4 Mathematics of Chance

Chances as numbers

There are ten beads in a box, nine black and one white. If we pick one from this, without looking, it's most likely to be black, can be white though.

There is another box, with eight black and two white beads. If we pick one from it, again it's more likely to be black than white.

A third box contains five black and five white beads. What if we pick one from this? Could be white or black; can't say anything more, can we?

These can be put in slightly different words: from the first and second, drawing a black is more probable; from the third, drawing a black and drawing a white are equally probable.

Think of a game with boxes and beads. A box contains five black beads and five white; another contains six black and four white. We have to choose a box and then pick a bead from it. If it's a black, we win. Which box should we choose?

The second box contains more black beads; so getting a black from it is more probable, right?

What if we take a black bead from the second box and put it in the first?

This is how the boxes and beads are now:

First box	:	6 black	5 white
Second box	:	5 black	4 white

Now if we play the game, which box is the better choice?

Now the first box has more black beads. Is the probability of getting a black from it also more?

Let's think in terms of totals.

The first box has 11 beads in all, 6 of them black. That is, $\frac{6}{11}$ of the total is black

What about the second box? $\frac{5}{9}$ of the total number of beads is black

Which is greater, $\frac{6}{11}$ or $\frac{5}{9}$?

$\frac{5}{9}$ is larger than $\frac{6}{11}$, right ?

(The section **Large and Small** of the lesson **Arithmetic of Parts** in the Class 6 textbook)

So, the second box is still the better choice, isn't it?

In other words, getting a black bead from the second box is more probable.

In fact we can say that the probability of a black bead from the first box is $\frac{6}{11}$, and the probability of a black bead from the second box is $\frac{5}{9}$.

What about the probability of getting a white bead?

$\frac{5}{11}$ from the first and $\frac{4}{9}$ from the second; which is greater?

So, if the winning draw is white, which box is the better choice?

We can tabulate the various probabilities in this problem like this:

		First box		Second box	
		Black	White	Black	White
Beginning	Number	5	5	6	4
	Probability	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Later	Number	6	5	5	4
	Probability	$\frac{6}{11}$	$\frac{5}{11}$	$\frac{5}{9}$	$\frac{4}{9}$

Another question: we have seen that both at the beginning and also after transferring a bead, the probability of getting a black from the second box is higher; is the probability less or more after the transfer?

(1) A box contains 6 black and 4 white balls. If a ball is picked from it, what is the probability that it is black? And the probability that it is white?

(2) A bag contains 3 red balls and 7 green balls. Another contains 8 red and 7 green

(i) If a ball is drawn from the first bag, what is the probability that it is red?

(ii) From the second bag?

(iii) The balls in both bags are put together in a single bag. If a ball is drawn from this, what is the probability that it is red?

(3) A bag contains 3 red beads and 7 green beads. Another bag contains one more of each. The probability of getting a red from which bag is greater?

Number probability

Let's look at another problem:

Each of the numbers from 1 to 25 is written in a slip and all these are put in a box. One slip is taken out. What is the probability that the number drawn is an even number ?

Among the 25 numbers, 13 are odd and 12 are even, right ? So, the probability of getting an even number is $\frac{12}{25}$.

What about the probability of drawing an odd number ?

What is the probability of getting a multiple of 3 from this box? And a multiple of 6?

Now look at this problem:

If a number is drawn from the numbers 1, 2, 3, ..., 1000, what is the probability that it is a factor of 1000?

To calculate it, we must first compute the number of factors of 1000 among the numbers from 1 to 1000.

All the factors of 1000 are among these, right ? So, we need only find the number of factors of 1000.

Remember doing such computations in Class 7 ? (The section **Number of factors** in the lesson **Number Relations**).

Writing 1000 as the product of powers of distinct primes, we get

$$1000 = 2^3 \times 5^3$$

So, the number of factors is

$$(3 + 1) \times (3 + 1) = 4 \times 4 = 16$$

Thus 16 of the numbers among the numbers 1, 2, 3, ..., 1000 are factors of 1000.

So, the probability of a number drawn from these numbers to be a factor of 1000 is

$$\frac{16}{1000} = \frac{2 \times 8}{125 \times 8} = \frac{2}{125}$$

Probabilities are sometimes given in decimal form or as a percent.

