6 TRIGONOMETRY

Angles and sides

See this triangle:

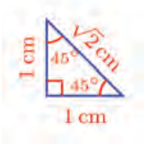
What are the lengths of its other two sides?

Since two of the angles are 90° and 45°, the third angle is also 45°

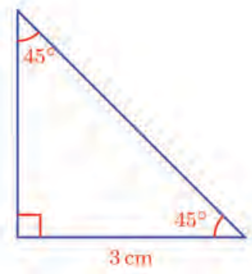
Now since two angles are equal, the sides opposite them are also equal (The section **Isosceles triangles** of the lesson **Equal Triangles** in the Class 8 textbook).

What about the third side?

We have seen that its length is $\sqrt{2}$ centimetres in Class 9 (the section **Lengths and numbers** in the lesson **New Numbers**).



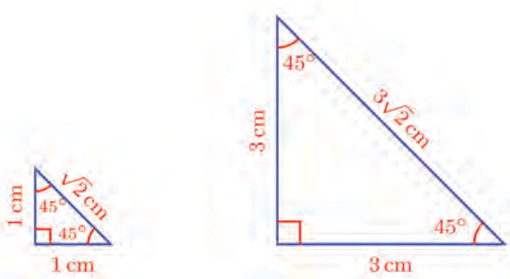
Now can you calculate the lengths of the other two sides of this triangle ?



This triangle has the same angles as the first one, right ?

We have seen, in the lesson **Similar Triangle** of the Class 9 textbook, that in triangles with the same angles, the sides are scaled by the same factor (that is, multiplied by the same number). Here, the side opposite the top 45° angle in the large triangle is 3 times the same angle in the small triangle.

So, the other sides of the large triangle must also be 3 times the other sides of the small triangle



Like this, if we know one side of a triangle with angles 45°, 45°, 90°, then we can calculate the lengths of the other two sides.

For example, if one of the perpendicular sides of such a triangle is $\frac{1}{2}$ metre, then the lengths of the other two sides are $\frac{1}{2}$ metre and $\frac{1}{2}\sqrt{2}$ metres.

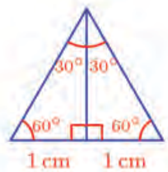
So, what can we say in general?

In any triangle with angles 45°, 45°, 90° both the shorter sides have the same length and the length of the longest side is $\sqrt{2}$ times this length

Now look at this triangle:

How do we calculate the lengths of its other two sides?

We can make an equilateral triangle by joining two of these like this:



Since the bottom side of this equilateral triangle is 2 centimetres, the length of each of the other two sides is also 2 centimetres.

We have also seen that its height is $\sqrt{3}$ centimetres (the section **Lengths and numbers** of the lesson **New Numbers** in the Class 9 textbook).

So, we have got the lengths of the sides of the first triangle:

Now look at another triangle with the same angles 30°, 60°, 90°:

In this, the length of the side opposite the 30° angle is double the length of the 30° angle of the small triangle. So, the lengths of the sides opposite other angles must also be double:

So, what can we say in general about the relation between sides of such a triangle?

In any triangle with angles 30°, 60°, 90°, the length of the longest side is 2 times the length of the shortest side; and the length of the side of medium size is $\sqrt{3}$ times the length of the shortest

Let's do a problem using this:

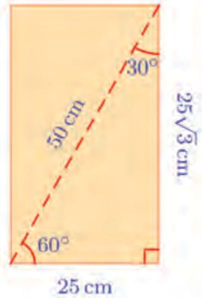
An equilateral triangle is to be made by cutting a rectangular board and rearranging the pieces as shown:

The sides of the triangle must be 50 centimetres each. What should be the lengths of the sides of the rectangle?

One side of the triangle is a diagonal of the rectangle. So, the diagonal of the rectangle must be 50 centimetres:

Since we must get an equilateral triangle on joining the triangles got by cutting the rectangle, the angles of these triangles must be 30°, 60°, 90°:

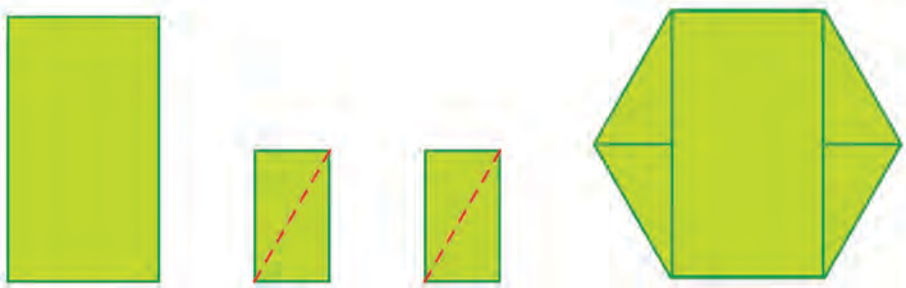
In such a triangle, the longest side is twice the shortest side. Here, the longest side is 50 centimetres. So, the shortest side must be 25 centimetres. And the third side?



Now we have the length of the sides of the rectangle, right?

Next try this problem:

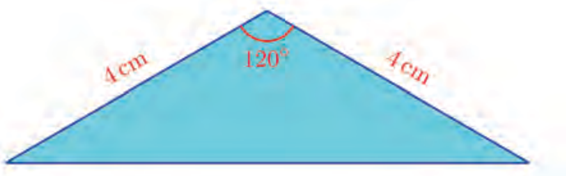
> Two rectangles are cut along their diagonals and joined to a third rectangle, to make a regular hexagon, as shown below:



The length of a side of the hexagon is to be 30 centimetres. What should be the lengths of the side of the rectangles?

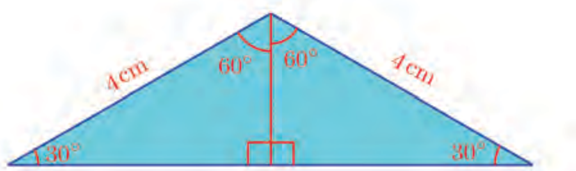
We can use these facts to compute some areas also. For example look at this problem:

What is the area of the triangle below?

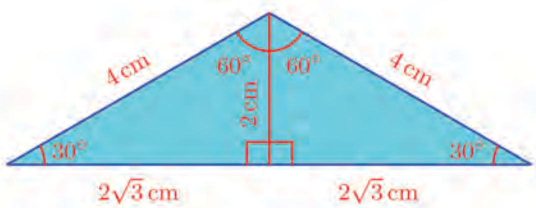


Two sides of this triangle are of the same length and so the angles opposite them must also be the same. Since the top angle is 120°, the sum of the other two angles is 60° and since they are equal, each must be 30°

The perpendicular from the top vertex to the bottom side bisects the top angle (the section **Isosceles triangles** of the lesson, **Equal Triangles** in the Class 8 textbook):



Now can't we compute the length of this perpendicular and the length of the bottom side?



