1, 1, $\sqrt{2}$ multiplied by the same number:

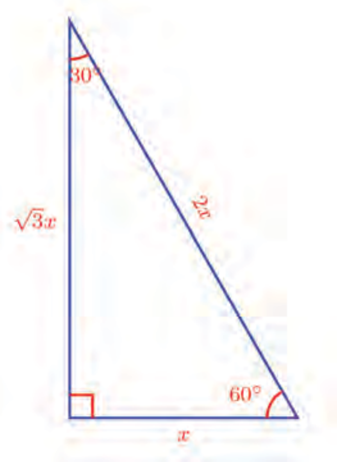


In other words,

In any triangle with angles 45°, 45°, 90° the sides are in the ratio 1: 1: $\sqrt{2}$

Similarly, we have seen that in a triangle with angles 30°, 60°, 90° the side of medium length is $\sqrt{3}$ times the shortest side and the longest side is twice the shortest side

This means, in any such triangle, the lengths of the sides are the numbers 1, $\sqrt{3}$, 2 multiplied by the same number:

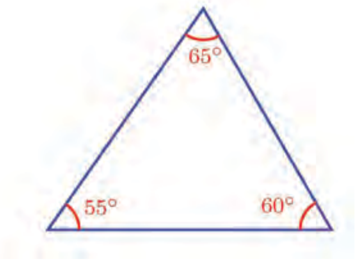


This can also be stated in terms of ratios:

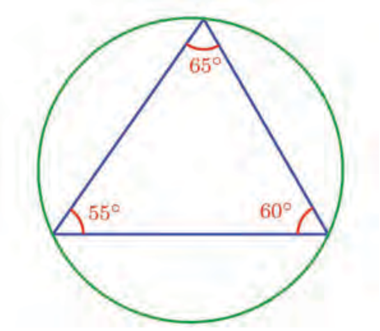
In any triangle with angles 30°, 60°, 90° the sides are in the ratio $1 : \sqrt{3} : 2$

To find the ratios of sides of other triangles, we can use the sine of the angles

For example, consider this triangle:



If we draw the circumcircle, the sides of the triangle would be chords of this circle:

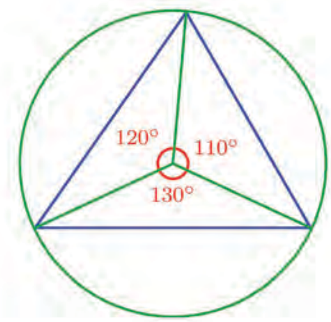


If the sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, would the angles be 30°, 60°, 90°? We can use GeoGebra to check this.

First we draw a triangles with sides in the ratio $1 : \sqrt{3} : 2$. For this create a slider with Min = 0. Select the Segment with Given Length and click on a point. In the dialogue window, give the length as a*sqrt(3) (This means $a\sqrt{3}$). With one end of this line as centre, draw a circle of radius a, and with the other end as centre, draw a circle of radius 2a. Draw the triangle with one of the points where these circles meet and the ends of the line as vertices. Mark the angles of this triangle and see, what happens if we change the value of *a*, using the slider?

In the same way draw triangles with sides in the ratio $2 : \sqrt{5} + 1 : \sqrt{5} + 1$ and see what the angles are.

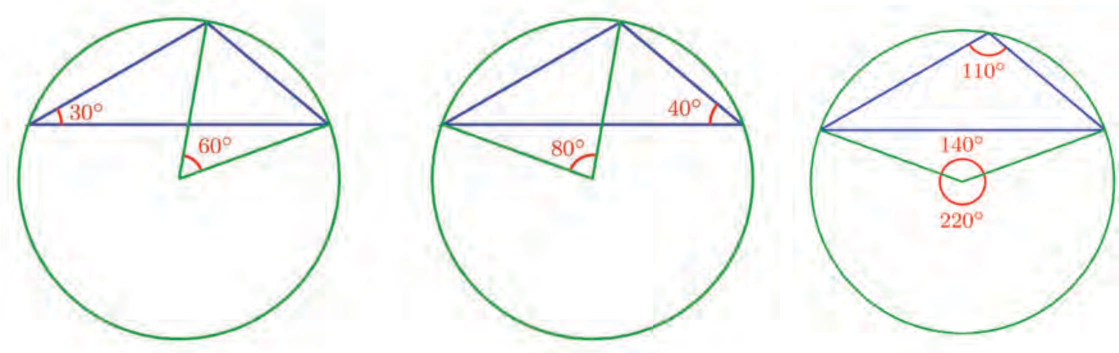
The central angle of each chord is twice the angle opposite to it in the triangle (the section **Arcs and angles** in the lesson **Circles and Angles**):



So, if we take the radius of the circumcircle as *r*, then the lengths of the chords are 2*r* sin 55°, 2*r* sin 60°, 2*r* sin 65°; and so the ratio of the sides of the triangle is sin 55°: sin 60°: sin 65°. We can calculate them using the table of sines. Note that in this, 55°, 60°, 65° are the angles of the triangle itself.

What if the triangle is like this?

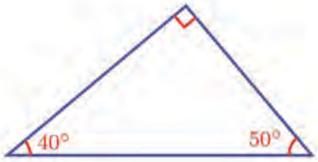
The circumcircle and the central angles of the chords which are sides of the triangle are then like this:



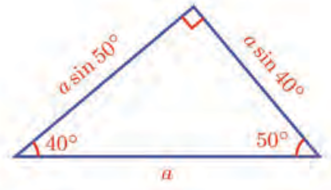
So, if we take the radius of the circumcircle as *r* as in the first case, the lengths of the chords are 2*r* sin 30°, 2*r* sin 40°, 2*r* sin 70°, and these are the lengths of the sides of the triangle. The ratio of the sides is then sin 30°: sin 40° : sin 70°

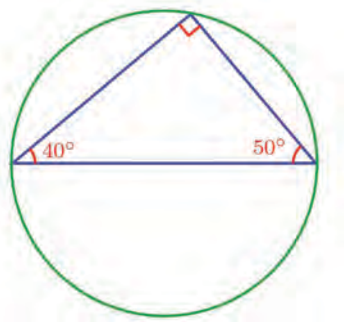
Note that here, the angles 30° and 40° are two angles of the triangle and 70° is the 110° angle of the triangle subtracted from 180°

What about a right triangle?



If we take the hypotenuse of this triangle as *a*, we can write the lengths of the other two sides:

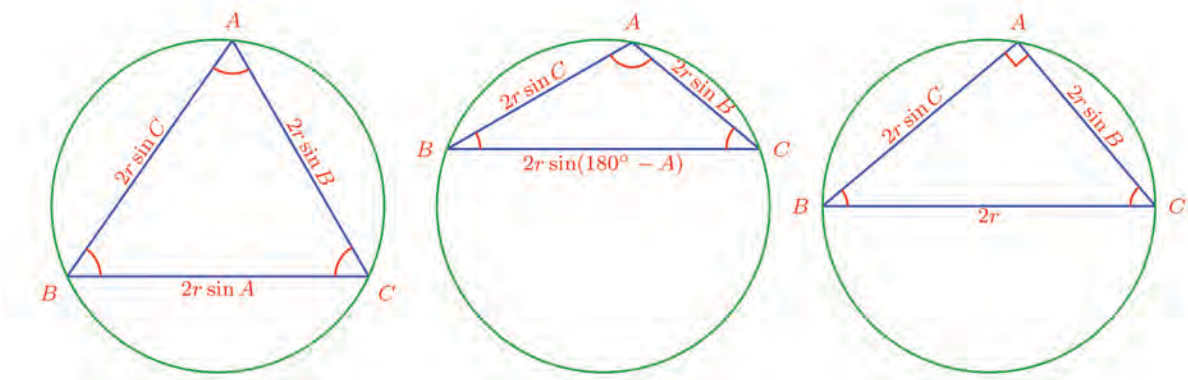




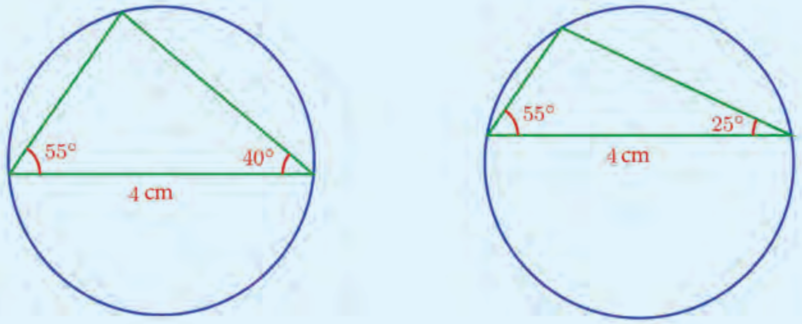
We have also seen that the diameter of the circumcircle is also *a* (Problem 1 (iii) at the end of the section **Triangle centres** of the lesson **Parallel lines** in the Class 9 textbook).

Note that here the ratio of sides is 1: sin 40°: sin 50°

We have seen in Class 9 that triangles with the same angles have sides in the same ratio. Now we know how to compute this ratio also:



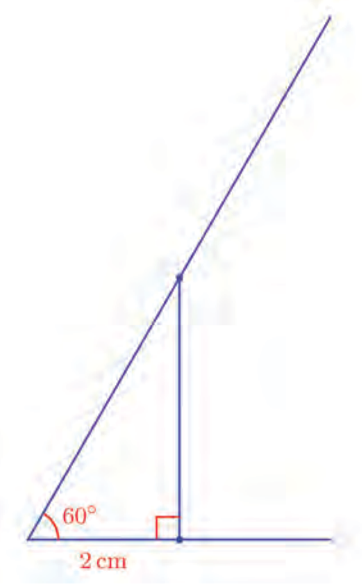
Two triangles and their circumcircles are shown in the figure.



Use the sine table to calculate the diameters of the circles and the other two sides of the triangles correct to a millimetre

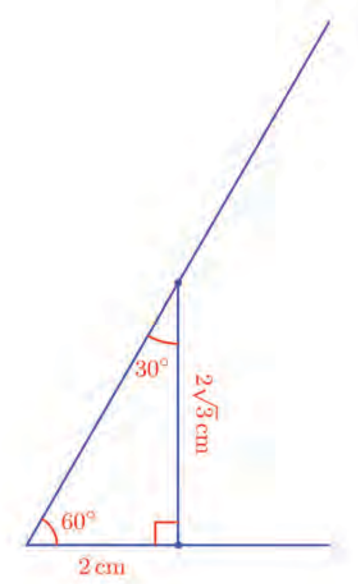
Another measure

See this picture:



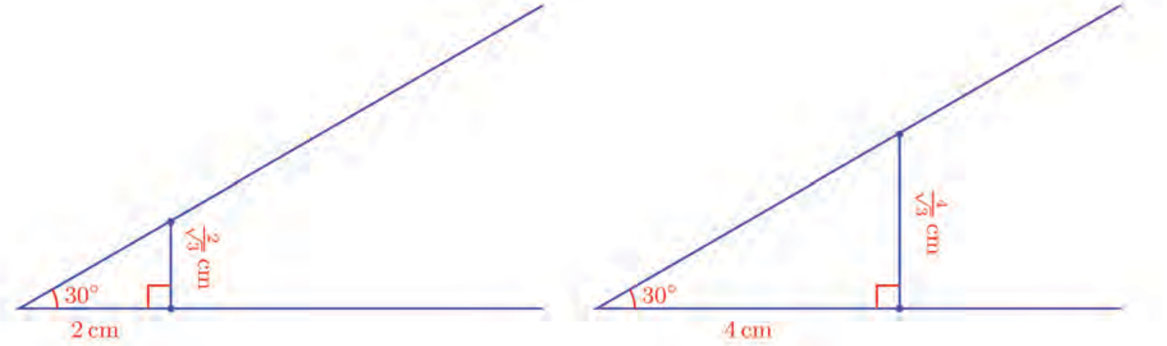
What is the height of the perpendicular in this?