The probability in this problem can be put in decimal form. Since

$$\frac{16}{1000} = 0.016$$

we can say that the probability is 0.016

Or we can convert the fraction into a percent. Since

$$\frac{16}{1000} \times 100 = \frac{16}{10} = 1.6$$

we can say that the probability is 1.6%

(1) In each of the problems below, compute the probability as a fraction and then write it in decimal form and as a percent

   (i) If one number from 1, 2, 3, ..., 10 is chosen, what is the probability that it is a prime number?

   (ii) If one number from 1, 2, 3, ..., 100 is chosen, what is the probability that it is a two-digit number?

   (iii) Every three digit number is written in a slip of paper, and all the slips are put in a box. If one slip is drawn, what is the probability that it is a palindrome ? (See Problem (4) at the end of the section **Growing numbers** of the lesson **Number World** in the Class 5 textbook).

(2) A person is asked to say a two-digit number. What is the probability that it is a perfect square?

(3) A person is asked to say a three-digit number.

   (i) What is the probability that all three digits of this number are the same?

   (ii) What is the probability that the digit in one's place of this number is zero?

   (iii) What is the probability that this number is a multiple of 3?

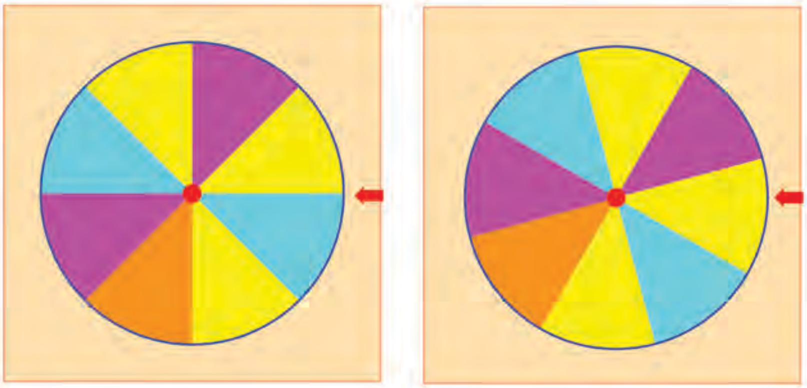
(4) Four cards with numbers 1, 2, 3, 4 on them, are joined to make a four-digit number.

   (i) What is the probability that the number is greater than four thousand?

   (ii) What is the probability that the number is less than four thousand?

Geometrical probability

A multicoloured disc is mounted on a board, so that it can freely rotate. What is the probability of getting yellow against the arrow when it stops rotating?



When the disc stops rotating, any of the eight sectors may be against the arrow mark. And three of these are yellow. So,the probability of getting yellow against the mark is $\frac{3}{8}$.

Now compute the probabilities of getting the other colours.

Let's look at another problem

Cut out a rectangular piece of cardboard and draw a triangle joining the midpoint of one side to the endpoints of the opposite side:

If we mark a point within the rectangle, with eyes closed, what is the probability that it would be within the triangle?

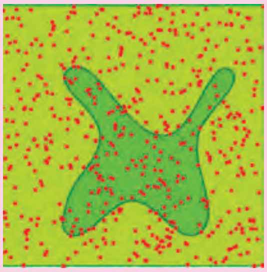
The triangle has the same base and height as the rectangle. So, it is half the rectangle.

Draw a circle in Geo Gebra and mark three points on it. Draw the triangle with these points as vertices (Use the Polygon tool). If we close our eyes and touch within the circle, what is the probability that it is within the triangle? If the circle is named c and the triangle t1, then to calculate this probability, type Area(t1)/Area(c) in the input bar.

Move the vertices of the triangle and see when this probability is maximum and when it is minimum. To get more decimal places in the probability value, choose Options → Rounding and increase number of decimal places.

Probability and area

We can use probability to estimate the area of complicated figures. The figure is drawn within a square and then a large number of dots are marked within the square, without any order or scheme.