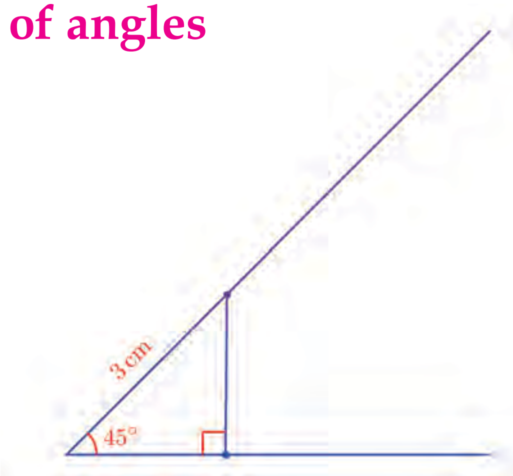
So, the area of the triangle is $2 \times 2\sqrt{3} = 4\sqrt{3}$ square centimetres

(1) Calculate the area of the parallelograms shown below:

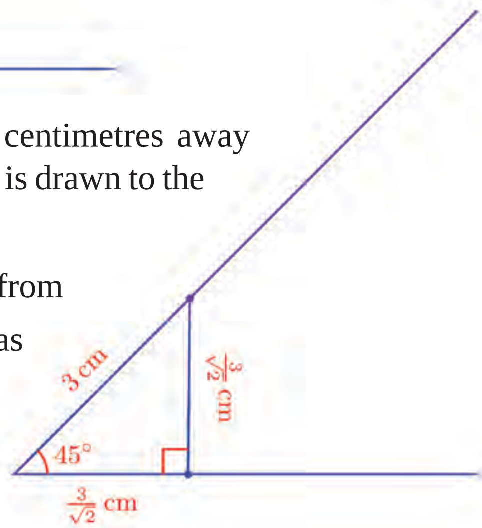
(2) Calculate the area of the triangles shown below:

New measures of angles

See this picture:

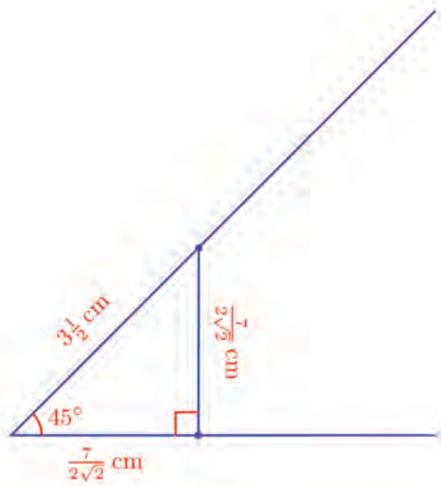


A point is chosen on one side of a 45° angle, 3 centimetres away from the vertex and a perpendicular from this point is drawn to the other side of the angle.



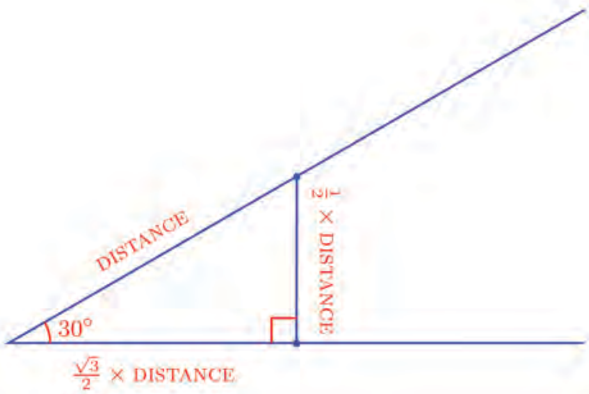
The height of this perpendicular and the distance from the vertex to its foot are both $\frac{3}{\sqrt{2}}$ centimetres, as seen in the last section.

What if the first point chosen is $3\frac{1}{2}$ centimetres away from the vertex?

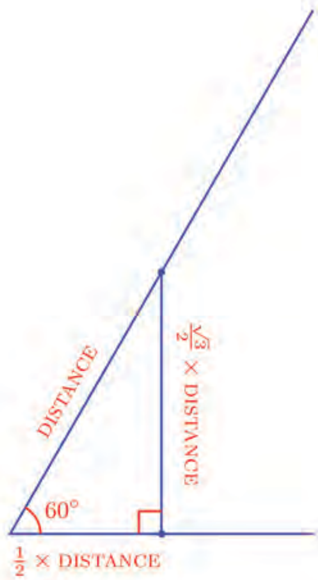


In general, at whatever distance from the vertex we take the first point, the height of the perpendicular to the other side and the distance to its foot from the vertex are both $\frac{1}{\sqrt{2}}$ distance.

What if we take the angle as 30° instead of 45°?

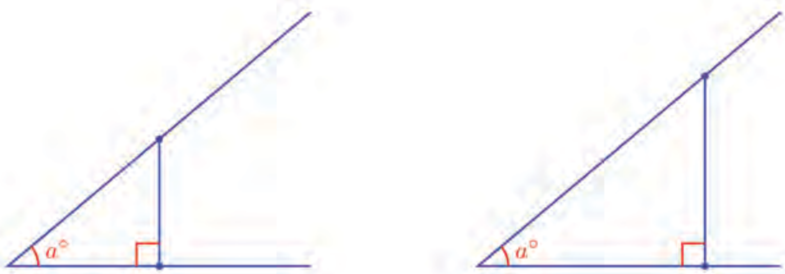


And if it is 60°?



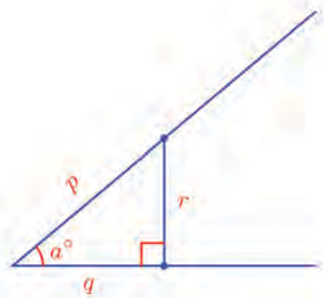
For other angles also, would the height of the perpendicular and the distance to its foot from the vertex be fixed multiples of the distance of the first point from the vertex?

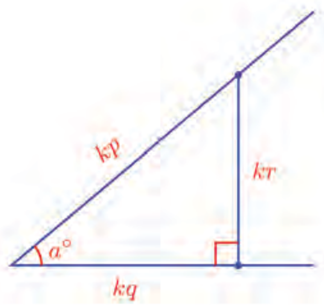
See these pictures:



On one side of an angle, two points are chosen at different distances from the vertex and perpendicular drawn to the other side

The two triangles got from these have equal angles and so their sides are scaled by the same factor. So, if the lengths of the sides of the triangle on the left are taken as p, q, r in decreasing order, then the lengths of the sides of the triangle on the right, in this order are p, q, r multiplied by the same number. Taking this number as k, the sides of the triangle on the right are kp, kq, kr:





In the triangle on the left, the height of the perpendicular is $\frac{r}{p}$ of the distance of the chosen point from the vertex and the distance of the foot of the perpendicular from the vertex is $\frac{q}{p}$ of this distance

And in the triangle on the right?

These fractions are $\frac{kr}{kp} = \frac{r}{p}$ and $\frac{kq}{kp} = \frac{q}{p}$

Thus these fractions do not change

So, what can we say in general?

> If on one side of an angle, points at different distances from the vertex are chosen and perpendiculars drawn to the other side, the heights of these perpendiculars and the distance of their feet from the vertex change; but these as fractions of the distances of the chosen points from the vertex do not change.

We have calculated these fractions for 45°, 60° and 30°. It is not easy to calculate these for other angles. But mathematicians have devised methods to compute these and tabulated them long ago.

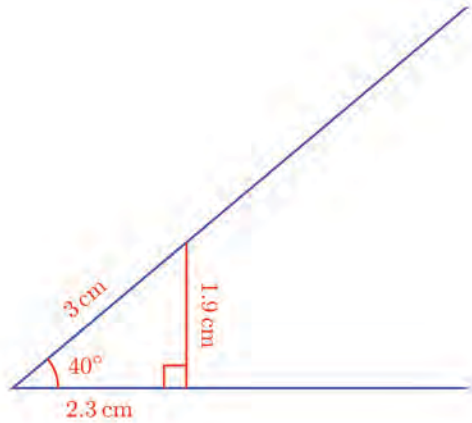
For example, we can see from this table that in a 40° angle, the height of the perpendicular from a point on one side to the other side is about 0.6428 times the distance of the point from the vertex and the foot of this perpendicular is at a distance of about 0.7660 times the distance of the point from the vertex.

Thus if in a 40° angle, we mark a point on one side, 3 centimetres from the vertex and draw a perpendicular to the other side, then we can calculate the height of the perpendicular as approximately

$$3 \times 0.6428 \approx 1.9 \text{ cm}$$

and the distance of the foot of the perpendicular from vertex as approximately

$$3 \times 0.7660 \approx 2.3 \text{ cm}$$



If the distance of the point from the vertex is 3 metres, we can compute the height of the perpendicular as approximately 1 metre, 928 millimetres and the distance of the foot of the perpendicular from the vertex as approximately 2 metres, 298 millimetres.