The angles of the triangle are 30°, 60°, 90° and the shortest side is 2 centimetres. So, the side of medium length is $2\sqrt{3}$ centimetres; and that is the height of the perpendicular:



Now if we take any point on one side of this angle and draw a perpendicular to the other side, the height of the perpendicular would be $\sqrt{3}$ times the distance from the vertex of the angle to the foot of the perpendicular.

What if we take a 30° angle?



The height of the perpendicular is $\frac{1}{\sqrt{3}}$ times the distance from the vertex to the foot.

What about a 45° angle?

Like this, the number which gives the height of the perpendicular from one side of an angle to the other as a fraction or multiple of the distance from the vertex of the angle to the foot of the perpendicular, is called the **tangent** of the angle and is written tan. So, in the examples seen above

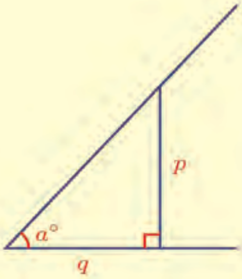
$$\tan 60° = \sqrt{3}$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$\tan 45° = 1$$

In general

Let's see how we can compute the tan of angles using GeoGebra, just as we did for sin and cos. Draw a line AB of length 1 and an angle slider a. Select Angle with Given Size and click on A and then on B. In the dialogue window, give Angle as a. We get a new point B'. Join A and B' and mark the point C where it meets the perpendicular drawn through B. Draw the triangle ABC. Now we can hide the other points and lines. Mark the length of BC. It is the tan of the angle at A (why?). What is the maximum value of tan?

Consider a point on one side of an angle of $a°$, and the perpendicular from this point to the other side; if the height of the perpendicular is p and the distance of the foot of the perpendicular from the vertex is q, then.

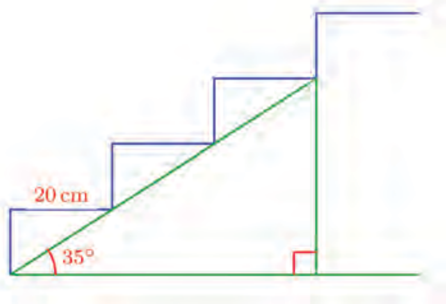
$$\tan a° = \frac{p}{q}$$



We can also describe tan of an angle in terms of a right triangle, as in the case of sin and cos:

In a right triangle with one angle $a°$,

$$\tan a° = \frac{\text{Opposite side}}{\text{Adjacent side}}$$



Let's look at a situation where we use the tan of an angle.

The picture shows part of a staircase with measurements.

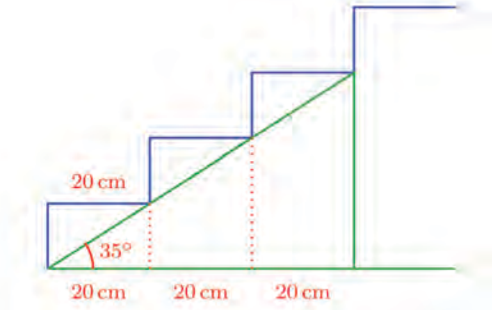
We want to calculate how high is the third step from the floor.

The height to be computed is the length of the side opposite the 35° in the right triangle shown in the picture.

For that we need only multiply the adjacent side of this angle by tan 35°

How do we calculate the adjacent side?

See this picture:



So, the height of the third step from the floor is $60 \times \tan 35°$

From the tables we find

$$\tan 35° \approx 0.7002$$

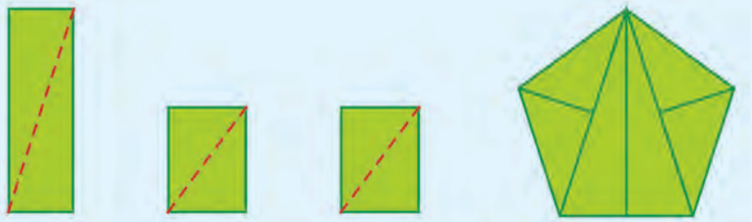
and so the height is approximately

$$60 \times 0.7002 = 42.012 \approx 42 \text{ cm}$$

(1) One angle of a rhombus is 50° and the shorter diagonal is 6 centimetres. What is its area?

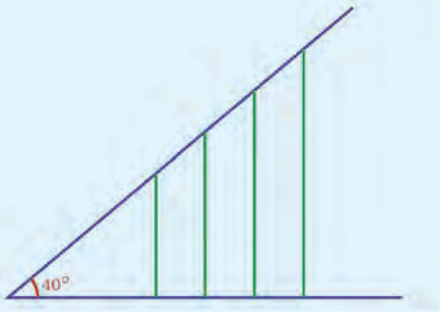
(2) A ladder leans against a wall with its foot 2 metres away from the wall. The angle between the ladder and the ground is 40°. How high is the top of the ladder from the ground?

(3) Three rectangles are to be cut along the diagonals to make triangles which are then rearranged to form a regular pentagon as shown below:



The sides of the pentagon must be 30 centimetres. What should be the lengths of the sides of the rectangles?

(4) The perpendiculars in the picture below are drawn 1 centimetre apart:

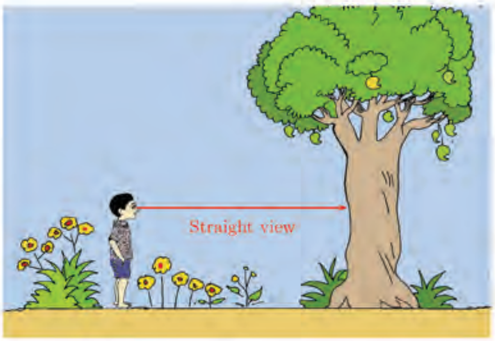


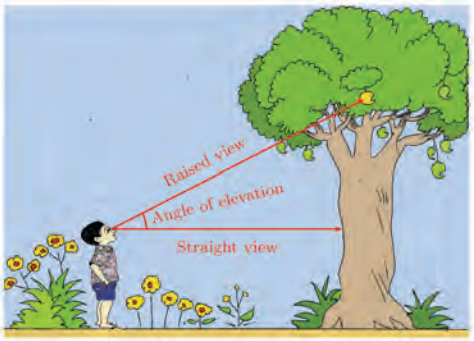
Prove that their heights are in an arithmetic sequence. What is the common difference?

(5) Calculate the area of a regular pentagon of sides 10 centimetres.

Heights and distances

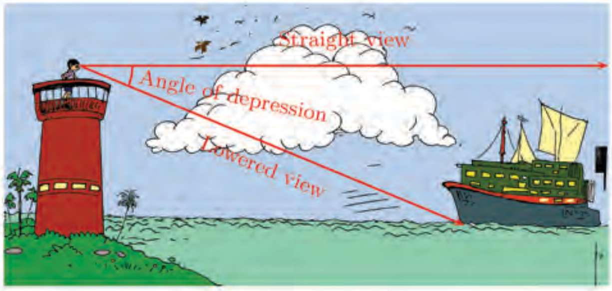
To see things above us, we need to look up by lifting our heads. See these pictures:





Usually our line of vision is parallel to the ground. To see things above, we have to raise (elevate) the line of vision. The angle between these two lines is called **angle of elevation**

Similarly when we stand at a high place, to see things below, we have to lower our line of vision:

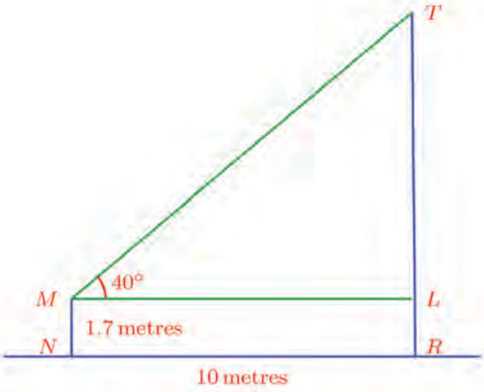


The angle produced thus is called the **angle of depression**. Such angles are measured using an instrument called **clinometer**. Distances and heights which cannot be directly measured are found out by measuring angles using a clinometer and computing using tan tables

Let's look at a few examples.

> A person, 1.7 metres tall, standing 10 metres away from the foot of a tree sees the top of the tree at an angle of elevation of 40°. What is the height of the tree?

In the picture below, MN is the person and TR is the tree:



From the picture we have

$$\tan 40° = \frac{TL}{ML}$$

So that

$$TL = ML \tan 40° \approx 10 \times 0.8391 = 8.391$$

using tables. So,

$$TR = TL + LR = TL + MN = 8.391 + 1.7 = 10.091$$

$$TR \approx 10.09$$

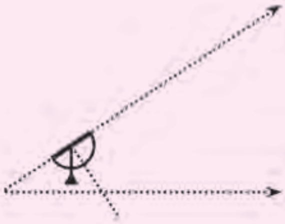
Thus the height of the tree is about 10.09 metres

Clinometer

We can make a simple clinometer to measure angles of elevation and depression:

Fix a hollow tube at the straight edge of a protractor, as shown in the picture. Hang a small weight from it, using a string stretched along the middle of the protractor.

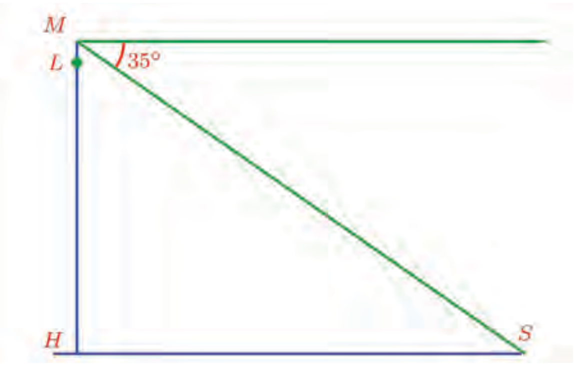
To see the top of a tree or a building through the tube, the protractor has to be tilted up. The string will still be straight down, because of the weight attached to it. The angle of elevation is the angle between the string and the 90° line of the protractor.