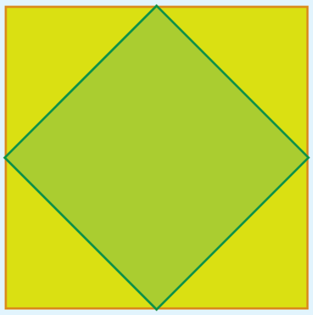


The number of dots falling within our figure, divided by the total number of dots gives an approximation of the area of the figure divided by the area of the square. And this approximation gets better, as we increase the number of dots. Both the geometric operation of marking the dots and the arithmetic operation of division can be done very fast, using computers. This is called the Monte Carlo Method

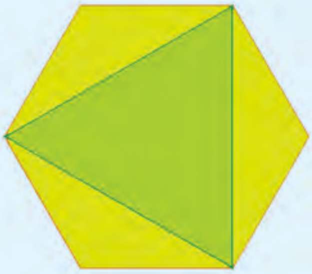
That is, the area of the triangle is half the area of the rectangle. So, the probability of the point being within the triangle is also $\frac{1}{2}$.

In each picture below, the description of the green region inside the yellow one is given. In each, find the probability that a dot put in the picture, without looking, falls within the green region:

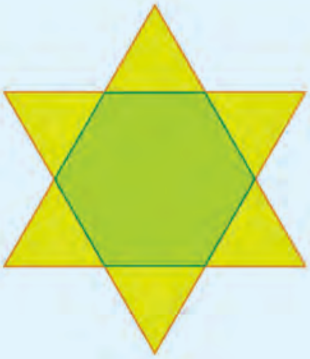
(1) The square got by joining the mid-points of a larger square



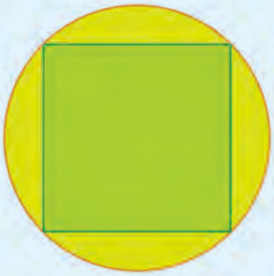
(2) The triangle got by joining alternate vertices of a regular hexagon



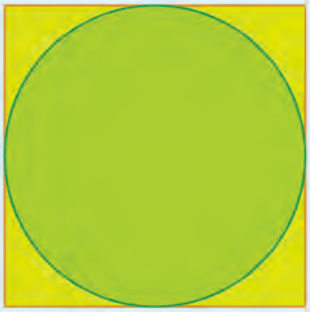
(3) The regular hexagon formed between two equal equilateral triangle



(4) A square drawn with vertices on a circle:



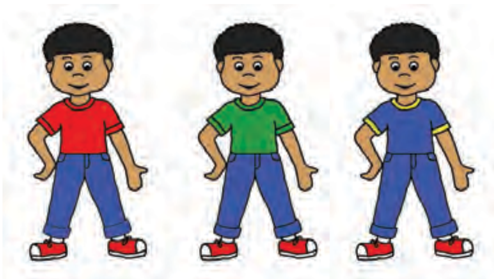
(5) The circle that just fits within a square



Draw a circle in GeoGebra and make an integer slider n. Draw a regular polygon of n sides as follows. Choose Angle with Given Size and click on a point B on the circle and the centre A of the circle in this order. In the dialogue window that opens, give the Angle as (360/n)°. We get a point B'. Choose Regular Polygon and click on B and B' and give Vertices as n. If the name of the circle is c and the name of the polygon is poly1, type Area (poly1)/ Area (c). This gives the probability of a dot put within the circle, without looking, actually falls within the polygon. When is this probability the least? What happens to the probability as the number of sides is increased?

Pairs

Searching for a clean dress, Johnny found a pair of blue pants and three shirts, red, green and blue. "in what all ways can I dress?", thought Johnny.



Searching again, He got a pair of green pants also. Now there are three more ways, wearing this with each of the three shirts, Johnny calculated.

Thus Johnny can dress in six different ways. In how many of these are the pants and shirt of the same colour?

$$\frac{2}{6} = \frac{1}{3}$$

A problem

The famous scientist Galileo writes about a problem asked by a gambler friend. He had computed that when three dice are rolled together, 9 or 10 can occur as the sum in 6 different ways:

	9	10
1.	1 + 2 + 6	1 + 3 + 6
2.	1 + 3 + 5	1 + 4 + 5
3.	1 + 4 + 4	2 + 2 + 6
4.	2 + 2 + 5	2 + 3 + 5
5.	2 + 3 + 4	2 + 4 + 4
6.	3 + 3 + 3	3 + 3+ 4

But then in actual experience, he found 10 occurring more often than 9 as the sum. He wanted an explanation of this.