Numbers computed like these have special names. In the problem just seen, the number 0.6428... shows what fraction of the distance between a point on one side of a 40° angle and the vertex of the angle, is the height of the perpendicular from this point to the other side. This number is called the **sine** of 40° and is written sin 40°. Thus

$$\sin 40° \approx 0.6428$$

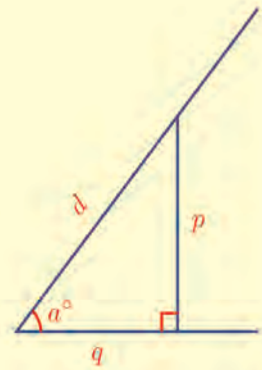
The second number 0.7660... shows what fraction of the distance between a point on one side of a 40° angle and the vertex of the angle, is the distance between the foot of the perpendicular from this point to the other side and the vertex of the angle. This number is called the **cosine** of 40° and is written cos 40°. Thus

$$\cos 40° \approx 0.7660$$

In general,

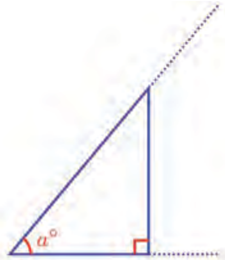
Consider a point on one side of an angle of $a°$, at distance d from the vertex, and the perpendicular from this point to the other side; if the height of the perpendicular is p and the the distance of the foot of the perpendicular from the vertex is q, then.

$$\sin a° = \frac{p}{d}$$
$$\cos a° = \frac{q}{d}$$



It must be noted that as we take different points on one side of the angle, the number d, p, q all change. But sin $a°$ and cos $a°$ do not change; only when the size $a°$ of the angle changes, do these change. In other words, they are numbers indicating the size of an angle.

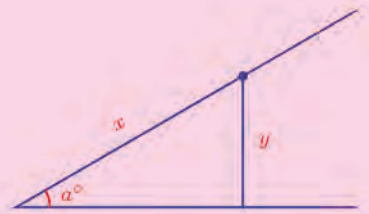
We can describe sin and cos in a slightly different way. When we draw a perpendicular from a point on one side of an angle to the other side, we get a right triangle:



The line joining the vertex of the angle and the point chosen is the hypotenuse of this triangle. The perpendicular from the point to the other side is the **opposite side** of the a° angle in the triangle. The line joining the vertex of the angle and the foot of the perpendicular is called the **adjacent side** of the $a°$ angle. So, sin $a°$ and cos $a°$ can be described like this:

Proportionality

We have seen in Class 9 that if we take all points on one side of an angle, the height of a point from the other side changes proportionally with respect to the distance of the point from the vertex of the angle.



(The section **Proportional changes** of the lesson, **Proportion**).

What is the proportionality constant in this change?

$\frac{y}{x}$ is what we call sin $a°$, isn't it?

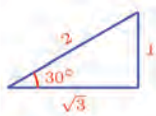
Thus sin $a°$ is the proportionality constant in the change of height of the perpendicular with respect to the distance of the point from the vertex.

Can you describe cos $a°$ also as a proportionality constant?

In a right triangle with one angle $a°$

$$\sin a° = \frac{\text{Opposite side}}{\text{Hypotenuse}} \qquad \cos a° = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

The relations between the sides of right triangles with one angle 45°, 60° or 30°, seen in the first section can be written in terms of sin and cos:

Angle	Picture	sin	cos
30°		$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60°		$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

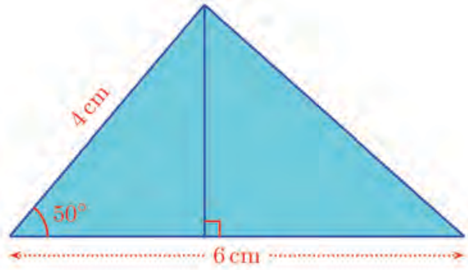
(The table at the end of the lesson gives the sine and cosine of all angles from 1° to 89° at intervals of 1°, correct to four decimal places).

We can use this to compute the areas of triangles, as done in the first section.

For example, let's find the area of the triangle shown below:

Draw a line AB of length 1 in GeoGebra and create an angle slider a. Select Angle with Given Size and click on B and then on A. In the dialogue window, give a as Angle. We get a new point B'. Draw the perpendicular to AB' through B and mark the point C where it meets AB'. Draw triangle ABC. Then AB' can be hidden. We can see that as we change the angle using the slider, the lengths of the sides of the triangle also change. In this the length of BC is equal to the sine of the angle and the length of AC is the cosine of the angle (why?). Make a table of sine and cosine of angles using this. What is the maximum value of sine and cosine?

To find the area, we draw the perpendicular from the top vertex to the bottom side:



We can directly find the sine and cosine of angles using GeoGebra. Typing sin(30°) in the input bar gives sin 30° (To type 30°, first type 30, then click the α symbol at the right of the input bar and select the ° symbol).

The area of the triangle is half the product of the lengths of the bottom side and the height of this perpendicular, isn't it?

How do we calculate the height of the perpendicular?

This perpendicular is the side opposite the 50° angle in the right triangle on the left. So, its height divided by the hypotenuse gives the sine of 50°. This means the height of the perpendicular is the hypotenuse multiplied by sin50°; that is $4 \times \sin 50°$

Now from the table we get

$$\sin 50° \approx 0.7660$$

From this we get the height as approximately

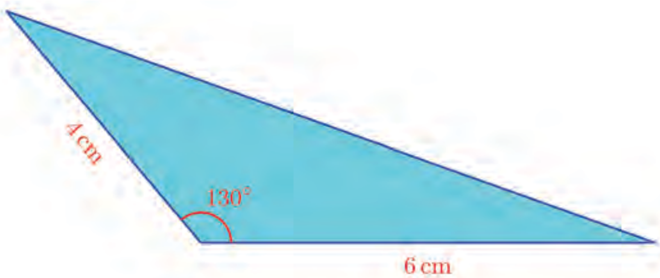
$$4 \times 0.7660 = 3.064$$

Now we can compute the area

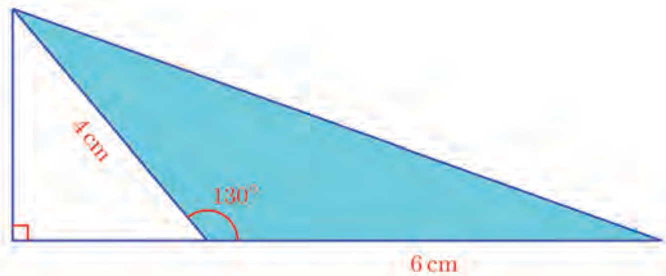
$$\frac{1}{2} \times 6 \times 3.064 \approx 9.19$$

Thus the area is about 9.19 square centimetres.

Suppose we change the angle to 130° instead of 50°



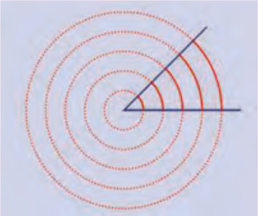
In this, the perpendicular from the top vertex is outside the triangle:



Degree measure of angles

What does it mean, when we say that an angle is 45°?

We can draw several circles centered at the vertex of such an angle. And the length of the arc of such circles within the sides of this angle are all different.

