7 COORDINATES

Position and number

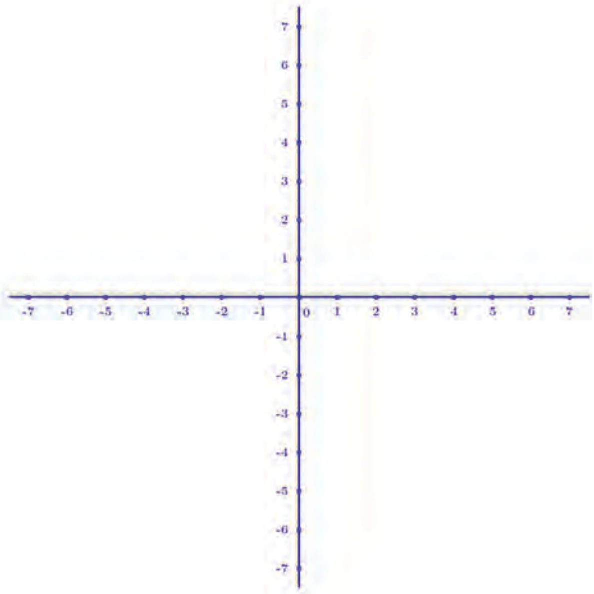
Remember drawing graphs of polynomials in Class 9? (The lesson **Polynomial Pictures**).

For example, how did we draw the graph of the polynomial $p(x) = 2x + 1$?

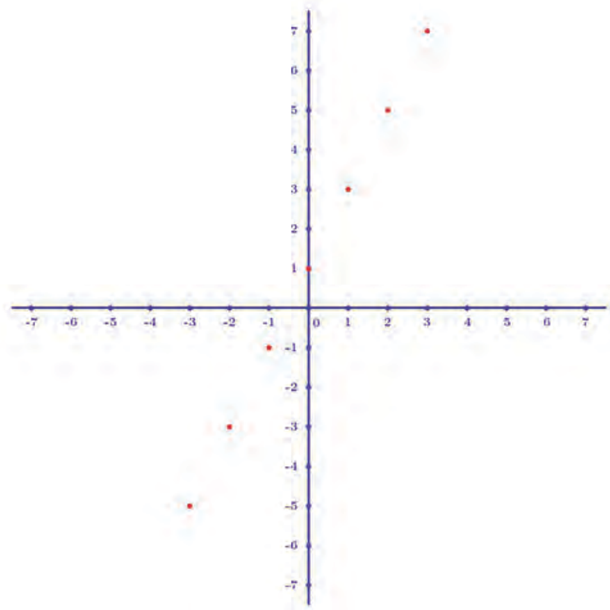
Take different numbers as x and calculate $p(x)$ for each of these:

x	-3	-2	-1	0	1	2	3
$p(x)$	-5	-3	-1	1	3	5	7

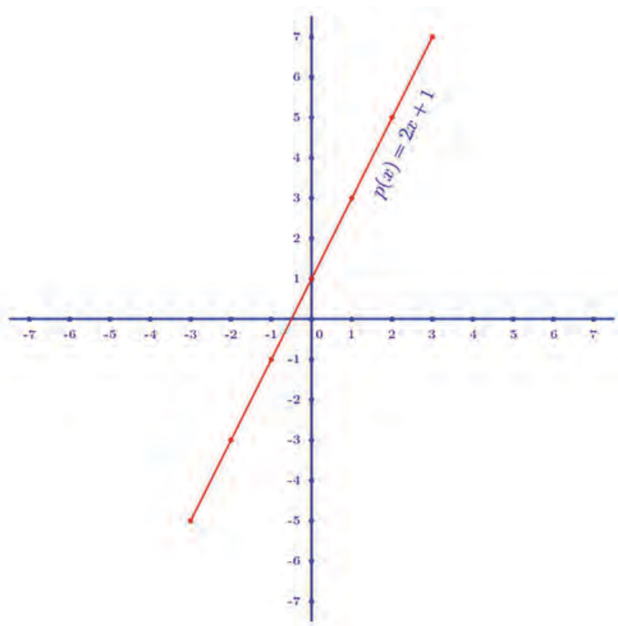
To mark them as points in a picture, first we drew a horizontal line and a vertical line and marked numbers on them at equal distances apart:



Then marked points at height $p(x)$ above each point x

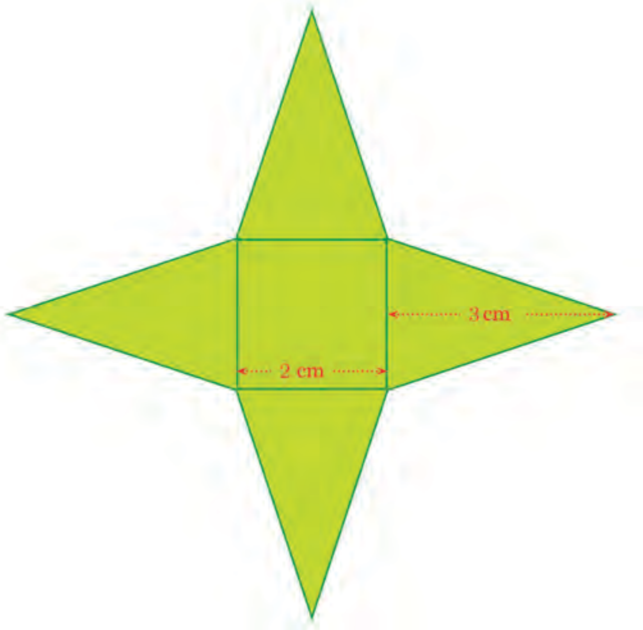


The line joining these points is the graph of the polynomial $p(x) = 2x + 1$

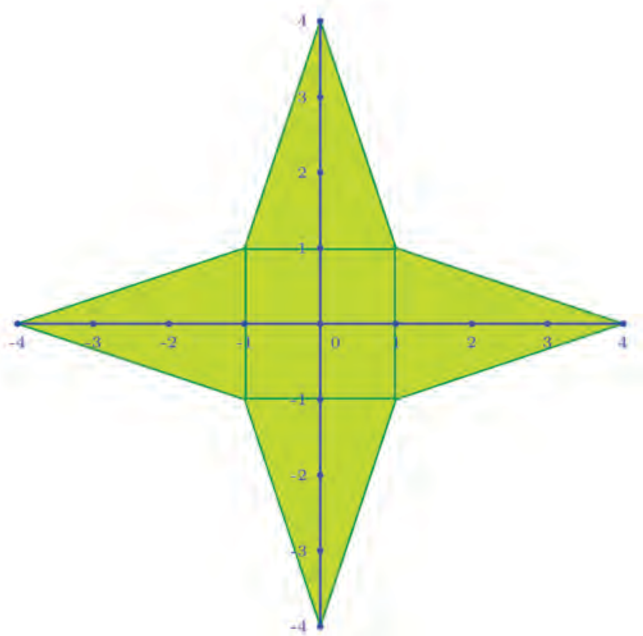


We can use this method to draw other figures also

For example, let's see how we can draw the picture below like this:

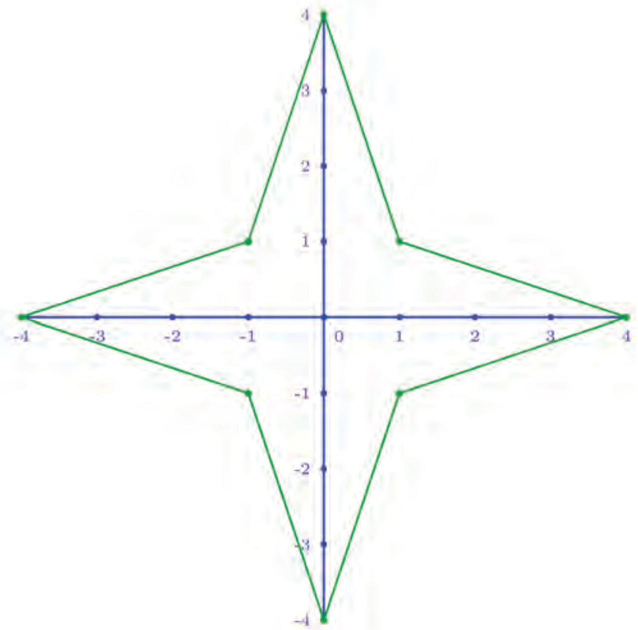


Imagine that horizontal and vertical lines are drawn through the middle of this picture, with distances marked 1 centimetre apart:



Thus by drawing horizontal and vertical lines and marking distances 1 centimetre apart on them, we can mark all eight corners of the figure; and joining them, we get the star

Draw this picture in your notebook, by drawing horizontal and vertical lines and marking the corners of the star

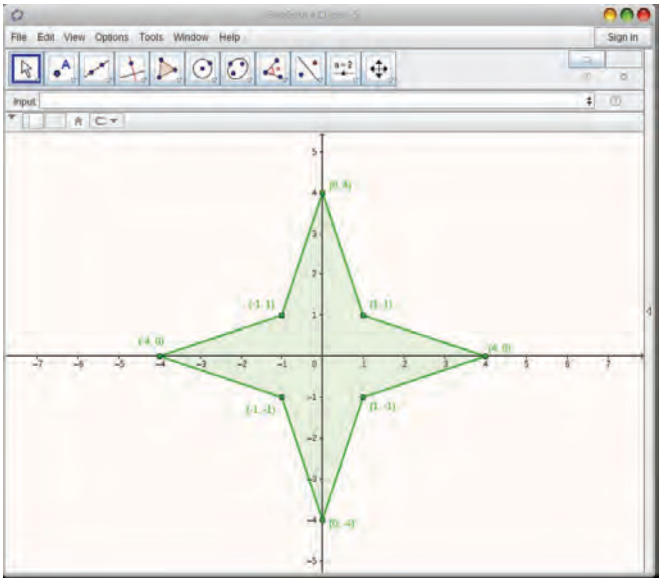


Now let's see how this can be drawn in GeoGebra. Remember how we plotted points as a pair of numbers?

For example, in the polynomial $p(x) = x^2$, if we take $x = 0.5$, we get $p(x) = 0.25$. So, the point at a height of 0.25 above the position marked 0.5 on the horizontal line, is a point on the graph of $p(x)$. We have seen that to mark this in GeoGebra we need only type (0.5,0.25) in the input bar (The section **Second degree polynomials** of the lesson, **Polynomial Pictures** in the Class 9 textbook).

Similarly, the top-right corner of the picture above is at a height of 1 centimetre from the point marking 1 centimetre on the horizontal line. So we can mark this point as (1, 1).

Thus we can mark all corners of the picture above:

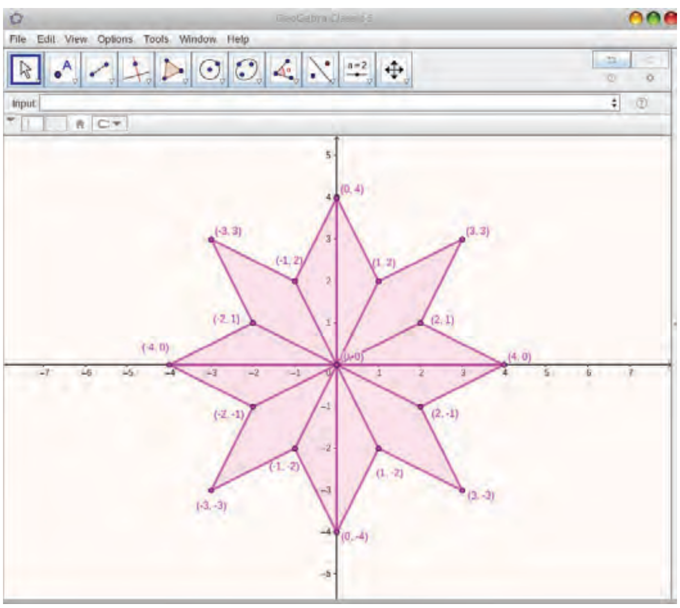


To plot many points in GeoGebra we need only input them one by one as pairs of numbers. To get the polygon joining them, we use the Polygon command.

For example, type this command in the input bar and see:

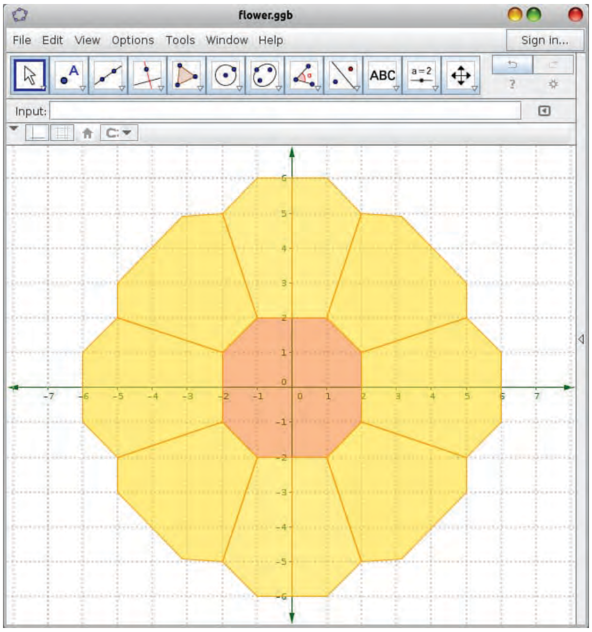
Polygon[(−1,−1),(1,−1),(1,1),(−1,1)]

Now can you draw the picture below in GeoGebra?



Open GeoGebra and input the number pairs (0,0), (4,0), (2,1), (3,3), (1,2), (0,4). Select the Polygon tool and click on the points A, B, C, D, E, F, A in this order to draw the top right quarter of the picture. Select the Segment tool and click on A, C and then A, E to draw the two lines inside this. Next select the Reflect about Line tool and click on the picture and the vertical line. Do you get the image of the picture on the left ? In the same way, the two lines in the right quarter can also be reflected onto the left. To get the bottom half of the picture, using the same tool, reflect each of the two quarters on the top of the horizontal line onto the bottom.

How about this picture?



For convenience, the Grid in GeoGebra is also shown in it.

Can you draw these in your notebook?

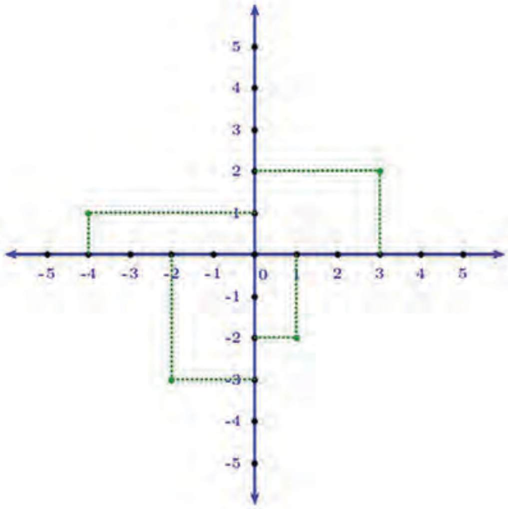
Geometry and numbers

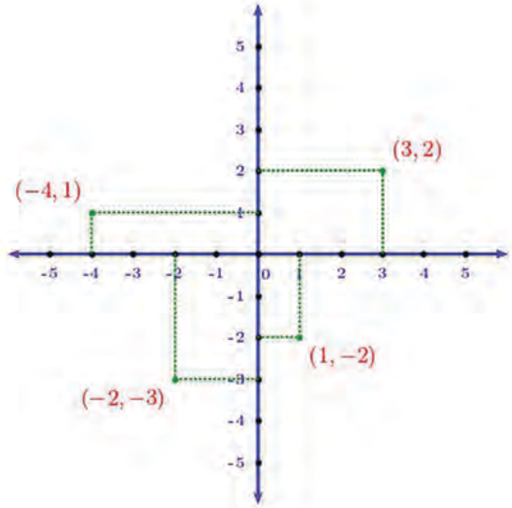
We have seen how we can draw pictures by representing points using pairs of numbers.