Another problem

A person, 1.8 metres tall, standing atop a lighthouse 25 metres high, sees a boat at an angle of depression 35°. How far is the boat from the foot of the light house?

Let's draw a picture:



In this, LH is the lighthouse and ML is the person standing on it. S is the boat. We have to calculate HS

From the given information, we have

$$MH = ML + LH = 25 + 1.8 = 26.8$$

and $\angle HMS = 55°$

So from the right triangle HMS,

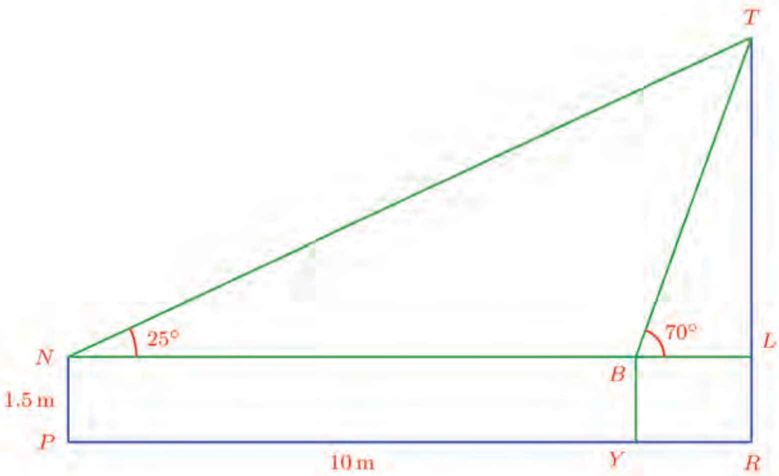
$$HS = MH \tan 55° \approx 26.8 \times 1.4281 \approx 38.27$$

That is, the boat is about 38.27 metres away from the foot of the lighthouse.

One more problem:

A boy, 1.5 metres tall, standing at the edge of a canal sees the top of a tree on the other edge at an angle of elevation of 70°. Stepping 10 metres back, he saw it at an angle of elevation of 25°. How wide is the canal and how tall is the tree?

In the picture below, TR is the tree, BY is the first position of the boy and NP is the later position of the boy.



What we have to calculate are YR and TR

From the picture we have

$$YR = BL$$

$$TR = TL + LR = TL + 1.5$$

So, we need to find BL and TL

Let

$$BL = x \quad TL = y$$

so that from the right triangle BTL

$$y = x \tan 70° \approx 2.7475x$$

And from the right triangle *NTL*

$$y = (x + 10) \tan 25° \approx 0.4663(x + 10) = 0.4663x + 4.663$$

So,

$$2.7475x \approx 0.4663x + 4.663$$

From this we can find (using a calculator)

$$x \approx \frac{4.663}{2.2812} \approx 2.044$$

and then

$$y \approx 2.7475 \times 2.044 \approx 5.616$$

Thus the width of the canal is about 2.04 metres and the height of the tree is about 5.62 +1.5 = 7.12 metres

(1) When the sun is seen at an angle of elevation of 40°, the length of a tree's shadow is 18 metres

   (i) What is the height of the tree?

   (ii) What would be the length of the shadow, when the sun is at an elevation of 80°?

(2) From top of a building, a person sees the foot of a shop 30 metres away at an angle of depression 25°. What is the height of the building?

(3) From the top of an electric post, two wires are stretched to either side and attached to the ground, making angles 55° and 45° with the ground. The distance between the feet of the wires is 25 metres. What is the height of the post?

(4) A person, 1.75 metres tall, standing at the foot of a tower sees the top of a hill 40 metres away at an elevation of 60°. From the top of the tower, he sees it at an elevation of 50°. Calculate the heights of the tower and the hill.

(5) A boy, 1.5 metres tall, standing at the edge of a canal sees the top of a tree on the other edge at an angle of elevation of 80°. Stepping 15 metres back, he saw it at an angle of elevation of 40°. How wide is the canal and how tall is the tree?

Trignometric tables

Angle	sin	cos	tan
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.364
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.404
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000

Angle	sin	cos	tan
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900

7

COORDINATES

Position and number

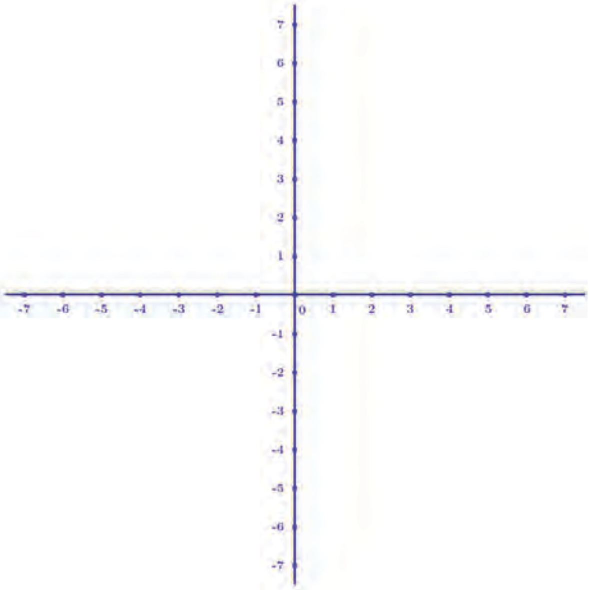
Remember drawing graphs of polynomials in Class 9? (The lesson **Polynomial Pictures**).

For example, how did we draw the graph of the polynomial $p(x) = 2x + 1$?

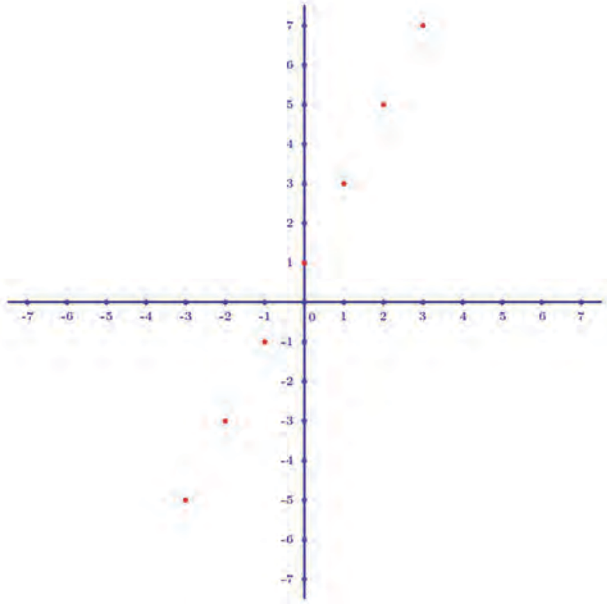
Take different numbers as x and calculate $p(x)$ for each of these:

x	–3	–2	–1	0	1	2	3
$p(x)$	–5	–3	–1	1	3	5	7

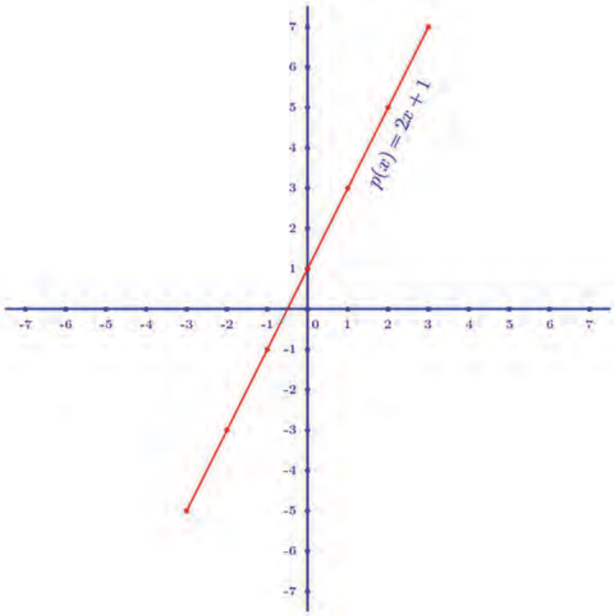
To mark them as points in a picture, first we drew a horizontal line and a vertical line and marked numbers on them at equal distances apart:



Then marked points at height $p(x)$ above each point x

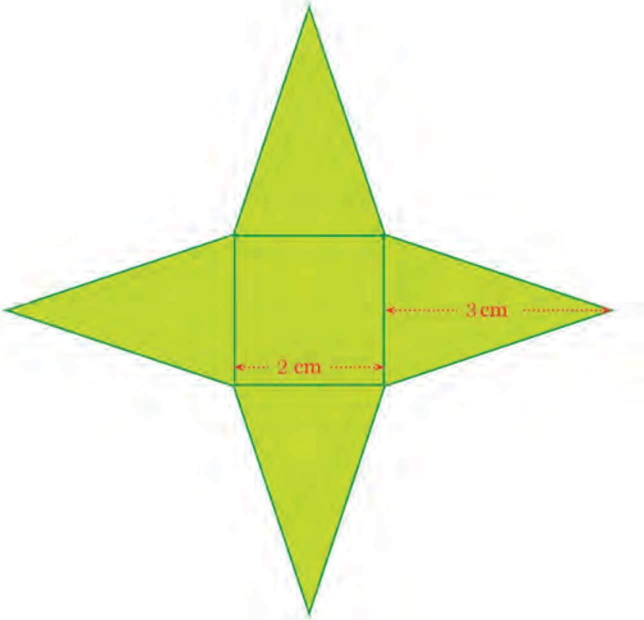


The line joining these points is the graph of the polynomial $p(x) = 2x + 1$

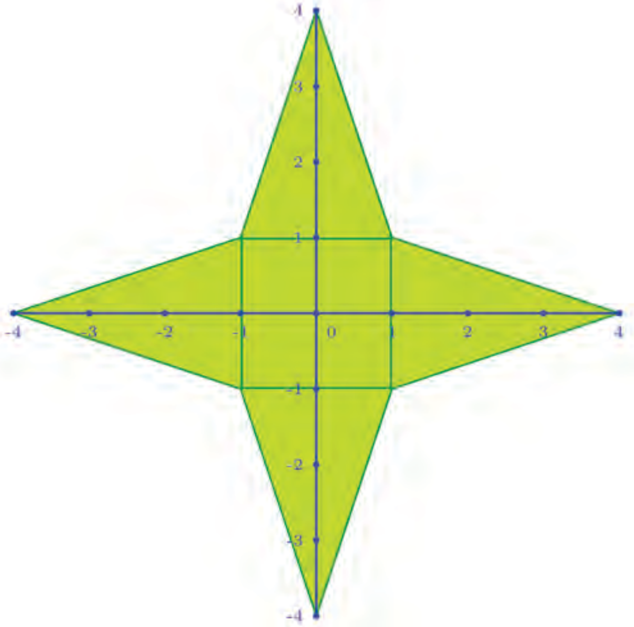


We can use this method to draw other figures also

For example, let's see how we can draw the picture below like this:

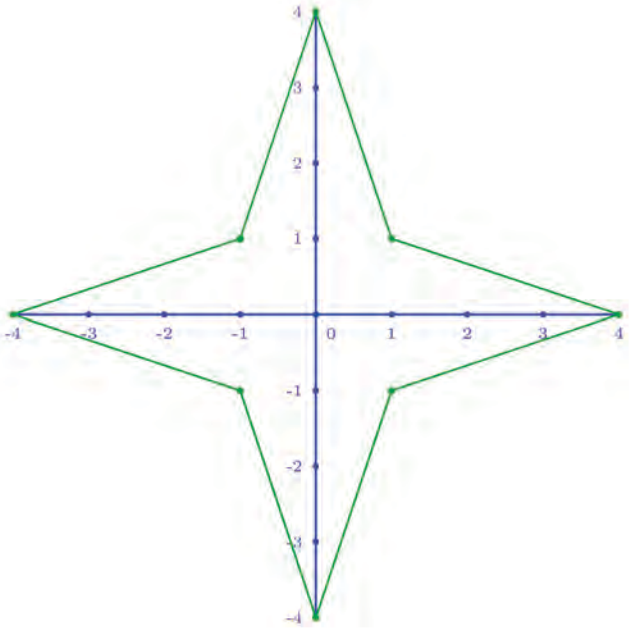


Imagine that horizontal and vertical lines are drawn through the middle of this picture, with distances marked 1 centimetre apart:



Thus by drawing horizontal and vertical lines and marking distances 1 centimetre apart on them, we can mark all eight corners of the figure; and joining them, we get the star

Draw this picture in your notebook, by drawing horizontal and vertical lines and marking the corners of the star

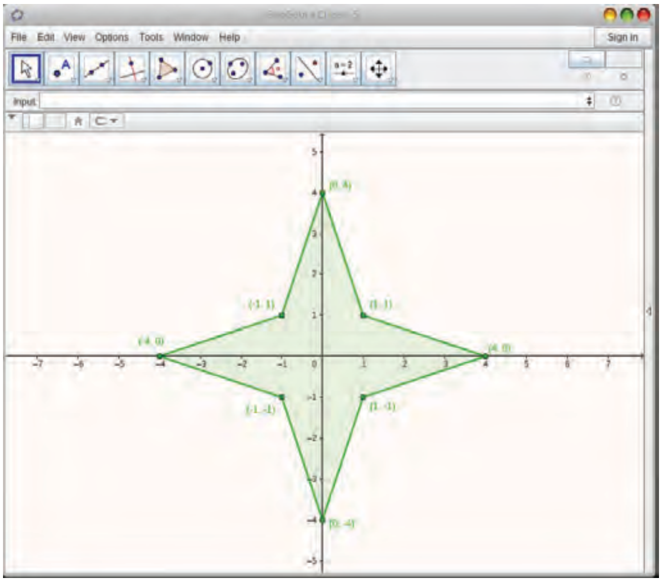


Now let’s see how this can be drawn in GeoGebra. Remember how we plotted points as a pair of numbers?

For example, in the polynomial $p(x) = x^2$, if we take $x = 0.5$, we get $p(x) = 0.25$. So, the point at a height of 0.25 above the position marked 0.5 on the horizontal line, is a point on the graph of $p(x)$. We have seen that to mark this in GeoGebra we need only type (0.5,0.25) in the input bar (The section **Second degree polynomials** of the lesson, **Polynomial Pictures** in the Class 9 textbook).

Similarly, the top-right corner of the picture above is at a height of 1 centimetre from the point marking 1 centimetre on the horizontal line. So we can mark this point as (1, 1).

Thus we can mark all corners of the picture above:

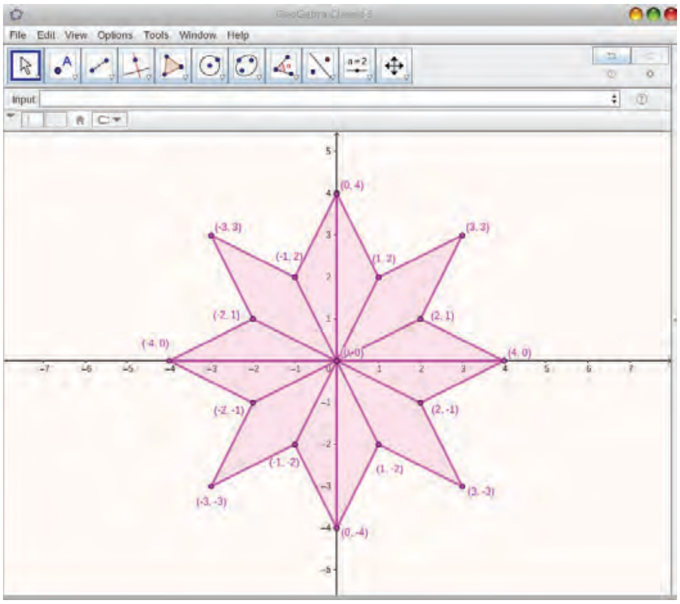


To plot many points in GeoGebra we need only input them one by one as pairs of numbers. To get the polygon joining them, we use the Polygon command.

For example, type this command in the input bar and see:

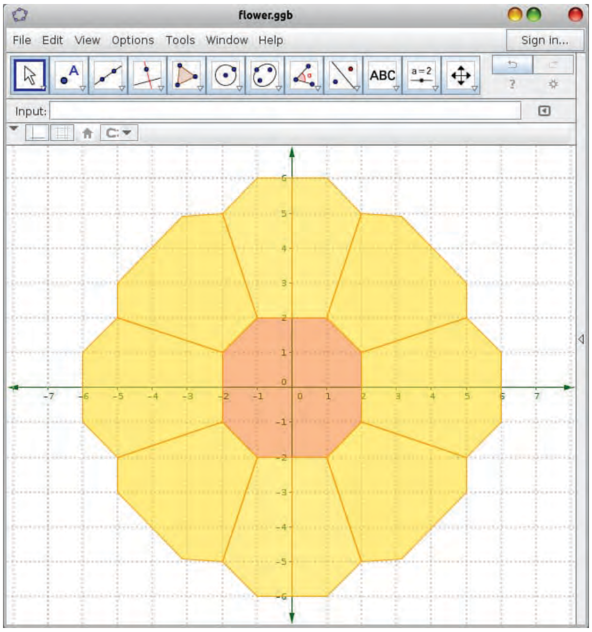
Polygon[(−1,−1),(1,−1),(1,1),(−1,1)]

Now can you draw the picture below in GeoGebra?



Open GeoGebra and input the number pairs (0,0), (4,0), (2,1), (3,3), (1,2), (0,4). Select the Polygon tool and click on the points A, B, C, D, E, F, A in this order to draw the top right quarter of the picture. Select the Segment tool and click on A, C and then A, E to draw the two lines inside this. Next select the Reflect about Line tool and click on the picture and the vertical line. Do you get the image of the picture on the left ? In the same way, the two lines in the right quarter can also be reflected onto the left. To get the bottom half of the picture, using the same tool, reflect each of the two quarters on the top of the horizontal line onto the bottom.

How about this picture?



For convenience, the Grid in GeoGebra is also shown in it.

Can you draw these in your notebook?

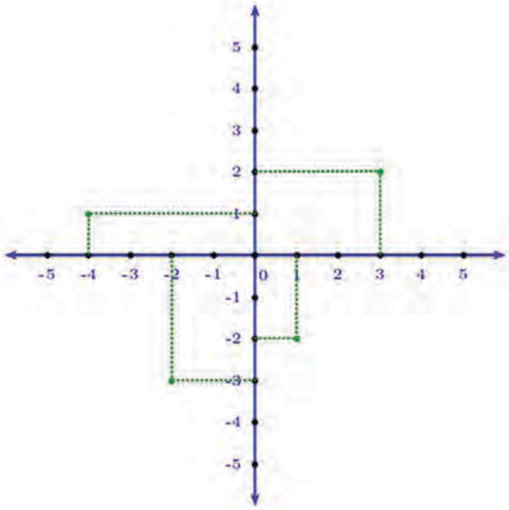
Geometry and numbers

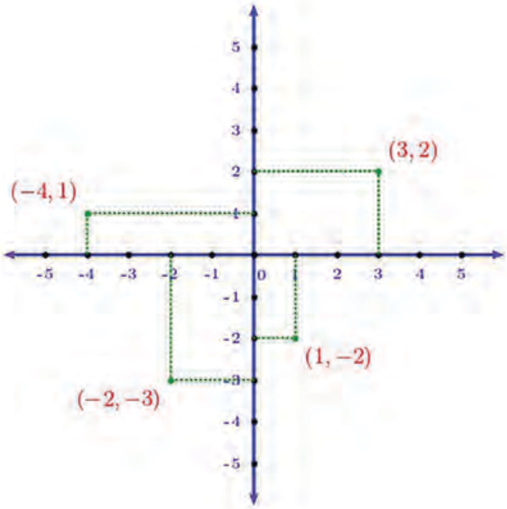
We have seen how we can draw pictures by representing points using pairs of numbers.

These two numbers are calculated using a horizontal line and a vertical line. These lines are called the **axes of coordinates**, the horizontal line being the ***x*-axis** and the vertical one the ***y*-axis**. The point of intersection of these lines is called **origin.**

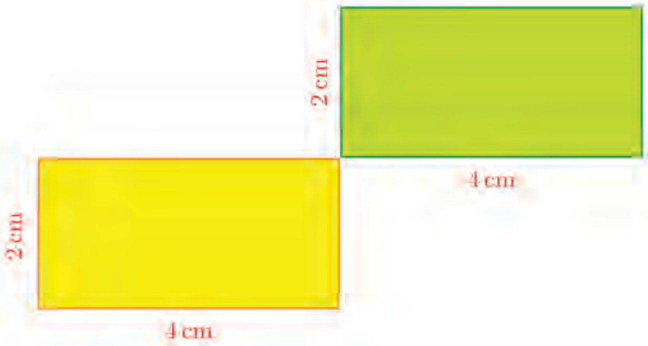
Once the axes are drawn, we can write the location of any point as a pair of numbers, as seen in the earlier examples. These numbers are called the **coordinates** of this point. The first number is the ***x*-coordinate** and the second is the ***y*-coordinate.**

To get the coordinates of a point, we need only draw perpendiculars to the axes. For example, see these pictures:

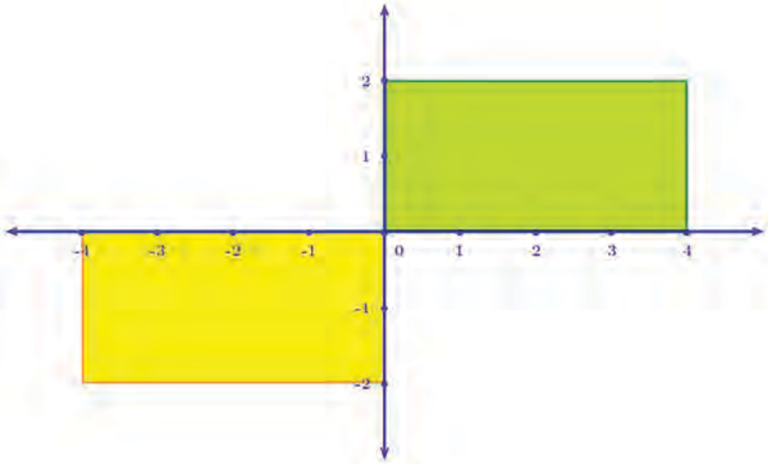




To draw a picture, we can take any two mutually perpendicular lines as axes. For example, see this picture:

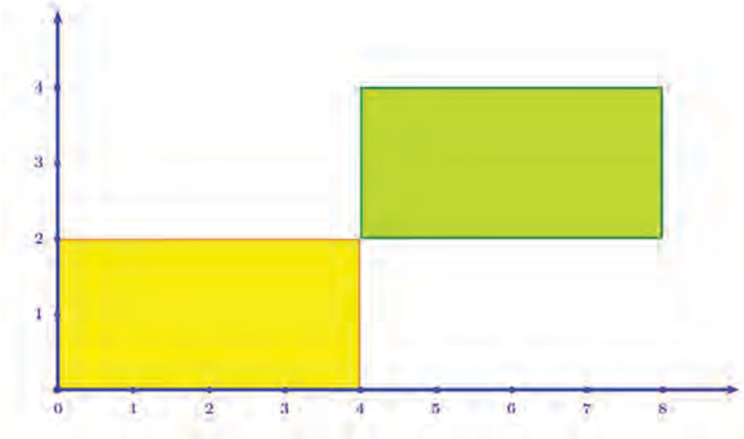


We can draw axes like this:



What are the coordinates of the vertices of the two rectangles?

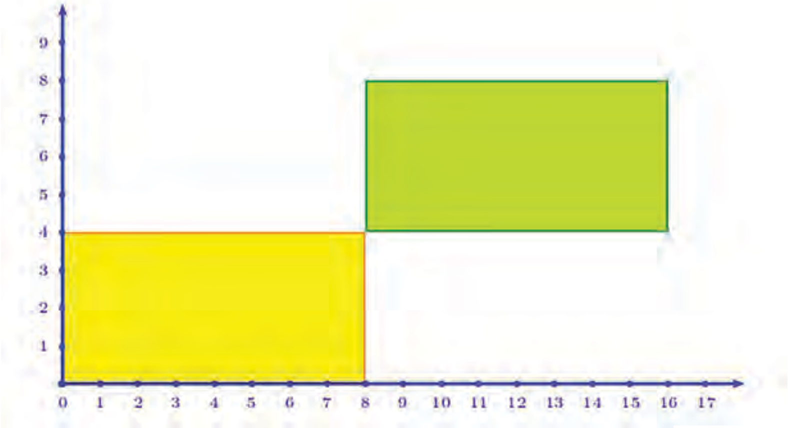
What if we draw axes like this?



With respect to these axes, what are the coordinates of the vertices of the rectangles?

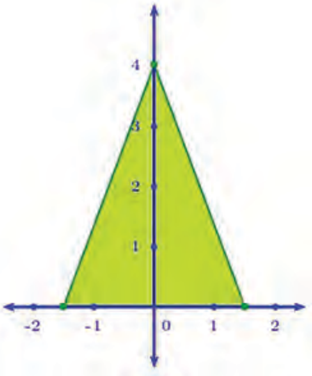
Again, the distance between the positions on the axes need not be 1 centimetre. Any convenient length would do.

For example, here is the last picture, with positions on the axes marked half a centimetre apart:



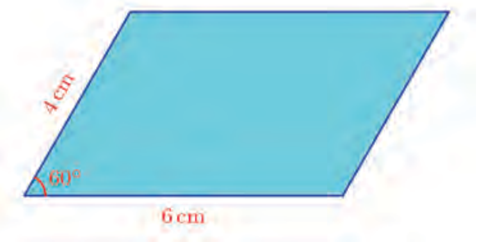
What are the coordinates of the vertices of the rectangles now?

Again, not all coordinates may be natural numbers. For example, to draw an isosceles triangle of base 3 centimetres and height 4 centimetres, the axes can be chosen like this:



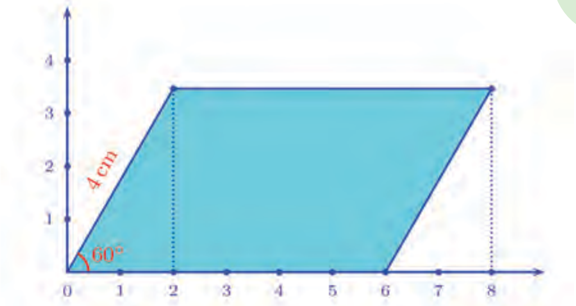
What are the coordinates of the vertices of the triangle?

What if we want to draw this parallelogram?



To get the point with coordinates $(2,\ 2\sqrt{3})$ in GeoGebra, type (2,2sqrt(3)) in the input bar

We can draw axes like this:



We know the ratio of the sides of a triangle with angles 30°, 60°, 90°. So the coordinates of the top-left vertex is $(2,\ 2\sqrt{3})$.

What about the top-right vertex?

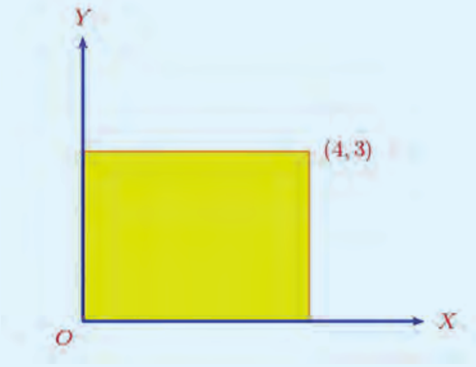
When we draw pictures using axes of coordinates, the *x*-axis, from left to right, is named *X′X* and the *y*-axis, from top to bottom, is named *YY′*; the origin as O.

(1) Find the following:

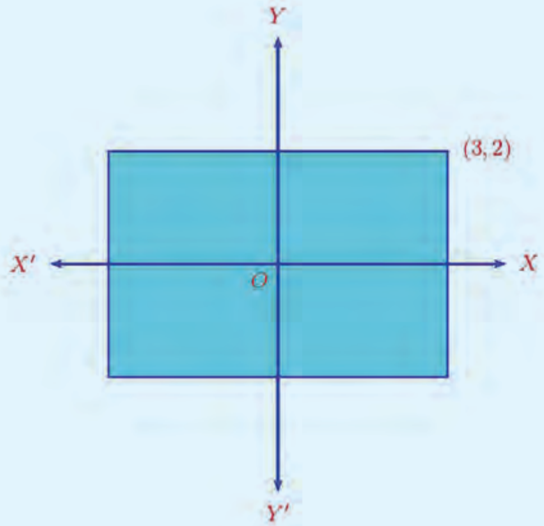
- (i) The *y*-coordinates of points on the *x*-axis
- (ii) The *x*-coordinates of points on the *y*-axis
- (iii) The coordinates of the origin
- (iv) The *y*-coordinates of points on the line parallel to the *x*-axis, through the point (0,1)
- (v) The *x*-coordinates of points on the line parallel to the *y*-axis, through the point (1,0)

Make a slider a in GeoGebra. Type (a,0) in the input bar. Move the slider; what is the path along which the point moves? Similarly mark each of the points (a,2), (a,1), (a,−2), (3,a), (−2,a) and see the path of travel of each as a changes. To see the entire path, check the option Trace on on the pop-up menu on right clicking the point.

(2) Find the coordinates of the other three vertices of the rectangle in the picture:



(3) The sides of the rectangle in the picture below are parallel to the axes and the origin is the midpoint (point of intersection of the diagonals) of the rectangle:



What are the coordinates of the other three vertices of the rectangle?

In GeoGebra, type

```
Sequence[(a,a+1),a,0,5]
```

in the input bar. This instructs the program to plot all points with coordinates (a,a+1) for all whole numbers from 0 to 5. That is the points with coordinates (0,1), (1,2), (2,3), (3,4), (4,5), (5, 6).

If we change the command slightly to

```
[Sequence(a,a+1),a,0,5,0.5]
```

In this the last number 0.5 instructs GeoGebra to plot all points with coordinates (a, a+1), with a starting at 0 as before, but increased by 0.5 (instead of 1 as in the first command) at every step; that is the points with coordinates (0,1), (0.5,1.5), (1,2), . . . , (5,6) (If the increment is to be 1, we need not specify it in the command)

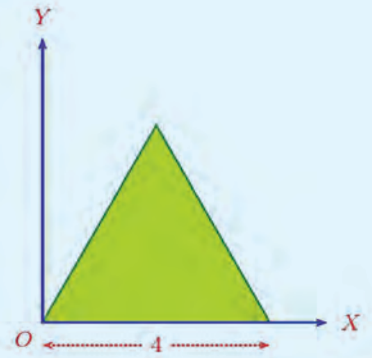
Use the commands below to plot points one by one and discuss the peculiarities of the points plotted in each case:

```
Sequence[(a,0),a,0,5,0.5]
Sequence[(a,2a),a,−3,4,0.25]
Sequence[(a,a),a,−3,3,0.2]
Sequence[(a,−a),a,−3.3,0.2]
Sequence[(a^2,a),a,−4,4,0.1]
```

In the last command a^2 stands for a^2

(4) The picture below shows an equilateral triangle:

Find the coordinates of the vertices of the triangle.

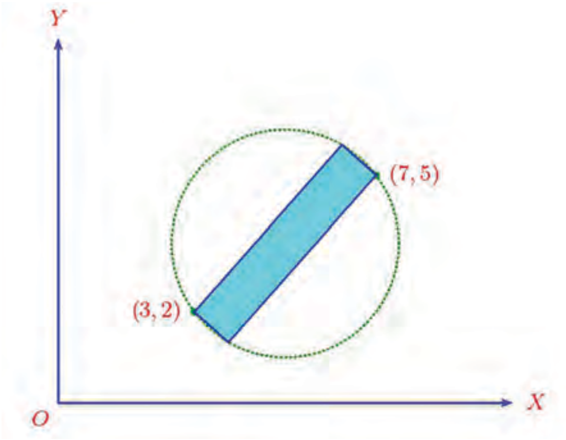


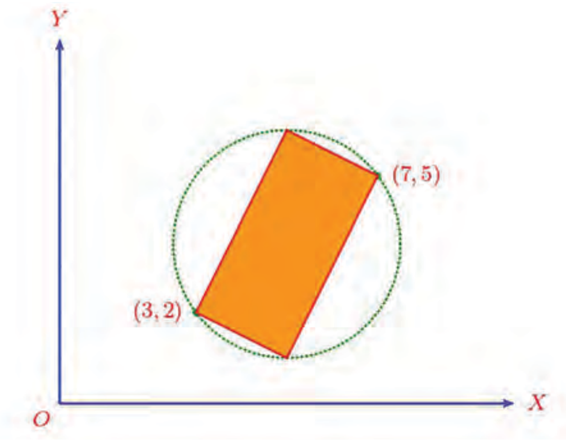
Rectangle math

See this picture:

We want to draw a rectangle with these as opposite vertices

We can draw several, can't we?