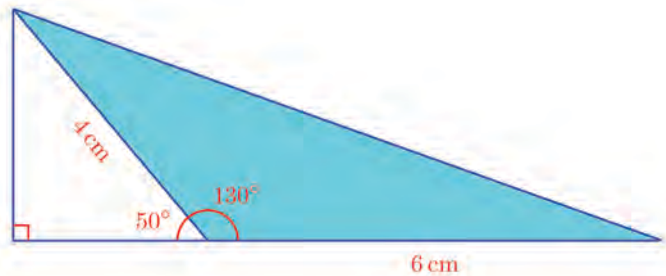


But each of these arcs is $\frac{1}{8}$ of the corresponding full circle. And 45 is the number got by multiplying this fraction $\frac{1}{8}$ by 360. What if the measure of the angle is 60°? For any circle centered at the vertex of an angle of this size, the length of the arc within the sides of the angle would be $\frac{1}{6}$ of the entire circle. And 60 is the number got by multiplying this $\frac{1}{6}$ by 360.

Generally speaking, the degree measure of any angle is the number got by first drawing a circle centered at its vertex, dividing the length of the arc within its sides by the circumference, and then multiplying this number by 360.

How do we calculate the height of this perpendicular?

One of the angles of right triangle outside our triangle is 50°; and the perpendicular we want is the side opposite the 50° angle in that triangle:



So, as in the first problem, the height of the perpendicular is again $4 \times \sin 50° \approx 4 \times 0.766 = 3.064$ centimetres. And the area is also approximately

$$\frac{1}{2} \times 6 \times 3.064 \approx 9.19 \text{ square centimetres}$$

(1) The lengths of two sides of a triangle are 8 centimetres and 10 centimetres and the angle between them 40°

(i) What is the area of this triangle?

(ii) What is the area of the triangle with lengths of two sides the same, but the angle between them 140°?

(2) The length of sides of a rhombus is 5 centimetres and one of its angles is 100°. Calculate its area.

(3) The lengths of the diagonals of a parallelogram are 8 centimetres and 12 centimetres and the angle between them 50°. Calculate its area.

(4) A triangle is to be drawn with one side 8 centimetres long and one of the angles on it 40°. What is the minimum length of the side opposite this angle?

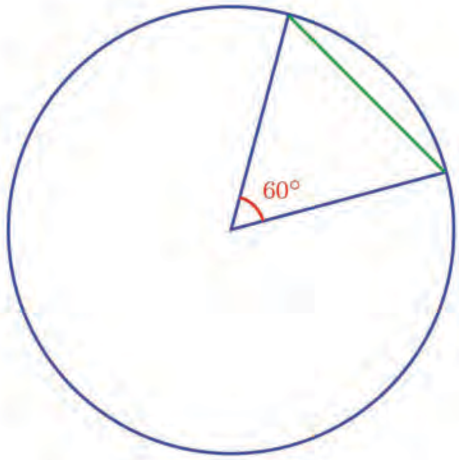
(5) The length of the sides of a rhombus is 5 centimetres and one of its angles is 70°. Calculate the length of its diagonals.

Triangles and circles

We have seen in Class 9 textbook that the length of an arc of a circle can be calculated using its central angle (the section **Length and angle** of the lesson, **Parts of Circles**).

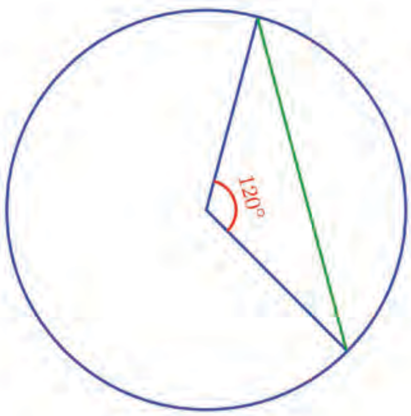
How do we compute the length of a chord?

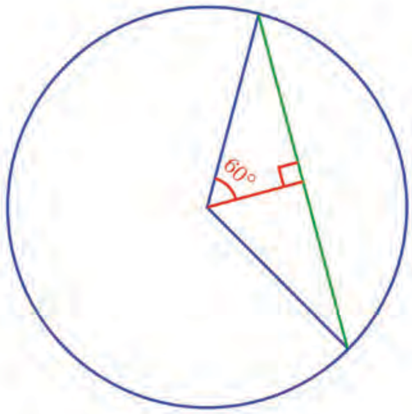
For example, we can easily see that the length of a chord of central angle 60° is equal to the radius of the circle (how?).



What about the length of a chord of central angle 120° ?

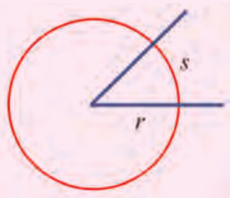
To calculate it, we draw the perpendicular from the centre of the circle to the chord. It bisects both the chord and the central angle (why?).





Another measure of an angle

We have seen what the degree measure of an angle means:



We noted that for circles of different radii, r and s in the picture above will change, but $\frac{s}{2\pi r}$ will not change and the degree measure of the angle is this fixed number multiplied by 360. In other words,

degree measure of the angle = $\frac{s}{2\pi r} \times 360$

In this, we can change r and s, but the numbers 2π and 360 don't. So, isn't it enough if we take $\frac{s}{r}$ as a measure of the angle?

That's right. This gives another measure of the angle, called its *radian measure*.

Thus

radian measure of the angle = $\frac{s}{r}$

We use the symbol ° to denote the degree measure, right? The radian measure is written rad.

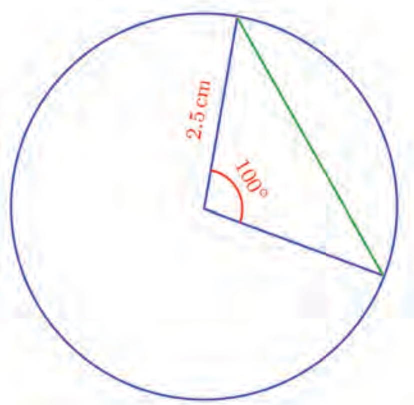
This idea was first proposed by the English mathematician, Roger Cotes in the eighteenth century. The name radian was first used by the English physicist, James Thomson in the nineteenth century

In the right triangle formed by the radius and half the chord, the hypotenuse is the radius; and the side opposite the 60° is half the chord.

Thus we can see that half the chord is $\frac{\sqrt{3}}{2}$ of the radius.

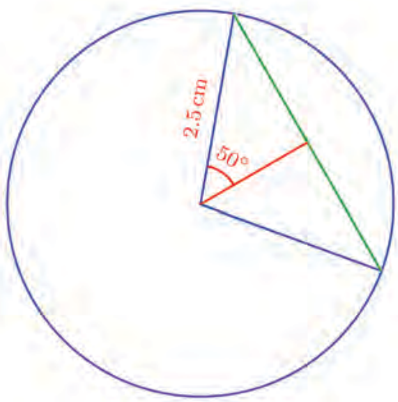
So, the length of the chord is $\sqrt{3}$ times the radius.

We have seen in Class 9 that as the central angle is doubled, the length of the arc is also doubled; but note that the chord is not doubled, as seen in this example. In other words, the length of a chord is not scaled by the same factor as the central angle.



Now how do we compute the length of the chord in this picture?

As before, let's draw the perpendicular from the centre to bisect the chord and the central angle



The hypotenuse of the right triangle thus got is 2.5 centimetres. So, the length of the side opposite the 50° angle is 2.5 × sin 50°. This is half the chord. So, the length of the chord is

$$5 \times \sin 50° \approx 5 \times 0.7660 \approx 3.8 \text{ cm}$$

We can use this method to find the length of any chord.

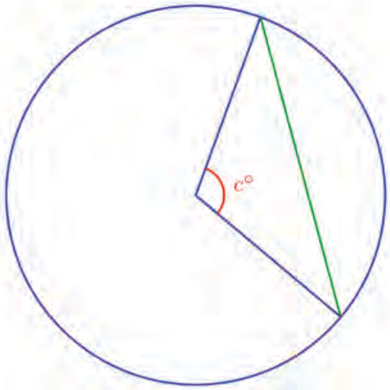
What all things did we do to compute the length of a chord?