These two numbers are calculated using a horizontal line and a vertical line. These lines are called the **axes of coordinates**, the horizontal line being the ***x*-axis** and the vertical one the ***y*-axis**. The point of intersection of these lines is called **origin.**

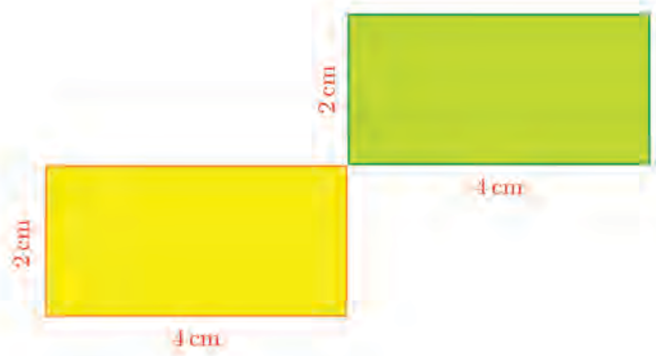
Once the axes are drawn, we can write the location of any point as a pair of numbers, as seen in the earlier examples. These numbers are called the **coordinates** of this point. The first number is the ***x*-coordinate** and the second is the ***y*-coordinate.**

To get the coordinates of a point, we need only draw perpendiculars to the axes. For example, see these pictures:

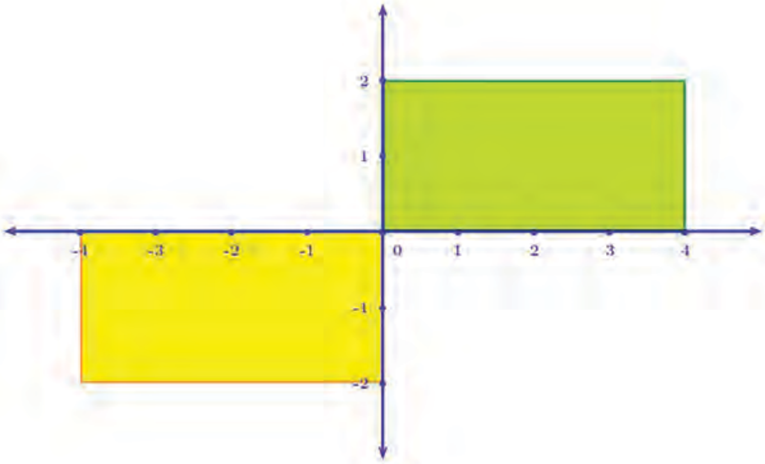




To draw a picture, we can take any two mutually perpendicular lines as axes. For example, see this picture:

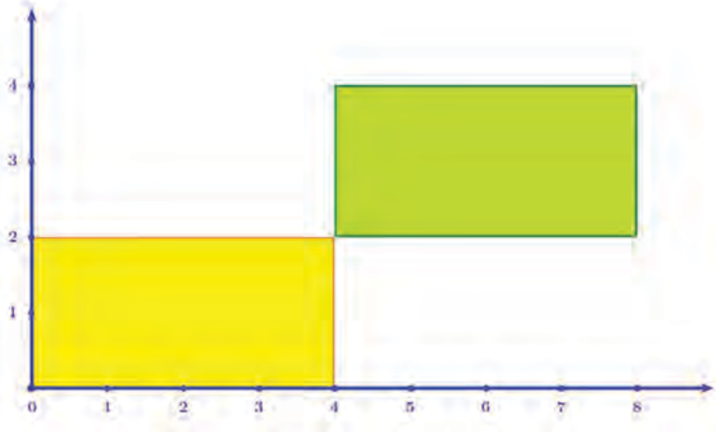


We can draw axes like this:



What are the coordinates of the vertices of the two rectangles?

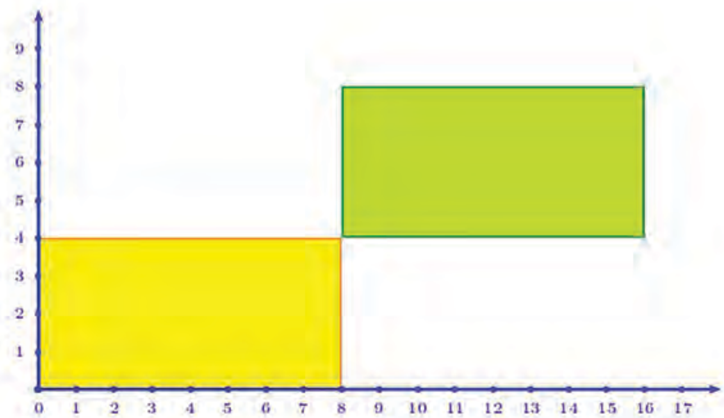
What if we draw axes like this?



With respect to these axes, what are the coordinates of the vertices of the rectangles?

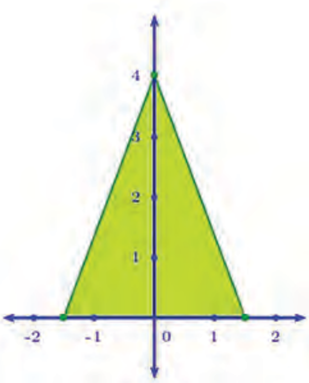
Again, the distance between the positions on the axes need not be 1 centimetre. Any convenient length would do.

For example, here is the last picture, with positions on the axes marked half a centimetre apart:



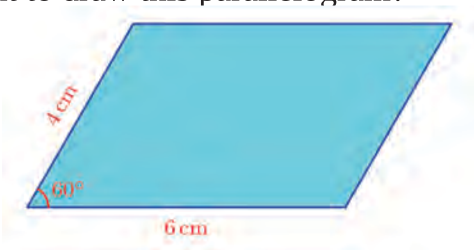
What are the coordinates of the vertices of the rectangles now?

Again, not all coordinates may be natural numbers. For example, to draw an isosceles triangle of base 3 centimetres and height 4 centimetres, the axes can be chosen like this:



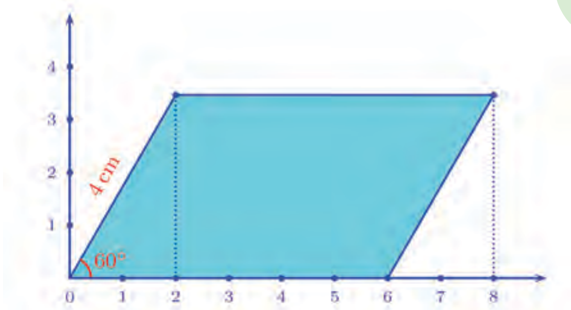
What are the coordinates of the vertices of the triangle?

What if we want to draw this parallelogram?



To get the point with coordinates $\left(2,\ 2\sqrt{3}\right)$ in GeoGebra, type (2,2sqrt(3)) in the input bar

We can draw axes like this:



We know the ratio of the sides of a triangle with angles 30°, 60°, 90°. So the coordinates of the top-left vertex is $\left(2,\ 2\sqrt{3}\right)$.

What about the top-right vertex?