In the list above 1,2,6, for example, stands for 1 coming up in some die, 2 in some other die and 6 in yet another die. Galileo argued instead of this, he must denote by the triple (1,2,6), the occurrence of 1 in the first die, 2 in the second die, 6 in the third die; by the triple (1,6,2), the occurrence of 1 in the first die, 6 in the second die and 2 in the third die and so on. This gives six different triples, (1,2,6), (1,6,2), (2,1,6), (2,6,1),(6,1,2), (6,2,1) all denoting the occurrence of the same numbers 1, 2, 6 in the three dice. Expanding other triples also like this, Galileo shows that the sum 9 can occur in 25 different ways, while 10 can occur in 27 different ways (Try this).

Let's look at another problem:

> A box contains four slips with numbers 1, 2, 3, 4 and another contains two slips with numbers 1 and 2. One slip is drawn from each box. What are the possible pairs of numbers that could be got?

Suppose 1 is drawn from the first box; the number got from the second may be 1 or 2, so that there are two possible pairs. Let's write these as (1,1) and (1,2).

Similarly, let's consider all number pairs, considering each number from the first box in turn and combining it with the numbers from the second:

(1, 1)	(1, 2)
(2, 1)	(2, 2)
(3, 1)	(3, 2)
(4, 1)	(4, 2)

8 pairs in all.

In how many of them are both the numbers odd?

Only two pairs, (1,1) and (3,1), right?

So, when one slip is drawn from each box, the probability of getting both the numbers odd is

$$\frac{2}{8} = \frac{1}{4}$$

Can you find like this, the probability of getting both numbers even and the probability of getting one even and the other odd? What about the probability of both number being the same ?

(1) Rajani has three necklaces and three earrings of green, blue and red stones. In how many different ways can she wear them ? What is the probability of her wearing a necklace and earrings of the same colour ? Of different colours?

(2) A box contains four slips numbered 1, 2, 3, 4 and another box contains two slips numbered 1 and 2. If one slip is drawn from each box, what is the probability of the sum of the numbers being odd? What is the probability of the sum being even?

(3) A box contains four slips numbered 1, 2, 3, 4 and another box contains three slips numbered 1, 2, 3. If one slip is drawn from each box, what is the probability of the product of the numbers being odd? What is the probability of the product being even?

(4) From all two-digit numbers using only the digits 1, 2 and 3, one number is chosen

(i) What is the probability of both digits being the same?

(ii) What is the probability of the sum of the digits being 4?

(5) A game for two players. Before starting, each player has to decide whether he wants an odd number or even number. Then both raise some fingers of one hand at the same time. If the sum of numbers of fingers is odd, the one who chose odd number at the beginning wins; if even, the one who chose even number wins. Which is the better choice at the beginning, odd or even?

More pairs

Again two boxes, one now containing ten slips numbered 1 to 10, and the other containing five slips numbered 1 to 5. One slip is drawn from each box, as usual. What is the probability of both being odd ?

The method of solution is simple. Compute all the possible pairs of numbers and then the number of pairs in which both the numbers are odd, as we want. Dividing the second number by the first gives the probability.

Easier said than done. It is tedious to list all pairs and count.

Let's think about this. The number drawn from the first box can be any of the ten in it. The number from the second can be any of the five in it. So, how many possible pairs are there with the first number 1? And how many with first number 2 ?

In short, once we fix the first number, we can make 5 different pairs, changing the second number. And the first number can be fixed in 10 different ways.

Probability and frequency

When a coin is tossed a large number of times, the number of heads and tails got are almost equal, and so we can take the probability of each coming up as $\frac{1}{2}$. But due to some reason, such as a manufacturing defect in the coin, it may happen that the probability of head coming up is higher. How do we recognize this?

We suspect such a case, if in a large number of tosses, one face comes up very much more than the other. Then we toss the coin more and more times and tabulate the number of times each face comes up. For example, see this table:

Tosses	Heads	Tails
10	6	4
100	58	42
1000	576	424
10000	5865	4135

This shows that, instead of taking the probability of each face as 0.5, it is better to take the probability of head as 0.6 and the probability of tail as 0.4.

There are mathematical methods for making such computations more accurate, which we will see in the further study of the branch of mathematics called *Probability Theory*.