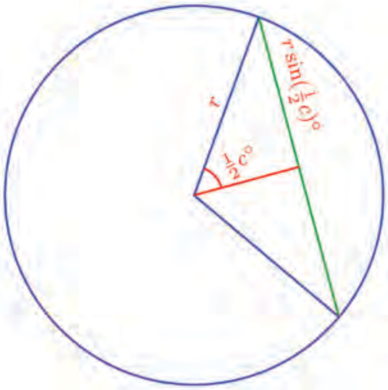
Straightening

In measuring an angle, we actually measure the length of an arc of a circle, whether we use degrees or radians. Instead of this, the Greek astronomer Hipparchus started using lengths of chords, in the second century BCE.

Later mathematicians often refer to a table of chords of various central angles computed by Hipparchus, but it has not been found. However, such a table of chords done by the Egyptian astronomer Claudius Ptolemy in the second century CE, has been found. He has computed accurately lengths of chords of central angles up to 180° in a circle of radius 60 units, at $\frac{1}{2}^{\circ}$ intervals.

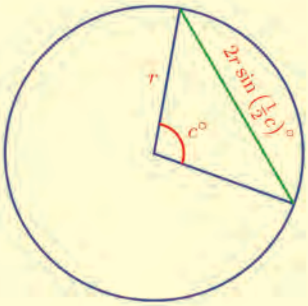
Multiplying the radius by the sine of half the central angle gave half the chord; doubling it gave the length of the chord





So what can we say in general?

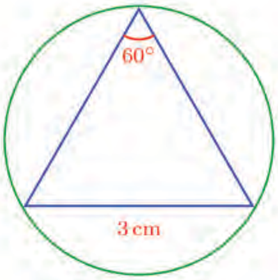
The length of any chord of a circle is twice the product of the radius and the sine of half the central angle



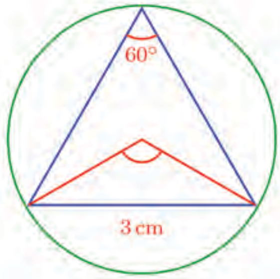
For example, look at this problem:

What is the radius of the circumcircle of an equilateral triangle of sides 3 centimetres

See this picture:



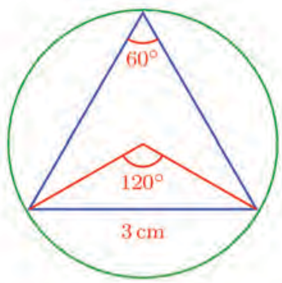
Each side of the triangle is a chord of the circle. What is its central angle?



The central angle of the chord, which is the bottom side of the triangle, is also the central angle of the arc joining the bottom two vertices of the triangle.

And the top vertex of the triangle is a point on the circle, which is not on this arc.

So, the central angle of this arc (which is also the central angle of the chord) is $2 \times 60° = 120°$, as seen in the section, **Arcs and angles** of the lesson **Circles and Angles**



As stated above, the length of this side is the product of the circumradius and 2 sin 60°

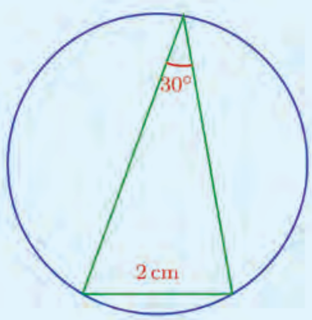
$$2 \sin 60^\circ = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

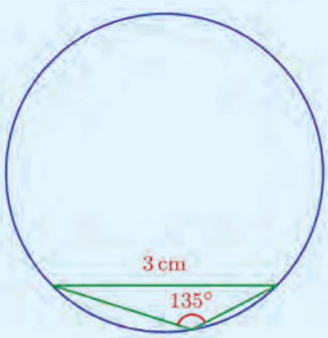
So the circumradius can be computed as

$$3 \div \sqrt{3} = \sqrt{3}$$

Thus the circumradius is $\sqrt{3}$ centimetres

(1) The pictures below show two triangles and their circumcircles:

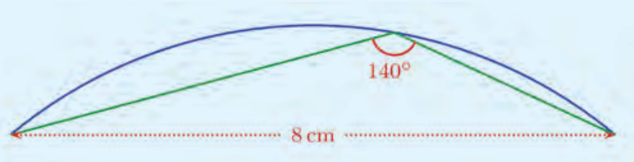




Calculate the radius of each circle

(2) A circle is to be drawn, passing through the ends of a line 5 centimetres long; and the angle in one of the segments made by the line should be 80°. What should be the radius of the circle?

(3) The picture shows part of a circle:



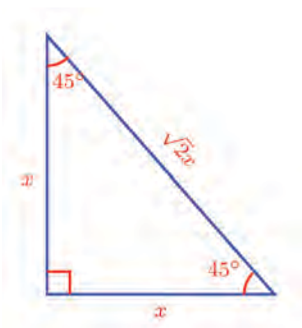
What is the radius of the circle?

Ratio of sides

We have seen in Class 9 textbook that in triangles with the same three angles, the sides are in the same ratio (the section **Scale and proportion** of the lesson **Proportion**). In other words, the angles of a triangle determine the ratio of the sides.

For example, we have seen at the beginning of this lesson that in any triangle with angles 45°, 45°, 90° both the shorter sides are of the same length and the longest side is $\sqrt{2}$ times this length.

That is, in any such triangle, the lengths of the sides are the numbers 1, 1, $\sqrt{2}$ multiplied by the same number:

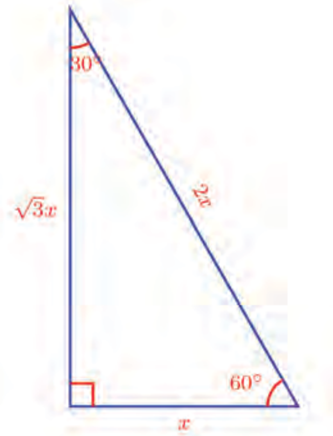


In other words,

In any triangle with angles 45°, 45°, 90° the sides are in the ratio 1: 1: $\sqrt{2}$

Similarly, we have seen that in a triangle with angles 30°, 60°, 90° the side of medium length is $\sqrt{3}$ times the shortest side and the longest side is twice the shortest side

This means, in any such triangle, the lengths of the sides are the numbers 1, $\sqrt{3}$, 2 multiplied by the same number:

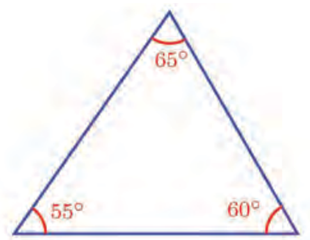


This can also be stated in terms of ratios:

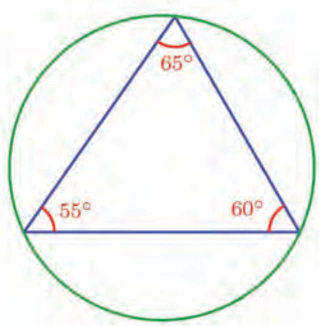
In any triangle with angles 30°, 60°, 90° the sides are in the ratio $1 : \sqrt{3} : 2$

To find the ratios of sides of other triangles, we can use the sine of the angles

For example, consider this triangle:



If we draw the circumcircle, the sides of the triangle would be chords of this circle:

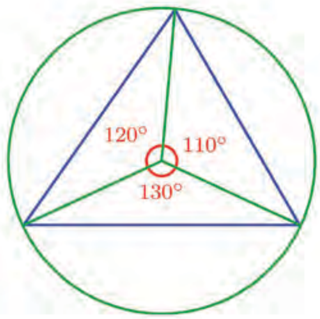


If the sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, would the angles be 30°, 60°, 90°? We can use GeoGebra to check this.