When we draw pictures using axes of coordinates, the *x*-axis, from left to right, is named $X'X$ and the *y*-axis, from top to bottom, is named YY'; the origin as O.

(1) Find the following:

(i) The *y*-coordinates of points on the *x*-axis

(ii) The *x*-coordinates of points on the *y*-axis

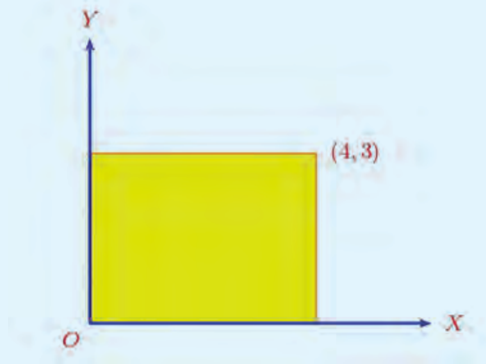
(iii) The coordinates of the origin

(iv) The *y*-coordinates of points on the line parallel to the *x*-axis, through the point (0,1)

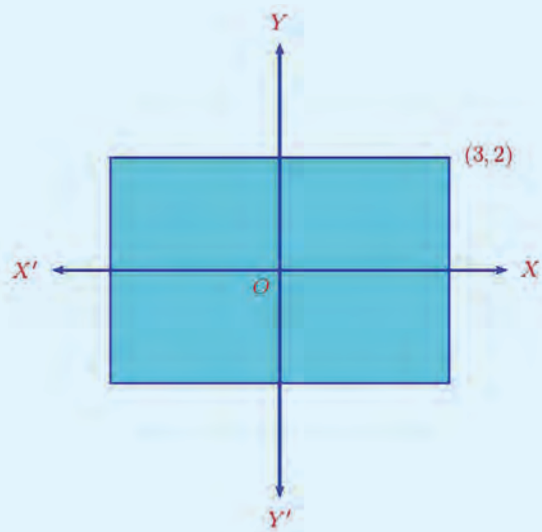
(v) The *x*-coordinates of points on the line parallel to the *y*-axis, through the point (1,0)

Make a slider a in GeoGebra. Type (a,0) in the input bar. Move the slider; what is the path along which the point moves? Similarly mark each of the points (a,2), (a,1), (a,−2), (3,a), (−2,a) and see the path of travel of each as a changes. To see the entire path, check the option Trace on on the pop-up menu on right clicking the point.

(2) Find the coordinates of the other three vertices of the rectangle in the picture:



(3) The sides of the rectangle in the picture below are parallel to the axes and the origin is the midpoint (point of intersection of the diagonals) of the rectangle:



What are the coordinates of the other three vertices of the rectangle?

In GeoGebra, type

Sequence[(a,a+1),a,0,5]

in the input bar. This instructs the program to plot all points with coordinates (a,a+1) for all whole numbers from 0 to 5. That is the points with coordinates (0,1), (1,2), (2,3), (3,4), (4,5), (5, 6).

If we change the command slightly to

[Sequence(a,a+1),a,0,5,0.5]

In this the last number 0.5 instructs GeoGebra to plot all points with coordinates (a, a+1), with a starting at 0 as before, but increased by 0.5 (instead of 1 as in the first command) at every step; that is the points with coordinates (0,1), (0.5,1.5), (1,2), . . . , (5,6) (If the increment is to be 1, we need not specify it in the command)

Use the commands below to plot points one by one and discuss the peculiarities of the points plotted in each case:

Sequence[(a,0),a,0,5,0.5]

Sequence[(a,2a),a,−3,4,0.25]

Sequence[(a,a),a,−3,3,0.2]

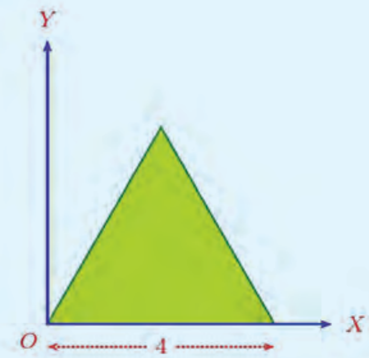
Sequence[(a,−a),a,−3.3,0.2]

Sequence[(a^2,a),a,−4,4,0.1]

In the last command a^2 stands for a^2

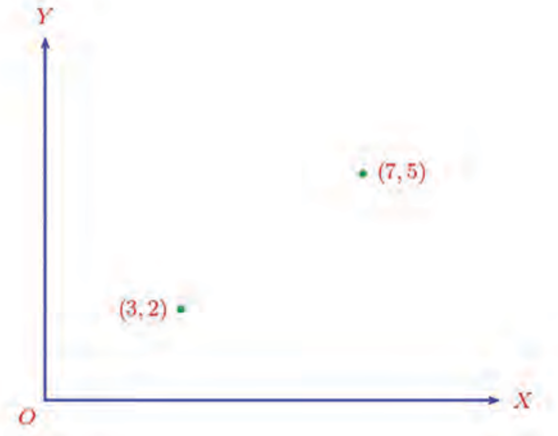
(4) The picture below shows an equilateral triangle:

Find the coordinates of the vertices of the triangle.



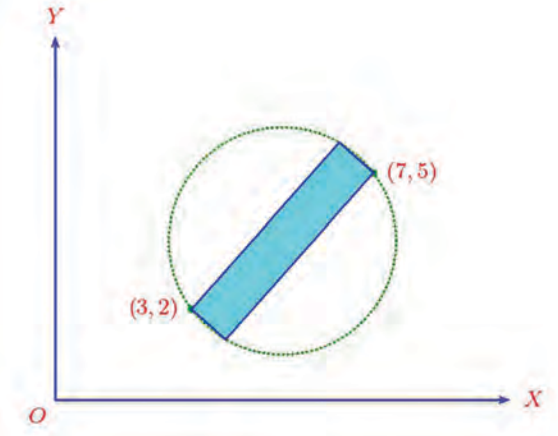
Rectangle math

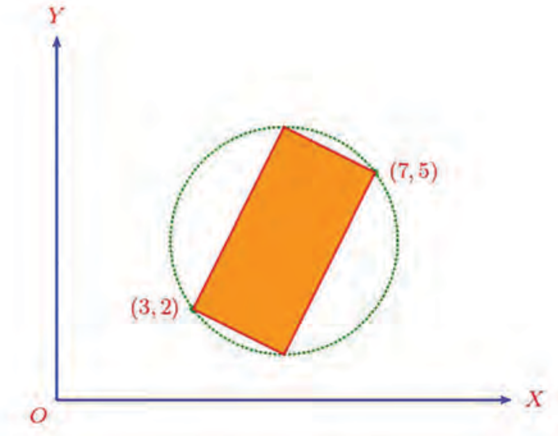
See this picture:



We want to draw a rectangle with these as opposite vertices

We can draw several, can't we?





Typing circle[(1,3),2] in the input of GeoGebra, we get the circle centred at (1,3) with radius 2.

If we type

Sequence[circle[(a,0),1],a,0,5,0.2]

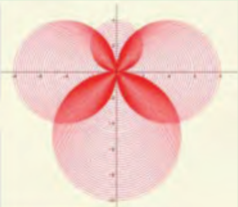
we get circles centred at the points (0,0), (0.2,0), (0.4,0), ..., (5,0) all with radius 1.Now try to visualize in mind the pictures we would get if we give the following commands.