Typing circle[(1,3),2] in the input of GeoGebra, we get the circle centred at (1,3) with radius 2.

If we type

Sequence[circle[(a,0),1],a,0,5,0.2]

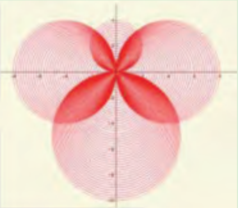
we get circles centred at the points (0,0), (0.2,0), (0.4,0), ..., (5,0) all with radius 1.Now try to visualize in mind the pictures we would get if we give the following commands.

Afterwards, you can actually draw them

Sequence[circle[(a,0),a],a,0,10,0.1]

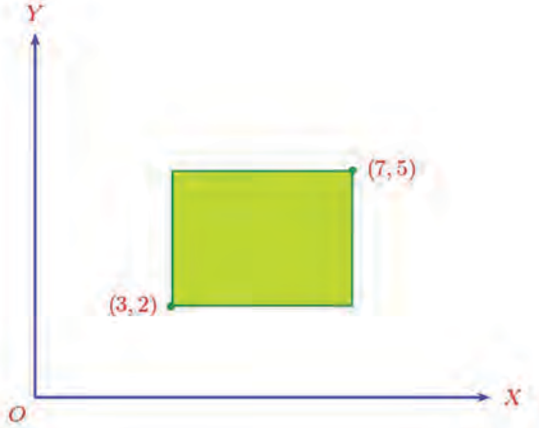
Sequence[circle[(a,0), $\frac{a}{4}$],a,0,10,0.1]

Now can you give the command to draw this picture?



Suppose we also want the sides of the rectangle to be parallel to the axes.

Then we can draw only one:

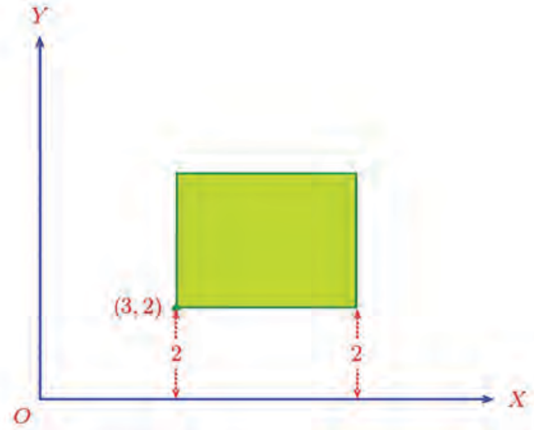


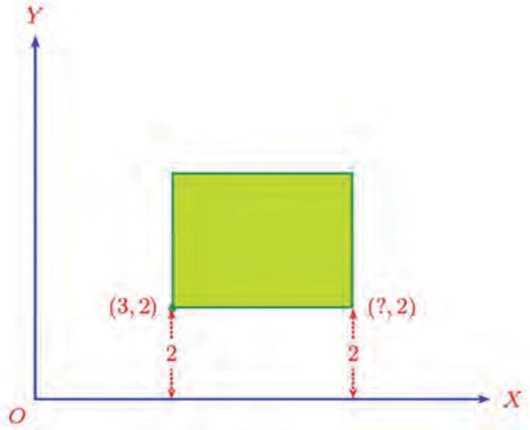
What are the coordinates of the other two vertices of this rectangle?

To find them, we have to explain the picture a little more.

The bottom-left corner of the rectangle has *y*-coordinate 2, which means its height from the *x*-axis is 2.

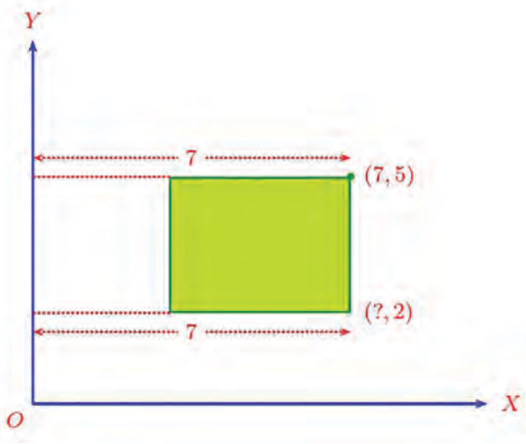
Since the bottom side of the rectangle is parallel to the *x*-axis, the other end of this side is also at a height 2 from the *x*-axis; that is its *y*-coordinate is also 2

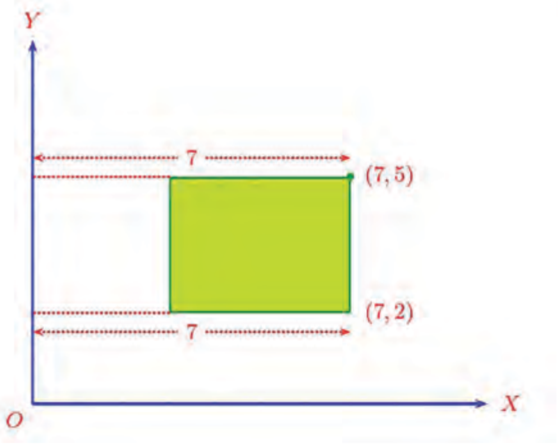




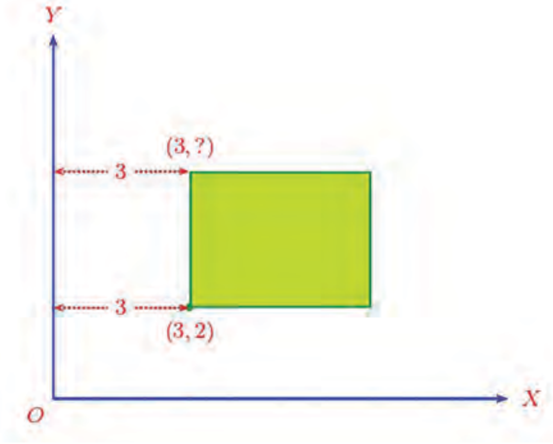
To find the *x*-coordinate of this point, look at the top-right corner of the rectangle. Since its *x*-coordinate is 7, it is at a distance of 7 from the *y*-axis.

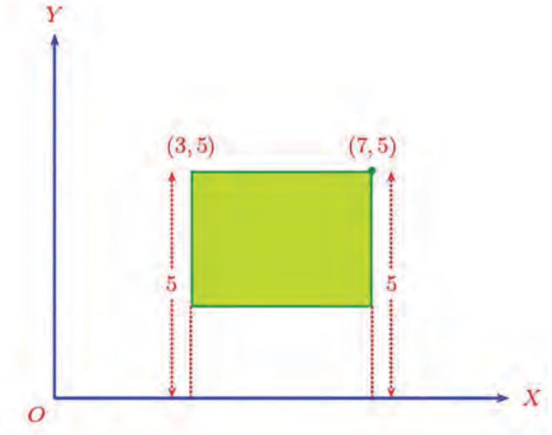
Since the right side of the rectangle is parallel to the *y*-axis, the other end of this side is also at the same distance from the *y*-axis. That is, its *x* coordinate is also 7





In the same way, we can find the coordinates of the top-left corner of the rectangle:

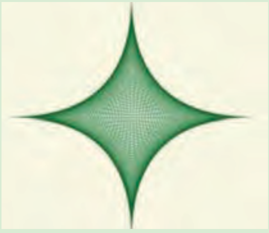




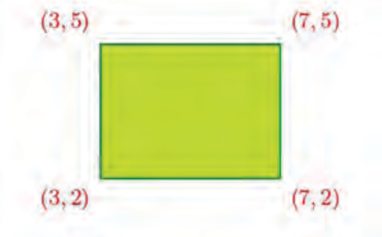
In GeoGebra, the command Segment[(2,−1),(3,5)] draws the line segment joining the points with coordinates (2,–1) and (3,5). Discuss the peculiarities of the line segments drawn by each of the command below:

```
Sequence[segment[(a,0),(a,3)],a,0,5,0.2]
Sequence[segment[(a,0),(a,a)],a,0,5,0.2]
Sequence[segment[(0,3),(a,0)],a,-4,4,0.1]
Sequence[segment[(a,0),(0,a)],a,-3,3,0.2]
Sequence[segment[(a,0),(0,5−a)],a,0,5,0.1]
```

Give the necessary commands to draw this picture:



Now look at the coordinates of all four vertices of the rectangle together:



And read again the method used to find the coordinates. What ideas did we use?

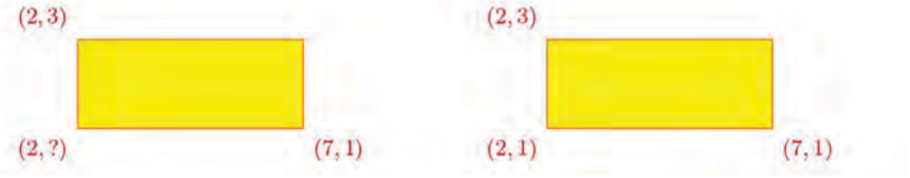
> Moving parallel to the *x*-axis does not change the *y*-coordinate; moving parallel to the *y*-axis does not change the *x*-coordinate.

See another rectangle with sides parallel to the axes:

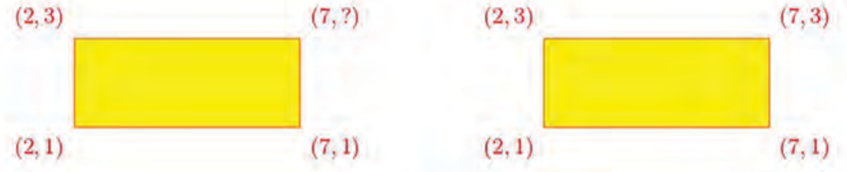
How do we find the coordinates of the other two vertices?

First let's look at the bottom-left corner. The top corner on the left side has *x*-coordinate 2; since this side is parallel to the *y*-axis, the *x*-coordinate of the bottom corner is also 2.

This corner is also on the bottom side of the rectangle. The *y*-coordinate of the right corner on this side is 1; since this side is parallel to the *x*-axis, the *y*-coordinate of the left corner is also 1.