First we draw a triangles with sides in the ratio $1 : \sqrt{3} : 2$. For this create a slider with Min = 0. Select the Segment with Given Length and click on a point. In the dialogue window, give the length as a*sqrt(3) (This means $a\sqrt{3}$). With one end of this line as centre, draw a circle of radius a, and with the other end as centre, draw a circle of radius 2a. Draw the triangle with one of the points where these circles meet and the ends of the line as vertices. Mark the angles of this triangle and see, what happens if we change the value of a, using the slider?

In the same way draw triangles with sides in the ratio $2 : \sqrt{5}+1 : \sqrt{5}+1$ and see what the angles are.

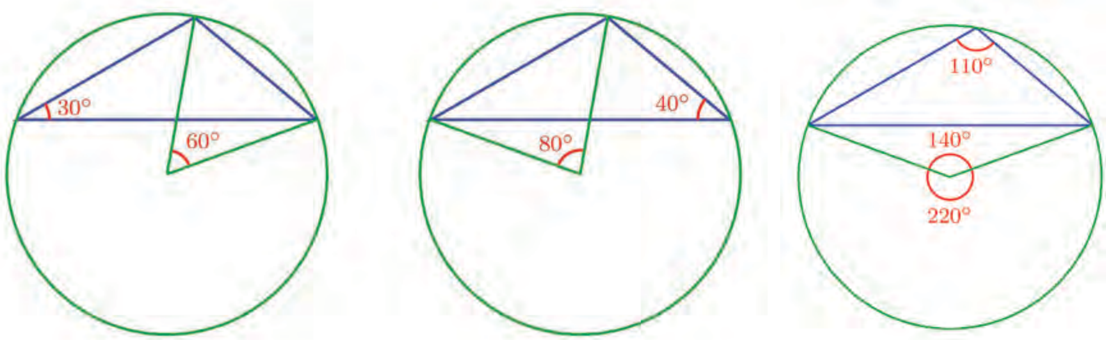
The central angle of each chord is twice the angle opposite to it in the triangle (the section **Arcs and angles** in the lesson **Circles and Angles**):



So, if we take the radius of the circumcircle as r, then the lengths of the chords are $2r$ sin 55°, $2r$ sin 60°, $2r$ sin 65°; and so the ratio of the sides of the triangle is sin 55°: sin 60°: sin 65°. We can calculate them using the table of sines. Note that in this, 55°, 60°, 65° are the angles of the triangle itself.

What if the triangle is like this?

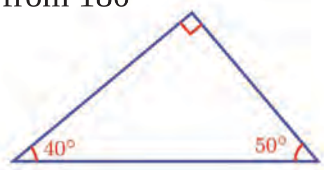
The circumcircle and the central angles of the chords which are sides of the triangle are then like this:



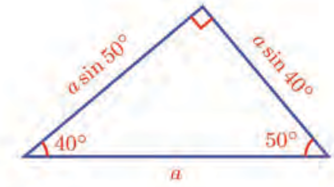
So, if we take the radius of the circumcircle as r as in the first case, the lengths of the chords are $2r$ sin 30°, $2r$ sin 40°, $2r$ sin 70°, and these are the lengths of the sides of the triangle. The ratio of the sides is then sin 30°: sin 40° : sin 70°

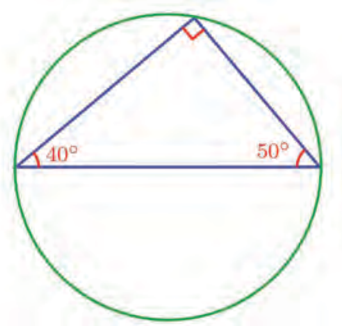
Note that here, the angles 30° and 40° are two angles of the triangle and 70° is the 110° angle of the triangle subtracted from 180°

What about a right triangle?



If we take the hypotenuse of this triangle as a, we can write the lengths of the other two sides:

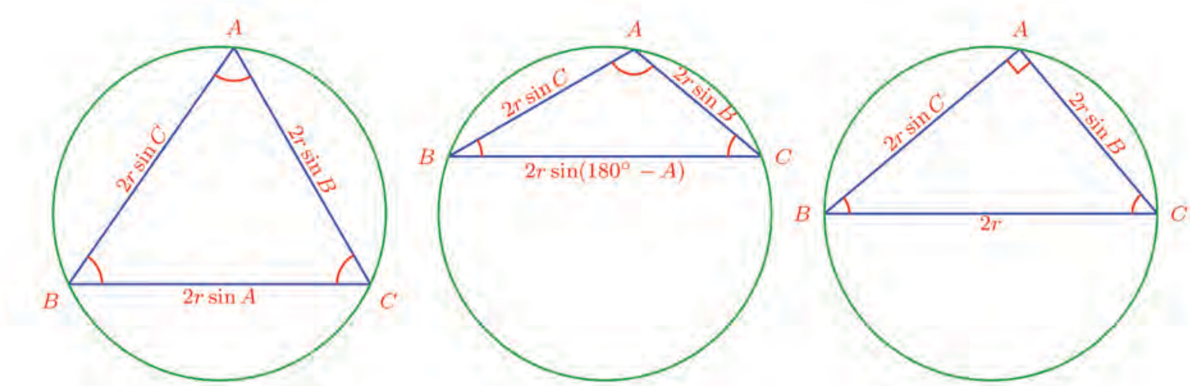




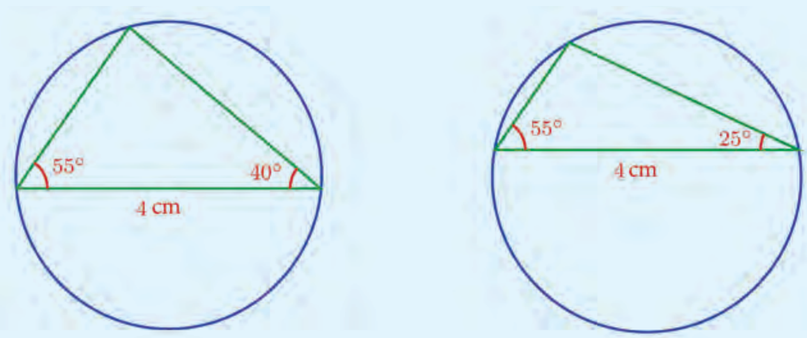
We have also seen that the diameter of the circumcircle is also a (Problem 1 (iii) at the end of the section **Triangle centres** of the lesson **Parallel lines** in the Class 9 textbook).

Note that here the ratio of sides is 1: sin 40°: sin 50°

We have seen in Class 9 that triangles with the same angles have sides in the same ratio. Now we know how to compute this ratio also:



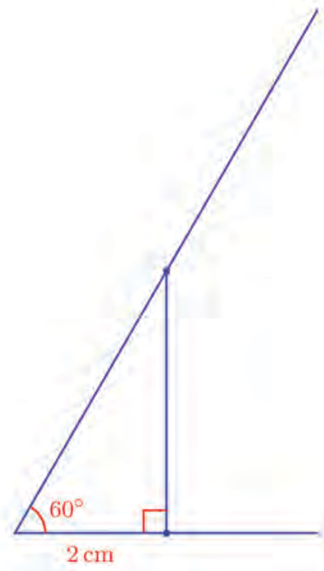
Two triangles and their circumcircles are shown in the figure.