Afterwards, you can actually draw them

Sequence[circle[(a,0),a],a,0,10,0.1]

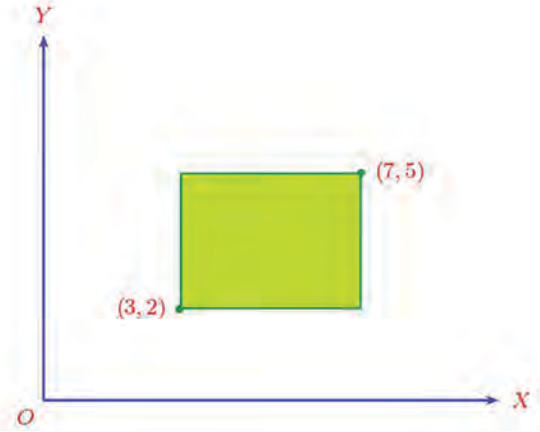
Sequence[circle[(a,0), $\frac{a}{4}$],a,0,10,0.1]

Now can you give the command to draw this picture?



Suppose we also want the sides of the rectangle to be parallel to the axes.

Then we can draw only one:

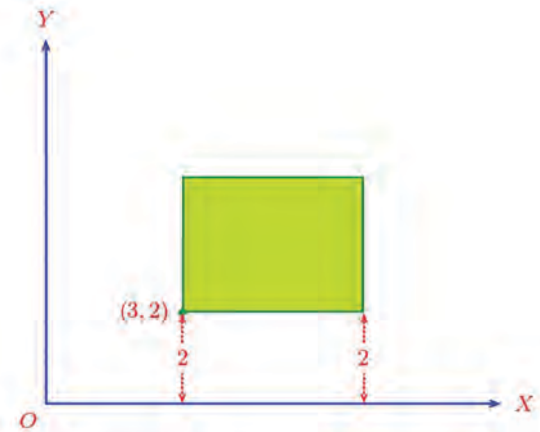


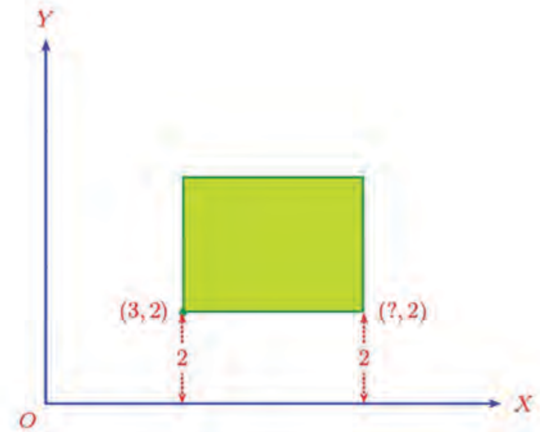
What are the coordinates of the other two vertices of this rectangle?

To find them, we have to explain the picture a little more.

The bottom-left corner of the rectangle has *y*-coordinate 2, which means its height from the *x*-axis is 2.

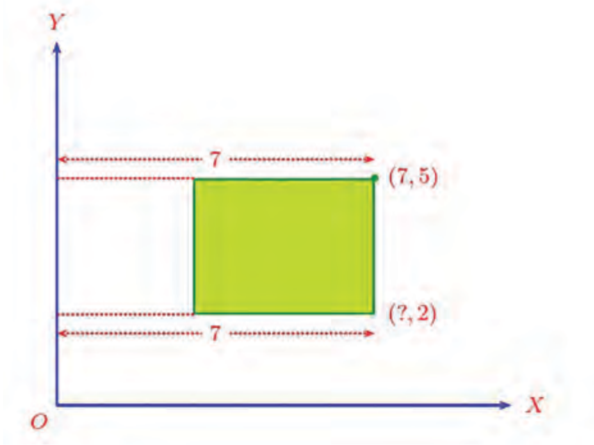
Since the bottom side of the rectangle is parallel to the *x*-axis, the other end of this side is also at a height 2 from the *x*-axis; that is its *y*-coordinate is also 2

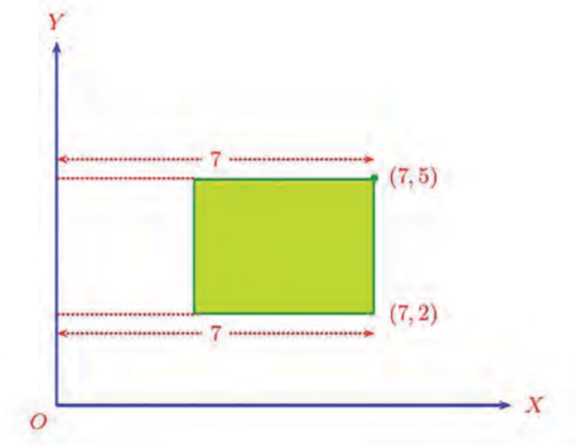




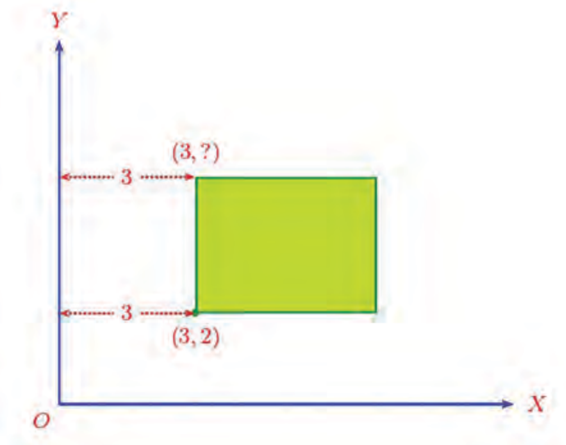
To find the *x*-coordinate of this point, look at the top-right corner of the rectangle. Since its *x*-coordinate is 7, it is at a distance of 7 from the *y*-axis.

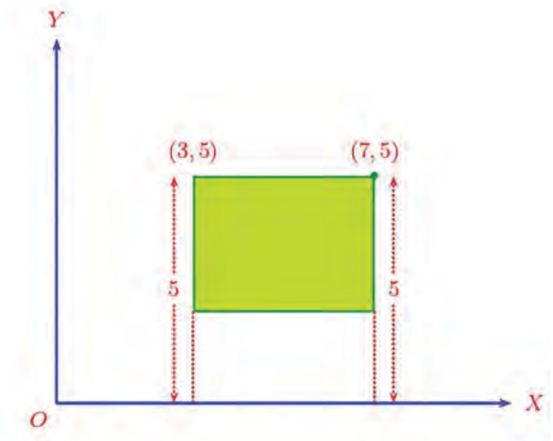
Since the right side of the rectangle is parallel to the *y*-axis, the other end of this side is also at the same distance from the *y*-axis. That is, its *x* coordinate is also 7





In the same way, we can find the coordinates of the top-left corner of the rectangle:





In GeoGebra, the command Segment[(2,−1),(3,5)] draws the line segment joining the points with coordinates (2,–1) and (3,5). Discuss the peculiarities of the line segments drawn by each of the command below:

Sequence[segment[(a,0),(a,3)],a,0,5,0.2]

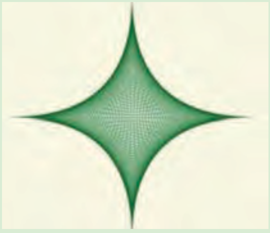
Sequence[segment[(a,0),(a,a)],a,0,5,0.2]

Sequence[segment[(0,3),(a,0)],a,-4,4,0.1]

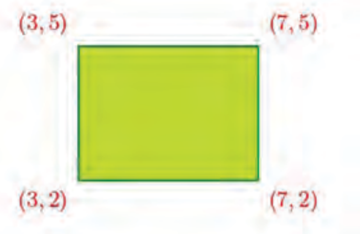
Sequence[segment[(a,0),(0,a)],a,-3,3,0.2]

Sequence[segment[(a,0),(0,5−a)],a,0,5,0.1]

Give the necessary commands to draw this picture:



Now look at the coordinates of all four vertices of the rectangle together:



And read again the method used to find the coordinates. What ideas did we use?