Thinking along similar lines, we can find the coordinates of the top-right corner also:

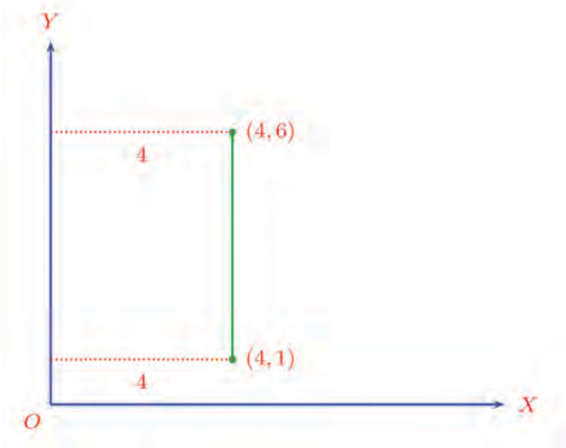


Given the coordinates of any two points, can we draw a rectangle with these as opposite vertices?

What if the points are as in one of these pictures:

Why can’t we draw a rectangle with either of these pairs as opposite vertices and sides parallel to the axes?

The line joining the first pair of points is parallel to the *y*-axis, and the line joining the second pair is parallel to the *x*-axis:



And in any rectangle with sides parallel to the axes, the diagonal cannot be parallel to either axis:

So, what can we say in general?

- If two points have the same *x*-coordinate, then the line joining them is parallel to the *y*-axis

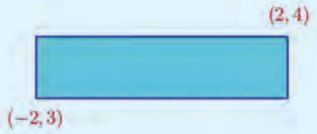
- If two points have the same *y*-coordinate, then the line joining them is parallel to the *x*-axis

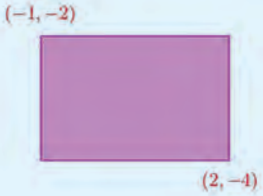
We cannot draw a rectangle with any such pair of points as opposite vertices and sides parallel to the axes.

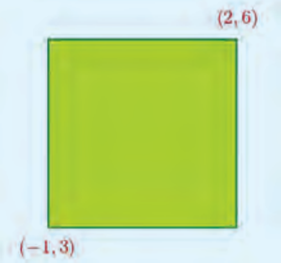
So, what can we say about those pair of points with which we can draw such a rectangle?

If a rectangle is to be drawn with a pair of points as opposite vertices and sides parallel to the axes, the *x*-coordinates of the points must be different and the *y*-coordinates of the points must be different.

(1) All these rectangles shown below have their sides parallel to the axes. Find the coordinates of the other two vertices of each:







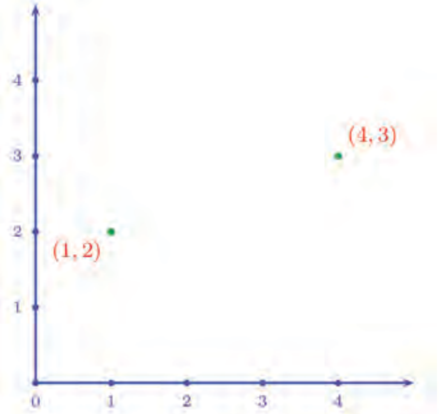
(2) Without drawing the axes, mark these points with the left and right, and top and bottom positions correct. Find the coordinates of the other two vertices of the rectangle with each pair as the coordinates of two opposite vertices and sides parallel to the axes.

(i) (3,5), (7,8) (ii) (6,2), (5, 4) (iii) (−3, 5), (−7,1) (iv) (−1, −2), (−5, −4)

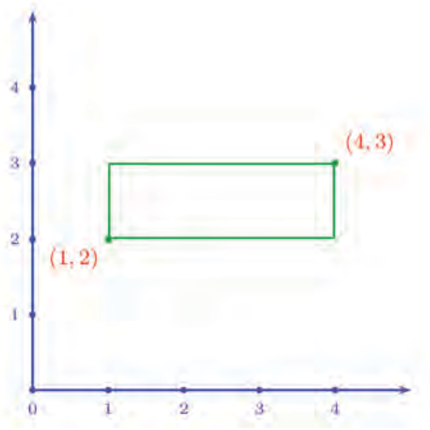
Lengths and distances

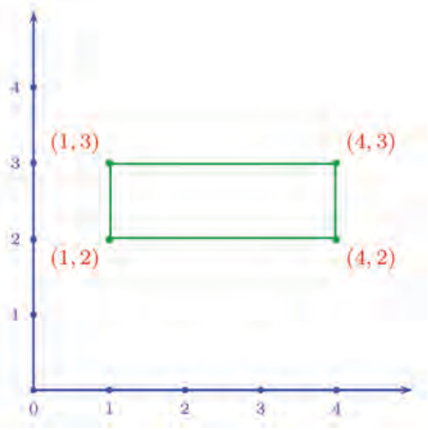
See this picture:

Two points are marked.



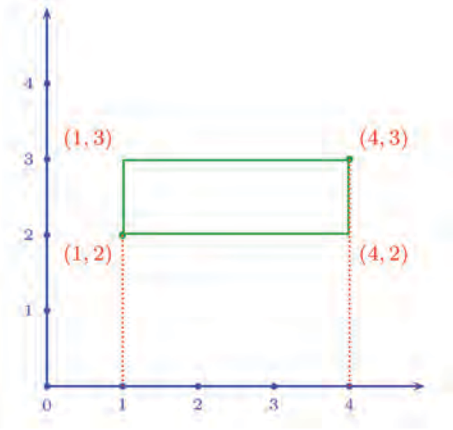
We can draw the rectangle with these as opposite vertices and sides parallel to the axes: also we can find the coordinates of the other two vertices.





Can you calculate the lengths of the sides of this rectangle?

The length of the top and bottom sides is the same as the distance between the points marked 1 and 4 on the *x*-axes, isn't it so?



What is the distance between these points?

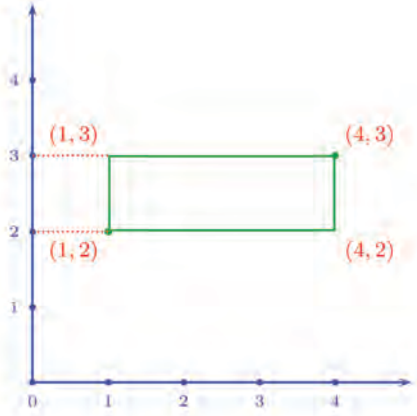
We have noted that to mark distance along the axes, we can choose any length as the unit. So, we can only say lengths and distances as how much of this unit they are.

So, the distance between the points marked 1 and 4 on the x-axis is 3 times this unit.

Usually we talk about these lengths and distances as numbers, without mentioning the unit.

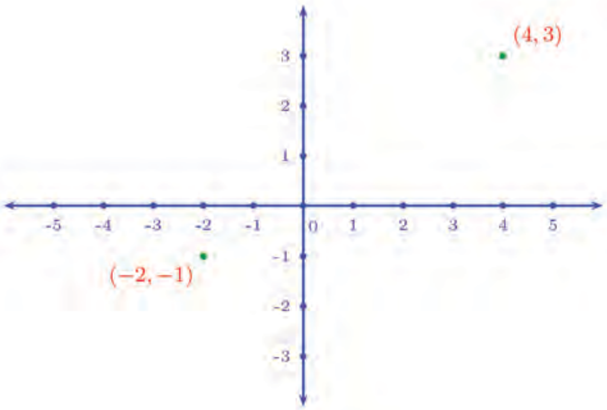
Thus we say that the lengths of the top and bottom sides of the rectangle is 3.

What about the lengths of the left and right sides?

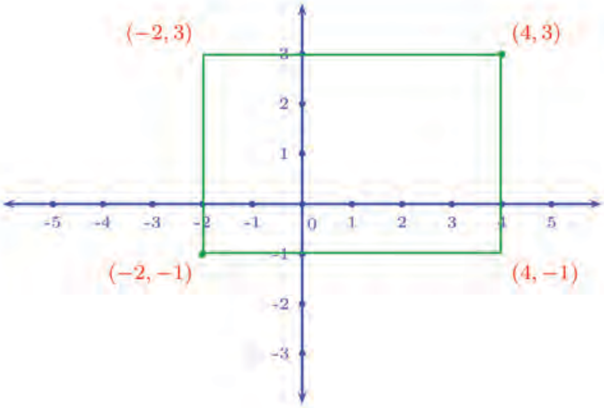


It is the distance between the points marked 2 and 3 on the *y*-axis; that is 1.

What if the opposite vertices are like this?



We can draw the rectangle and find the coordinates of the other two vertices:



The length of the top and bottom sides is the distance between which two points on the x-axis?

What is this distance?

Can you find the length of the left and right sides like this?

In general, what can we say about the lengths of the sides of the rectangle with the coordinates of a pair of opposite vertices (x_1, y_1) and (x_2, y_2) and sides parallel to the axes?

- The length of the top and bottom sides is the distance between the points marked x_1 and x_2 on the x-axis
- The length of the left and right sides is the distance between the points marked y_1 and y_2 on the y-axis

How do we calculate such distances?

We can think of each axis as a number line. Then, as seen in Class 9, each of these distances is found from the numbers marking the points, by subtracting the smaller from the larger; and this operation can be written as an absolute value (The section, **Distances** of the lesson, **Real Numbers**).

So, what we have said above can be written using algebra like this:

In the rectangle with opposite vertices (x_1, y_1), (x_2, y_2) and sides parallel to the axes

- the length of the top and bottom sides is $|x_1 - x_2|$
- the length of the left and right sides is $|y_1 - y_2|$

Now once we know the coordinates of the opposite vertices, we can calculate the lengths of the sides of a rectangle, without drawing the axes.

For example, suppose the opposite vertices are $(-1, 3)$ and $(5, -2)$

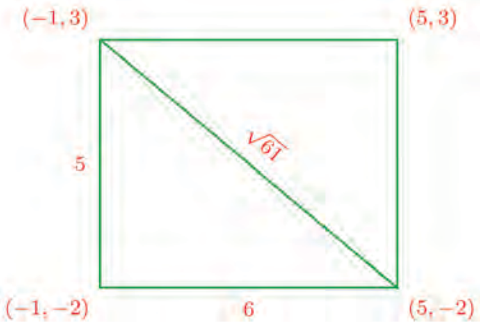
The length of the top and bottom sides is $5 - (-1) = 6$

The length of the left and right sides is $3 - (-2) = 5$

The line joining the opposite vertices is the diagonal of the rectangle, isn't it ?

What is its length?

$$\sqrt{6^2 + 5^2} = \sqrt{61}$$



Any pair of points with different x-coordinates and different y-coordinates can be the opposite vertices of a rectangle

So, the distance between any such pair of points can be computed like this

For example, how do we calculate the distance between the points (2, 5) and (6, 6) ?

The lengths of the sides of the rectangles with these as opposite vertices are 6 – 2 = 4 and 6 – 5 = 1; so the length of its diagonal is

$$\sqrt{4^2+1^2}=\sqrt{17}$$

How do we state this as a general principle?