Use the sine table to calculate the diameters of the circles and the other two sides of the triangles correct to a millimetre

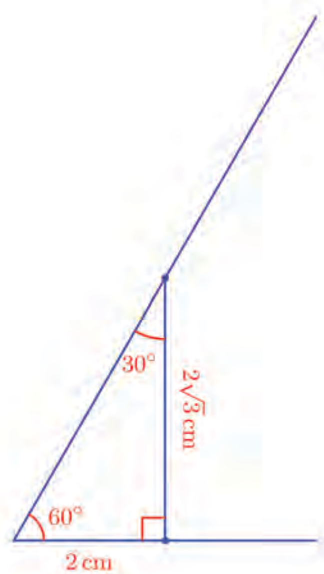
Another measure

See this picture:



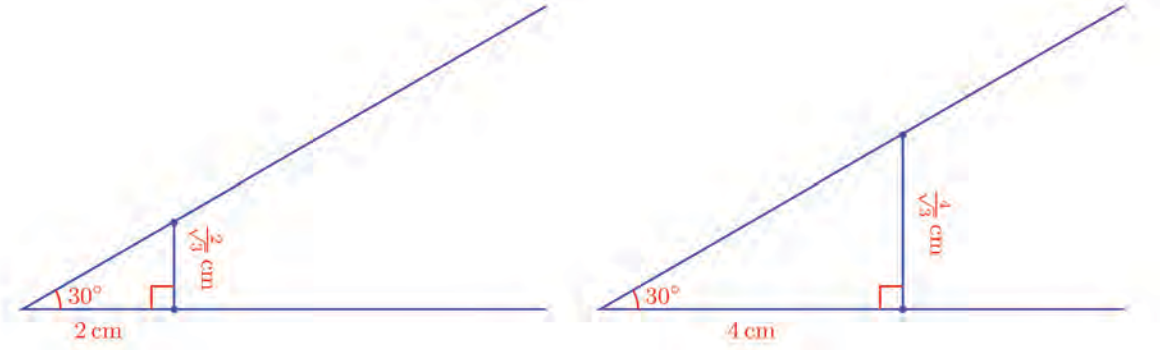
What is the height of the perpendicular in this?

The angles of the triangle are 30°, 60°, 90° and the shortest side is 2 centimetres. So, the side of medium length is $2\sqrt{3}$ centimetres; and that is the height of the perpendicular:



Now if we take any point on one side of this angle and draw a perpendicular to the other side, the height of the perpendicular would be $\sqrt{3}$ times the distance from the vertex of the angle to the foot of the perpendicular.

What if we take a 30° angle?



The height of the perpendicular is $\frac{1}{\sqrt{3}}$ times the distance from the vertex to the foot.

What about a 45° angle?

Like this, the number which gives the height of the perpendicular from one side of an angle to the other as a fraction or multiple of the distance from the vertex of the angle to the foot of the perpendicular, is called the **tangent** of the angle and is written tan. So, in the examples seen above

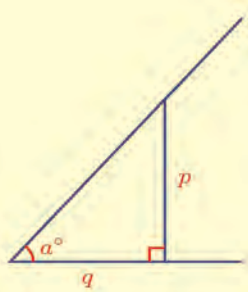
$$\tan 60° = \sqrt{3}$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$\tan 45° = 1$$

Let's see how we can compute the tan of angles using GeoGebra, just as we did for sin and cos. Draw a line AB of length 1 and an angle slider a. Select Angle with Given Size and click on A and then on B. In the dialogue window, give Angle as a. We get a new point B'. Join A and B' and mark the point C where it meets the perpendicular drawn through B. Draw the triangle ABC. Now we can hide the other points and lines. Mark the length of BC. It is the tan of the angle at A (why?). What is the maximum value of tan?

In general

Consider a point on one side of an angle of a°, and the perpendicular from this point to the other side; if the height of the perpendicular is p and the distance of the foot of the perpendicular from the vertex is q, then.

$$\tan a° = \frac{p}{q}$$



We can also describe tan of an angle in terms of a right triangle, as in the case of sin and cos:

In a right triangle with one angle $a°$,

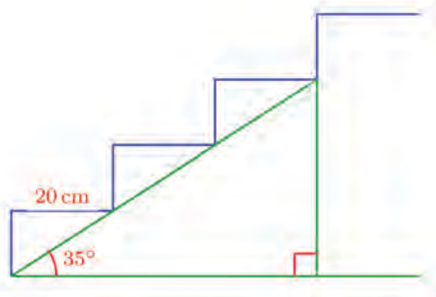
$$\tan a° = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

Let's look at a situation where we use the tan of an angle.

The picture shows part of a staircase with measurements.

We want to calculate how high is the third step from the floor.

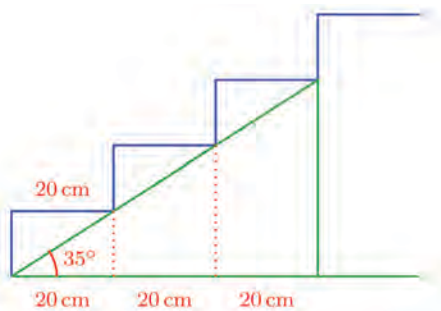
The height to be computed is the length of the side opposite the 35° in the right triangle shown in the picture.



For that we need only multiply the adjacent side of this angle by tan 35°

How do we calculate the adjacent side?

See this picture:



So, the height of the third step from the floor is $60 \times \tan 35°$

From the tables we find

$$\tan 35° \approx 0.7002$$

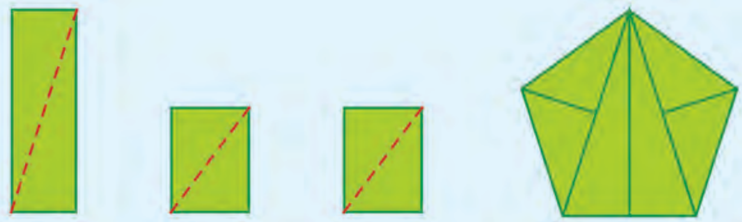
and so the height is approximately

$$60 \times 0.7002 = 42.012 \approx 42 \text{ cm}$$

(1) One angle of a rhombus is 50° and the shorter diagonal is 6 centimetres. What is its area?

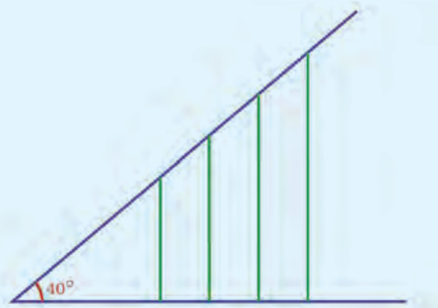
(2) A ladder leans against a wall with its foot 2 metres away from the wall. The angle between the ladder and the ground is 40°. How high is the top of the ladder from the ground?

(3) Three rectangles are to be cut along the diagonals to make triangles which are then rearranged to form a regular pentagon as shown below:



The sides of the pentagon must be 30 centimetres. What should be the lengths of the sides of the rectangles?

(4) The perpendiculars in the picture below are drawn 1 centimetre apart:

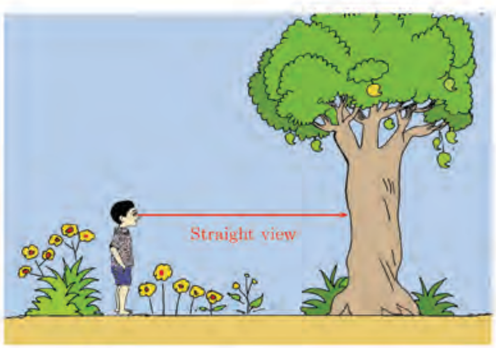


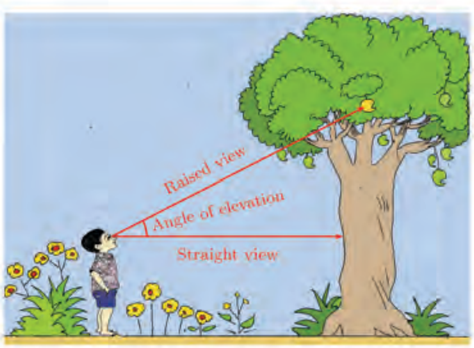
Prove that their heights are in an arithmetic sequence. What is the common difference?