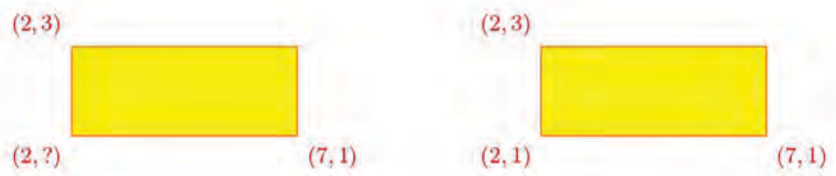
Moving parallel to the x-axis does not change the y-coordinate; moving parallel to the y-axis does not change the x-coordinate.

See another rectangle with sides parallel to the axes:

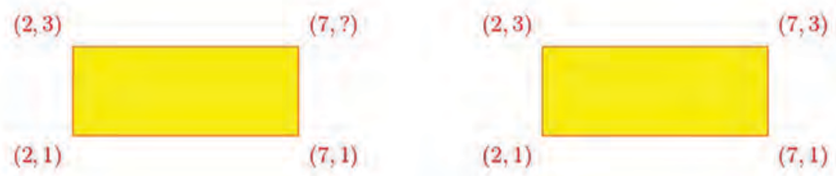
How do we find the coordinates of the other two vertices?

First let's look at the bottom-left corner. The top corner on the left side has x-coordinate 2; since this side is parallel to the y-axis, the x-coordinate of the bottom corner is also 2.

This corner is also on the bottom side of the rectangle. The y-coordinate of the right corner on this side is 1; since this side is parallel to the x-axis, the y-coordinate of the left corner is also 1.

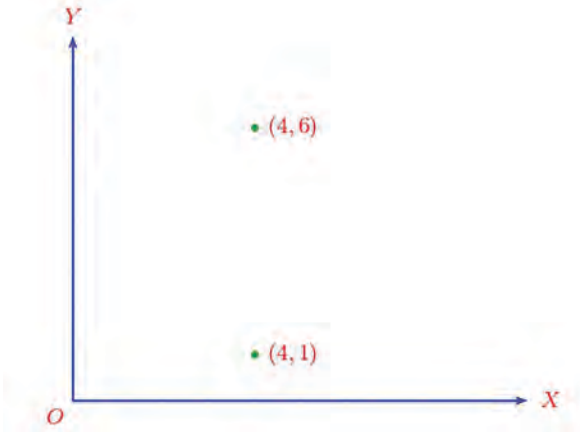


Thinking along similar lines, we can find the coordinates of the top-right corner also:



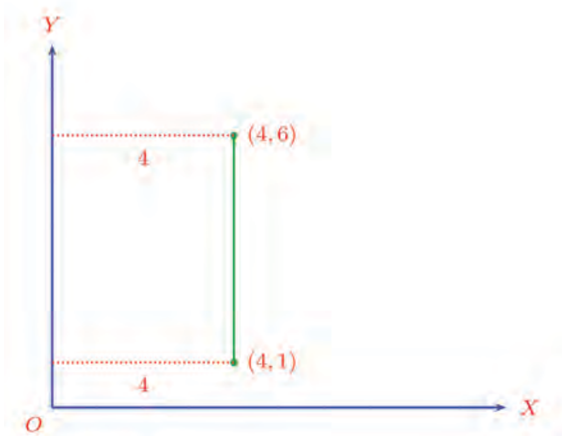
Given the coordinates of any two points, can we draw a rectangle with these as opposite vertices?

What if the points are as in one of these pictures:



Why can't we draw a rectangle with either of these pairs as opposite vertices and sides parallel to the axes?

The line joining the first pair of points is parallel to the *y*-axis, and the line joining the second pair is parallel to the *x*-axis:



And in any rectangle with sides parallel to the axes, the diagonal cannot be parallel to either axis:

So, what can we say in general?

- If two points have the same *x*-coordinate, then the line joining them is parallel to the *y*-axis

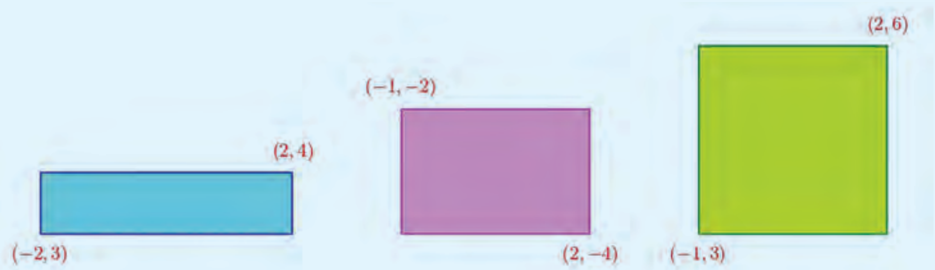
- If two points have the same y-coordinate, then the line joining them is parallel to the x-axis

We cannot draw a rectangle with any such pair of points as opposite vertices and sides parallel to the axes.

So, what can we say about those pair of points with which we can draw such a rectangle?

If a rectangle is to be drawn with a pair of points as opposite vertices and sides parallel to the axes, the x-coordinates of the points must be different and the y-coordinates of the points must be different.

(1) All these rectangles shown below have their sides parallel to the axes. Find the coordinates of the other two vertices of each:



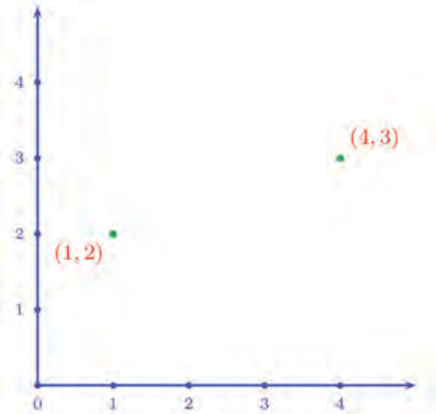
(2) Without drawing the axes, mark these points with the left and right, and top and bottom positions correct. Find the coordinates of the other two vertices of the rectangle with each pair as the coordinates of two opposite vertices and sides parallel to the axes.

(i) (3,5), (7,8) (ii) (6,2), (5, 4) (iii) (−3, 5), (−7,1) (iv) (−1, −2), (−5, −4)

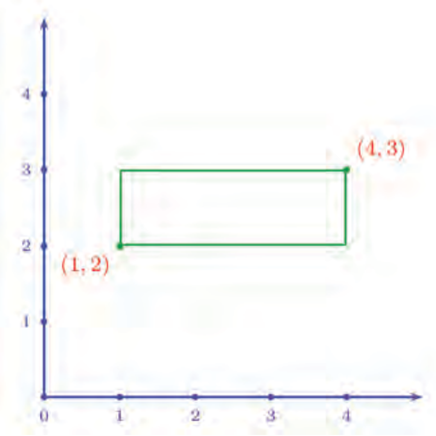
Lengths and distances

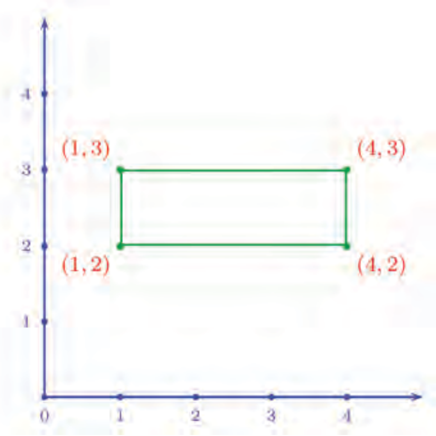
See this picture:

Two points are marked.