(i) If two points with coordinates (x_1, y_1) and (x_2, y_2) are to be coordinates of the opposite vertices of a rectangle, we must have $x_1 \neq x_2$ and $y_1 \neq y_2$

(ii) The lengths of the sides of the rectangle are $|x_1 - x_2|$, $|y_1 - y_2|$

(iii) Since the distance between the points is the diagonal of this rectangle, this distance is $\sqrt{|x_1-x_2|^2+|y_1-y_2|^2}$

We have seen in Class 9 that the square of the absolute value of a number is just the square of the number itself (the section, **Absolute value** of the lesson, **Real Numbers**).

So,

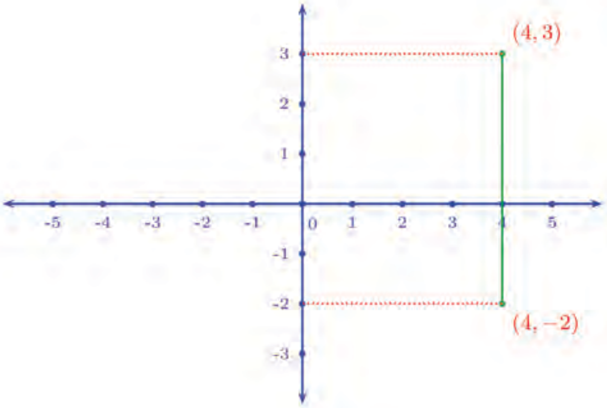
$$\sqrt{|x_1-x_2|^2+|y_1-y_2|^2} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Combining all this we have this

> If the coordinates of two points are (x_1, y_1) and (x_2, y_2) such that $x_1 \neq x_2$, and $y_1 \neq y_2$, then the distance between these points is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Now what can we say about the distance between two points with one (or both) of the coordinates the same?

For example, consider the points (4,–2) and (4, 3)



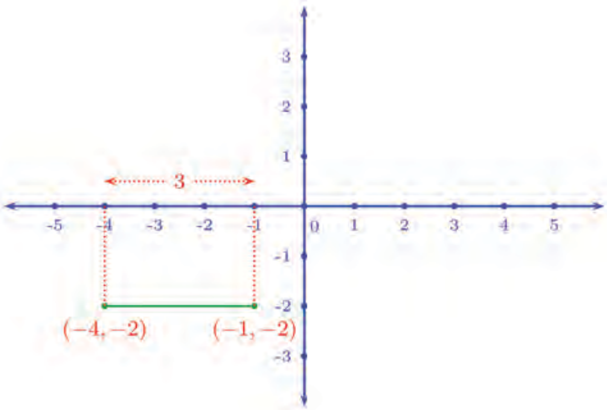
The distance between them is equal to the distance between the points marked 3 and –2 on the *y*-axis, right?

And this is 3 – (–2) = 5

In general,

> If two points have the same *x*-coordinate, they can be denoted (x, y_1) and (x, y_2) and the distance between them is $|y_1 - y_2|$

What about points with the same *y*-coordinate?



> If two points have the same *y*-coordinate, they can be denoted $(x_1, y)$and (x_2, y); and the distance between them is $|x_1 - x_2|$.

Now what happens if we put $x_1 = x_2$ in the algebraic expression $\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$ used to compute the distance between points with different *x* and *y* coordinates?

We get

$$\sqrt{|y_1 - y_2|^2} = |y_1 - y_2|$$

And if we take $y_1 = y_2$ instead, we get

$$\sqrt{|x_1 - x_2|^2} = |x_1 - x_2|$$

So, we can use the same operation to compute the distance between points with one of the coordinates the same

The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

For example, the distance between points with coordinates (4, –2) and (–3, –1) is

$$\sqrt{(4-(-3))^2 + (-2-(-1))^2} = \sqrt{7^2 + (-1)^2} = \sqrt{50} = 5\sqrt{2}$$

What is the distance between the point with coordinates (–2, 1) and the origin?

$$\sqrt{(-2-0)^2+(1-0)^2} = \sqrt{5}$$

In general,

The distance between the point with coordinates (x, y) and the origin is $\sqrt{x^2+y^2}$

Now look at this problem:

Are the points with coordinates (–1, 2), (3, 5), (9, – 3) on the same line?

If three points are on the same line, then the largest of the distances must be the sum of the other two distances.

Let's name the three points in this problem, *A*, *B*, *C*. Then

$$AB = \sqrt{(-1-3)^2+(2-5)^2} = \sqrt{16+9} = 5$$

$$BC = \sqrt{(3-9)^2+(5-(-3))^2} = \sqrt{36+64} = 10$$

$$AC = \sqrt{(-1-9)^2+(2-(-3))^2} = \sqrt{100+25} = 5\sqrt{5}$$

The largest of these is the distance AC (how do we get that ?).

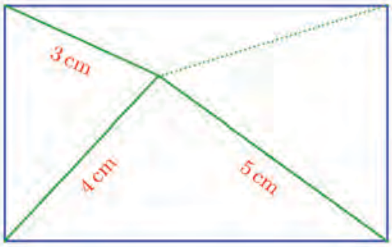
The sum of the other two distances *AB* and *BC* is 15, which is not equal to the distance *AC*.

So, *A*, *B*, *C* are not on the same line

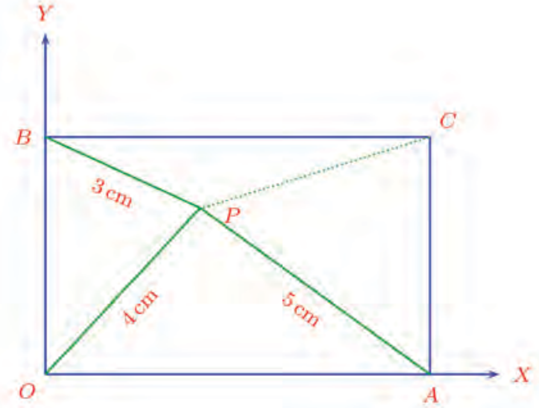
Let's look at another problem:

The distances of a point within a rectangle to three vertices are 3 centimetres, 4 centimetres and 5 centimetres. What is the distance to the fourth vertex?

Let's draw a picture:



We can take the bottom-left corner of the rectangle as the origin and axes along the two sides through it:

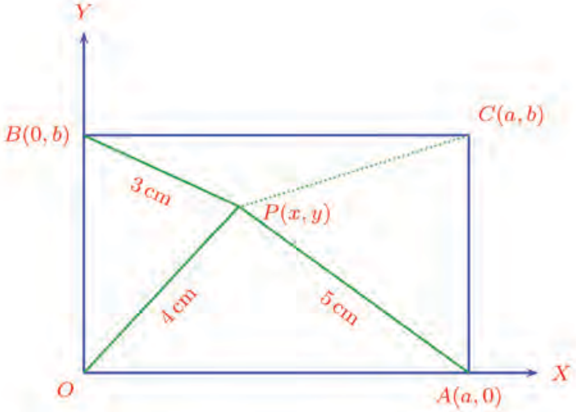


In the picture, the point A is on the x-axis; so if we take its x-coordinate as a, then its coordinates are $(a, 0)$

Similarly, if the y-coordinate of B is taken as b, its coordinates are $(0, b)$

So, the coordinates of C must be (a, b)

Let's take the coordinates of P as (x, y)



Now let's write the known distances using coordinates:

$$x^2 + (y - b)^2 = 9$$
$$x^2 + y^2 = 16$$
$$(x - a)^2 + y^2 = 25$$

What we want to find is the distance PC. It's square is

$$(x - a)^2 + (y - b)^2$$

Can we filter out this from the three equations above?

Adding the first and the last of these equations, we get

$$(x^2 + (y - b)^2) + (x - a)^2 + y^2 = 9 + 25$$

That is

$$x^2 + y^2 + (x - a)^2 + (y - b)^2 = 34$$

According to the second equation of our first three equations, $x^2 + y^2 = 16$. Using this in the equation we got now,

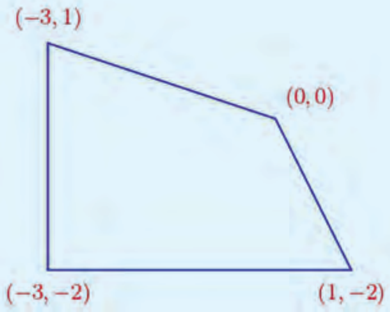
$$16 + (x - a)^2 + (y - b)^2 = 34$$

and from this we get

$$(x - a)^2 + (y - b)^2 = 34 - 16 = 18$$

Thus the distance PC is $\sqrt{18} = 3\sqrt{2}$ centimetres.

(1) Calculate the lengths of the sides and the diagonals of the quadrilateral in the picture below:



(2) Prove that by joining the points (2, 1), (3, 4), (–3, 6), we get a right triangle

(3) A circle is drawn with centre at the origin and with radius 10.

- (i) Check whether each of the points with coordinates (6, 9), (5, 9), (6, 8) is within the circle,outside the circle or on the circle.
- (ii) Write the coordinates of 8 points on this circle.

(4) Calculate the coordinates of the points where the circle centred on (1, 1) and with radius $\sqrt{2}$ intersects the coordinate axes.

(5) The vertices of a triangle are the points (0,0), (4,0), (1,3). Calculate the centre and radius of its circumcircle.

CONSTITUTION OF INDIA

Part IV A

FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

CHILDREN'S RIGHTS

Dear Children,

Wouldn't you like to know about your rights? Awareness about your rights will inspire and motivate you to ensure your protection and participation, thereby making social justice a reality. You may know that a commission for child rights is functioning in our state called the ***Kerala State Commission for Protection of Child Rights.***

Let's see what your rights are:

- Right to freedom of speech and expression.
- Right to life and liberty.
- Right to maximum survival and development.
- Right to be respected and accepted regardless of caste, creed and colour.
- Right to protection and care against physical, mental and sexual abuse.
- Right to participation.
- Protection from child labour and hazardous work.
- Protection against child marriage.
- Right to know one's culture and live accordingly.
- Protection against neglect.
- Right to free and compulsory education.
- Right to learn, rest and leisure.
- Right to parental and societal care, and protection.

Major Responsibilities

- Protect school and public facilities.
- Observe punctuality in learning and activities of the school.
- Accept and respect school authorities, teachers, parents and fellow students.
- Readiness to accept and respect others regardless of caste, creed or colour.

Contact Address:

Kerala State Commission for Protection of Child Rights
'Sree Ganesh', T. C. 14/2036, Vanross Junction
Kerala University P. O., Thiruvananthapuram - 34, Phone : 0471 - 2326603
Email: childrights.cpcr@kerala.gov.in, rte.cpcr@kerala.gov.in
Website : www.kescpcr.kerala.gov.in

Child Helpline - 1098, Crime Stopper - 1090, Nirbhaya - 1800 425 1400

Kerala Police Helpline - 0471 - 3243000/44000/45000

Online ***R. T. E Monitoring : www.nireekshana.org.in***