(5) Calculate the area of a regular pentagon of sides 10 centimetres.

Heights and distances

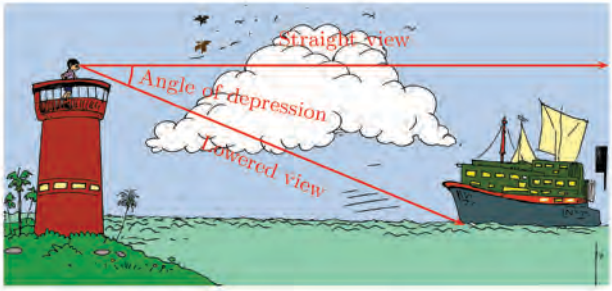
To see things above us, we need to look up by lifting our heads. See these pictures:





Usually our line of vision is parallel to the ground. To see things above, we have to raise (elevate) the line of vision. The angle between these two lines is called **angle of elevation**

Similarly when we stand at a high place, to see things below, we have to lower our line of vision:

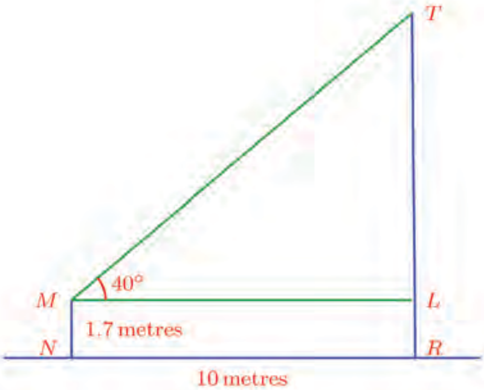


The angle produced thus is called the **angle of depression**. Such angles are measured using an instrument called **clinometer**. Distances and heights which cannot be directly measured are found out by measuring angles using a clinometer and computing using tan tables

Let's look at a few examples.

A person, 1.7 metres tall, standing 10 metres away from the foot of a tree sees the top of the tree at an angle of elevation of 40°. What is the height of the tree?

In the picture below, MN is the person and TR is the tree:



From the picture we have

$$\tan 40° = \frac{TL}{ML}$$

So that

$$TL = ML \tan 40° \approx 10 \times 0.8391 = 8.391$$

using tables. So,

$$TR = TL + LR = TL + MN = 8.391 + 1.7 = 10.091$$

$$TR \approx 10.09$$

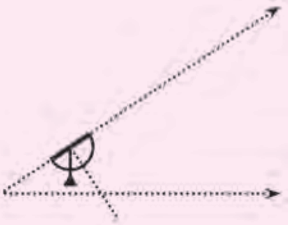
Thus the height of the tree is about 10.09 metres

Clinometer

We can make a simple clinometer to measure angles of elevation and depression:

Fix a hollow tube at the straight edge of a protractor, as shown in the picture. Hang a small weight from it, using a string stretched along the middle of the protractor.

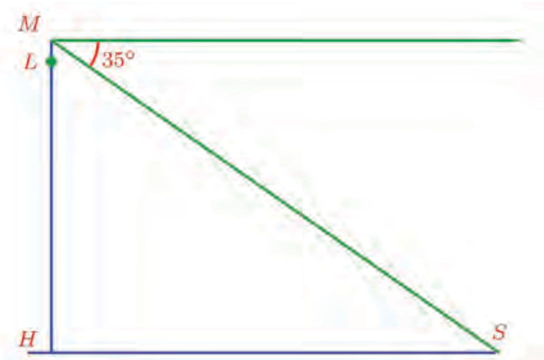
To see the top of a tree or a building through the tube, the protractor has to be tilted up. The string will still be straight down, because of the weight attached to it. The angle of elevation is the angle between the string and the 90° line of the protractor.



Another problem

A person, 1.8 metres tall, standing atop a lighthouse 25 metres high, sees a boat at an angle of depression 35°. How far is the boat from the foot of the light house?

Let's draw a picture:



In this, LH is the lighthouse and ML is the person standing on it. S is the boat. We have to calculate HS

From the given information, we have

$$MH = ML + LH = 25 + 1.8 = 26.8$$

and $\angle HMS = 55°$

So from the right triangle HMS,

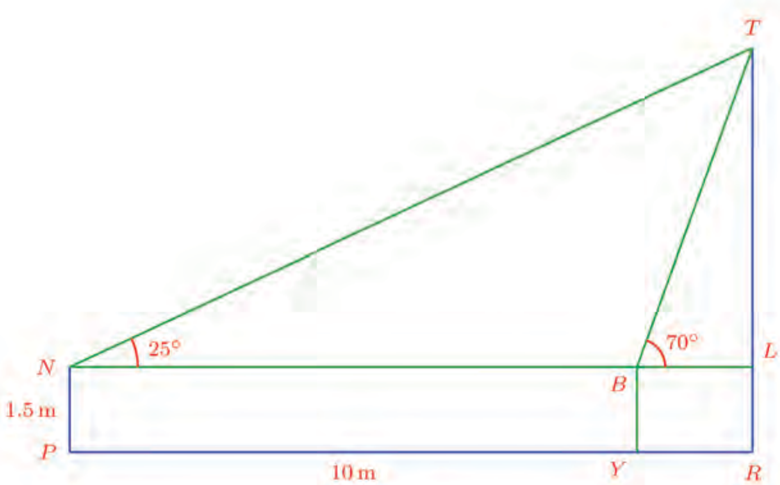
$$HS = MH \tan 55° \approx 26.8 \times 1.4281 \approx 38.27$$

That is, the boat is about 38.27 metres away from the foot of the lighthouse.

One more problem:

A boy, 1.5 metres tall, standing at the edge of a canal sees the top of a tree on the other edge at an angle of elevation of 70°. Stepping 10 metres back, he saw it at an angle of elevation of 25°. How wide is the canal and how tall is the tree?

In the picture below, TR is the tree, BY is the first position of the boy and NP is the later position of the boy.



What we have to calculate are YR and TR

From the picture we have

$$YR = BL$$

$$TR = TL + LR = TL + 1.5$$

So, we need to find BL and TL

Let

$$BL = x \quad TL = y$$

so that from the right triangle BTL

$$y = x \tan 70° \approx 2.7475x$$

And from the right triangle *NTL*

$$y = (x + 10)\tan 25° \approx 0.4663(x + 10) = 0.4663x + 4.663$$

So,

$$2.7475x \approx 0.4663x + 4.663$$

From this we can find (using a calculator)

$$x \approx \frac{4.663}{2.2812} \approx 2.044$$

and then

$$y \approx 2.7475 \times 2.044 \approx 5.616$$

Thus the width of the canal is about 2.04 metres and the height of the tree is about 5.62 +1.5 = 7.12 metres

(1) When the sun is seen at an angle of elevation of 40°, the length of a tree's shadow is 18 metres

(i) What is the height of the tree?

(ii) What would be the length of the shadow, when the sun is at an elevation of 80°?

(2) From top of a building, a person sees the foot of a shop 30 metres away at an angle of depression 25°. What is the height of the building?

(3) From the top of an electric post, two wires are stretched to either side and attached to the ground, making angles 55° and 45° with the ground. The distance between the feet of the wires is 25 metres. What is the height of the post?

(4) A person, 1.75 metres tall, standing at the foot of a tower sees the top of a hill 40 metres away at an elevation of 60°. From the top of the tower, he sees it at an elevation of 50°. Calculate the heights of the tower and the hill.

(5) A boy, 1.5 metres tall, standing at the edge of a canal sees the top of a tree on the other edge at an angle of elevation of 80°. Stepping 15 metres back, he saw it at an angle of elevation of 40°. How wide is the canal and how tall is the tree?

Trignometric tables

Angle	sin	cos	tan
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.364
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.404
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900