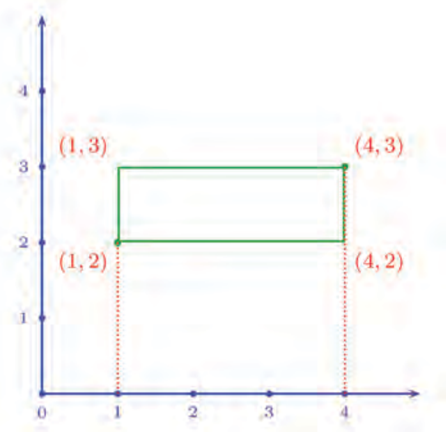
We can draw the rectangle with these as opposite vertices and sides parallel to the axes: also we can find the coordinates of the other two vertices.





Can you calculate the lengths of the sides of this rectangle?

The length of the top and bottom sides is the same as the distance between the points marked 1 and 4 on the *x*-axes, isn't it so?



What is the distance between these points?

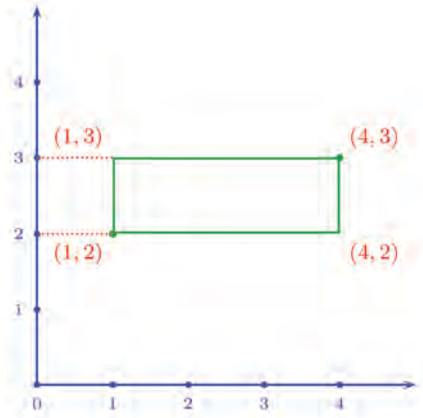
We have noted that to mark distance along the axes, we can choose any length as the unit. So, we can only say lengths and distances as how much of this unit they are.

So, the distance between the points marked 1 and 4 on the x-axis is 3 times this unit.

Usually we talk about these lengths and distances as numbers, without mentioning the unit.

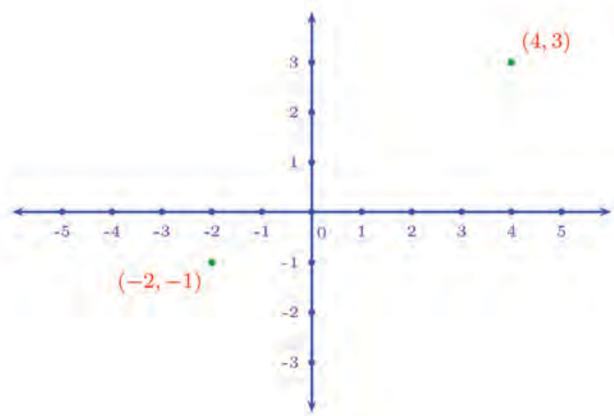
Thus we say that the lengths of the top and bottom sides of the rectangle is 3.

What about the lengths of the left and right sides?

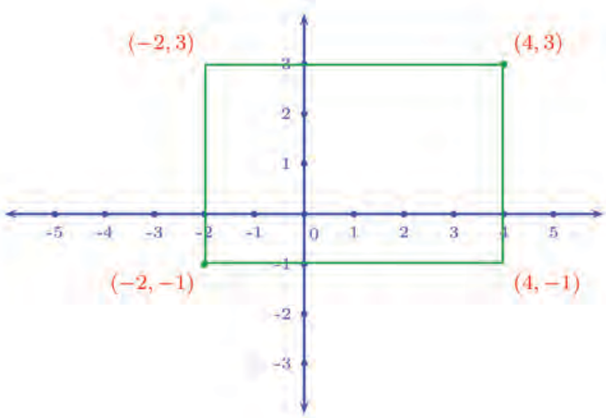


It is the distance between the points marked 2 and 3 on the *y*-axis; that is 1.

What if the opposite vertices are like this?



We can draw the rectangle and find the coordinates of the other two vertices:



The length of the top and bottom sides is the distance between which two points on the x-axis?

What is this distance?

Can you find the length of the left and right sides like this?

In general, what can we say about the lengths of the sides of the rectangle with the coordinates of a pair of opposite vertices (x_1, y_1) and (x_2, y_2) and sides parallel to the axes?

- The length of the top and bottom sides is the distance between the points marked x_1 and x_2 on the x-axis
- The length of the left and right sides is the distance between the points marked y_1 and y_2 on the y-axis

How do we calculate such distances?

We can think of each axis as a number line. Then, as seen in Class 9, each of these distances is found from the numbers marking the points, by subtracting the smaller from the larger; and this operation can be written as an absolute value (The section, **Distances** of the lesson, **Real Numbers**).

So, what we have said above can be written using algebra like this:

In the rectangle with opposite vertices (x_1, y_1), (x_2, y_2) and sides parallel to the axes

- the length of the top and bottom sides is $|x_1 - x_2|$
- the length of the left and right sides is $|y_1 - y_2|$

Now once we know the coordinates of the opposite vertices, we can calculate the lengths of the sides of a rectangle, without drawing the axes.

For example, suppose the opposite vertices are $(-1, 3)$ and $(5, -2)$

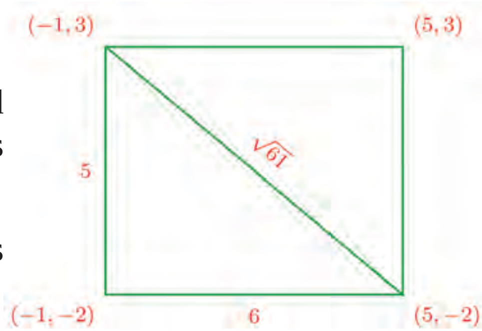
The length of the top and bottom sides is $5 - (-1) = 6$

The length of the left and right sides is $3 - (-2) = 5$

The line joining the opposite vertices is the diagonal of the rectangle, isn't it ?

What is its length?

$$\sqrt{6^2 + 5^2} = \sqrt{61}$$



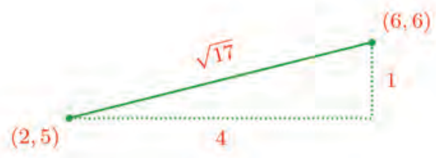
Any pair of points with different x-coordinates and different y-coordinates can be the opposite vertices of a rectangle

So, the distance between any such pair of points can be computed like this

For example, how do we calculate the distance between the points (2, 5) and (6, 6) ?

The lengths of the sides of the rectangles with these as opposite vertices are 6 – 2 = 4 and 6 – 5 = 1; so the length of its diagonal is

$$\sqrt{4^2 + 1^2} = \sqrt{17}$$



How do we state this as a general principle?

(i) If two points with coordinates (x_1, y_1) and (x_2, y_2) are to be coordinates of the opposite vertices of a rectangle, we must have $x_1 \neq x_2$ and $y_1 \neq y_2$

(ii) The lengths of the sides of the rectangle are $|x_1 - x_2|$, $|y_1 - y_2|$

(iii) Since the distance between the points is the diagonal of this rectangle, this distance is $\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$

We have seen in Class 9 that the square of the absolute value of a number is just the square of the number itself (the section, **Absolute value** of the lesson, **Real Numbers**).

So,

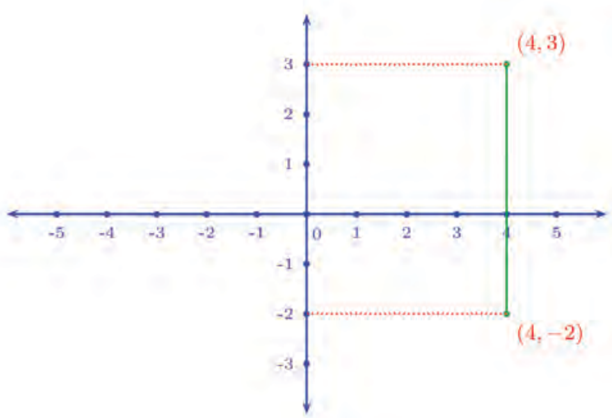
$$\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Combining all this we have this

If the coordinates of two points are (x_1, y_1) and (x_2, y_2) such that $x_1 \neq x_2$, and $y_1 \neq y_2$, then the distance between these points is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Now what can we say about the distance between two points with one (or both) of the coordinates the same?

For example, consider the points (4,–2) and (4, 3)



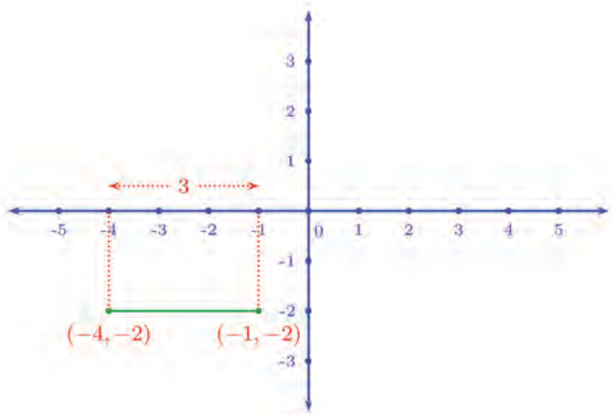
The distance between them is equal to the distance between the points marked 3 and –2 on the y-axis, right?

And this is $3 - (-2) = 5$

In general,

If two points have the same x-coordinate, they can be denoted (x, y_1) and (x, y_2) and the distance between them is $|y_1 - y_2|$

What about points with the same y-coordinate?



If two points have the same y-coordinate, they can be denoted $(x_1, y)$and (x_2, y); and the distance between them is $|x_1 - x_2|$.

Now what happens if we put $x_1 = x_2$ in the algebraic expression $\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$ used to compute the distance between points with different x and y coordinates?

We get

$$\sqrt{|y_1 - y_2|^2} = |y_1 - y_2|$$

And if we take $y_1 = y_2$ instead, we get

$$\sqrt{|x_1 - x_2|^2} = |x_1 - x_2|$$

So, we can use the same operation to compute the distance between points with one of the coordinates the same

The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

For example, the distance between points with coordinates (4, –2) and (–3, –1) is

$$\sqrt{\left(4-(-3)\right)^2 + \left(-2-(-1)\right)^2} = \sqrt{7^2 + (-1)^2} = \sqrt{50} = 5\sqrt{2}$$

What is the distance between the point with coordinates (–2, 1) and the origin?

$$\sqrt{(-2-0)^2+(1-0)^2} = \sqrt{5}$$

In general,

The distance between the point with coordinates (x, y) and the origin is $\sqrt{x^2+y^2}$

Now look at this problem:

Are the points with coordinates (–1, 2), (3, 5), (9, – 3) on the same line?

If three points are on the same line, then the largest of the distances must be the sum of the other two distances.

Let's name the three points in this problem, *A*, *B*, *C*. Then

$$AB = \sqrt{(-1-3)^2+(2-5)^2} = \sqrt{16+9} = 5$$

$$BC = \sqrt{(3-9)^2+(5-(-3))^2} = \sqrt{36+64} = 10$$

$$AC = \sqrt{(-1-9)^2+(2-(-3))^2} = \sqrt{100+25} = 5\sqrt{5}$$

The largest of these is the distance AC (how do we get that ?).

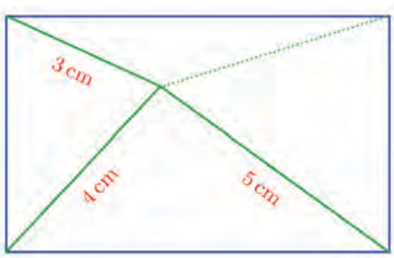
The sum of the other two distances *AB* and *BC* is 15, which is not equal to the distance *AC*.

So, *A*, *B*, *C* are not on the same line

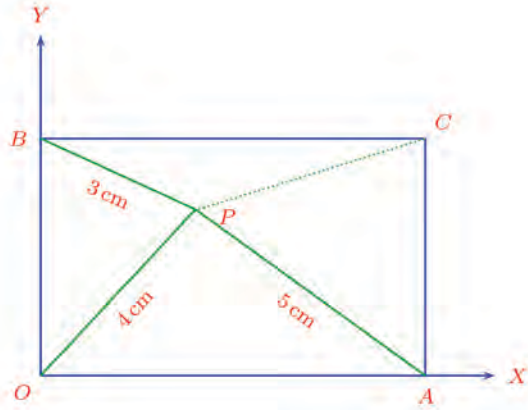
Let's look at another problem:

The distances of a point within a rectangle to three vertices are 3 centimetres, 4 centimetres and 5 centimetres. What is the distance to the fourth vertex?

Let's draw a picture:



We can take the bottom-left corner of the rectangle as the origin and axes along the two sides through it:

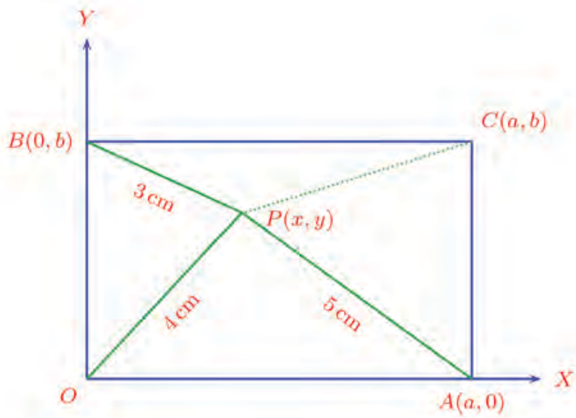


In the picture, the point A is on the x-axis; so if we take its x-coordinate as a, then its coordinates are $(a, 0)$

Similarly, if the y-coordinate of B is taken as b, its coordinates are $(0, b)$

So, the coordinates of C must be (a, b)

Let's take the coordinates of P as (x, y)



Now let's write the known distances using coordinates:

$$x^2 + (y - b)^2 = 9$$
$$x^2 + y^2 = 16$$
$$(x - a)^2 + y^2 = 25$$

What we want to find is the distance PC. It's square is

$$(x - a)^2 + (y - b)^2$$

Can we filter out this from the three equations above?

Adding the first and the last of these equations, we get

$$(x^2 + (y - b)^2) + (x - a)^2 + y^2 = 9 + 25$$

That is

$$x^2 + y^2 + (x - a)^2 + (y - b)^2 = 34$$

According to the second equation of our first three equations, $x^2 + y^2 = 16$. Using this in the equation we got now,

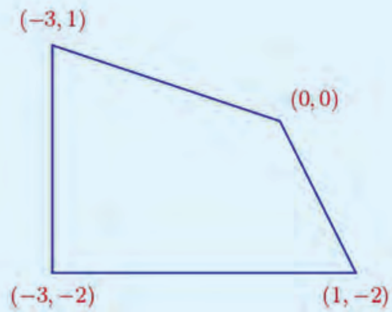
$$16 + (x - a)^2 + (y - b)^2 = 34$$

and from this we get

$$(x - a)^2 + (y - b)^2 = 34 - 16 = 18$$

Thus the distance PC is $\sqrt{18} = 3\sqrt{2}$ centimetres.

(1) Calculate the lengths of the sides and the diagonals of the quadrilateral in the picture below:



(2) Prove that by joining the points (2, 1), (3, 4), (–3, 6), we get a right triangle

(3) A circle is drawn with centre at the origin and with radius 10.

(i) Check whether each of the points with coordinates (6, 9), (5, 9), (6, 8) is within the circle,outside the circle or on the circle.

(ii) Write the coordinates of 8 points on this circle.

(4) Calculate the coordinates of the points where the circle centred on (1, 1) and with radius $\sqrt{2}$ intersects the coordinate axes.

(5) The vertices of a triangle are the points (0,0), (4,0), (1,3). Calculate the centre and radius of its circumcircle.