So, we can imagine all the number pairs arranged like this:

(1,1)	(1,2)	. . .	(1,5)
(2,1)	(2,2)	. . .	(2,5)
. . .	. . .	. . .	. . .
(10,1)	(10,2)	. . .	(10,5)

- Pairs with first number 1 in the 1^{st} row, pairs with first number 2 in the 2^{nd} row and so on till the first number is 10, giving 10 rows
- 5 pairs in each row, with the second number 1 to 5
- Altogether, $5 \times 10 = 50$ pairs

In how many of these pairs are both the numbers odd?

For that, the first number must be one among the 5 numbers 1, 3, 5, 7, 9 and the second number one among the 3 numbers 1, 3, 5

$5 \times 3 = 15$ pairs in all

(1,1)	(1,3)	(1,5)
(3,1)	(3,3)	(3,5)
(5,1)	(5,3)	(5,5)
(7,1)	(7,3)	(7,5)
(9,1)	(9,3)	(9,5)

So the probability of getting odd numbers from both the boxes is

$$\frac{15}{50} = \frac{3}{10}$$

Can you compute like this, the probability of getting both the numbers even and the probability of getting one even and other odd ?

Measuring uncertainty

Have you noticed that calendars show the time of sun-rise and sun-set for each day? It is possible to compute these, since the earth and sun move according to definite mathematical laws.

Because of this, we can also predict the months of rain and shine; but we may not be able to predict a sudden shower during summer. It is the largeness of the number of factors affecting rainfall and the complexity of their inter-relations that makes such predictions difficult.

But in such instances also, we can analyze the context mathematically and compute probabilities. This is the reason why weather predictions are often given as possibilities. And unexpected changes in the circumstances sometimes make these predictions wrong.

If we look at such situations rationally, we can see that such probability predictions are more reliable than predictions sounding exact, but made without any scientific basis.

One more problem:

> A basket contains 50 mangoes and 20 of them are not ripe. Another contains 40 mangoes and 15 of them are not ripe. If one mango is picked from each basket, what is the probability of both being ripe?

In how many different ways can we choose a pair of mangoes, one from each basket? (If necessary, imagine the mangoes in the first basket numbered 1 to 50 and the mangoes in the second basket numbered 1 to 40).

So, there are $40 \times 50 = 2000$ ways of choosing two mangoes, one from each basket.

In how many of these pairs are both mangoes ripe?

In the first basket, there are $50 - 20 = 30$ ripe mangoes, and in the second basket, $40 - 15 = 25$ are ripe

If we pair each ripe mango in the first with each ripe mango in the second, we would have $25 \times 30 = 750$ pairs.

So, the probability of both being ripe is $\frac{750}{2000} = \frac{3}{8}$

Can't you compute similarly, the probability of both being unripe?

What is the probability of at least one of the mangoes being ripe?

At least one ripe means, one ripe and the other unripe, or both ripe

One ripe and the other unripe can occur in two ways:

(i) ripe from the first, unripe from the second

(ii) unripe from the first, ripe from the second

How many pairs are there of each possibility?

So the total number of pairs with only one ripe is

$$\begin{aligned}(15 \times 30) + (25 \times 20) &= 450 + 500 \\ &= 950\end{aligned}$$

We have already seen there are 750 possible pairs with both ripe. Together with this, the total number of pairs with at least one ripe is

$$950 + 750 = 1700$$

So, the probability of at least one ripe is $\frac{1700}{2000} = \frac{17}{20}$

We can also think like this: at least one ripe means both cannot be unripe, and among all 2000 possible pairs, only in $20 \times 15 = 300$ pairs are both unripe; so in $2000 - 300 = 1700$ pairs, at least one should be ripe.

This again gives the probability of at least one being ripe as $\frac{1700}{2000} = \frac{17}{20}$.

(1) There are 30 boys and 20 girls in Class 10 A and 15 boys and 25 girls in Class 10 B. One student is to be chosen from each Class.

(i) What is the probability of both being girls ?

(ii) What is the probability of both being boys ?

(iii) What is the probability of one being a boy and one being a girl ?

(iv) What is the probability of at least one being a boy ?

(2) One is asked to say a two-digit number

(i) What is the probability of both digits being the same ?

(ii) What is the probability of the first digit being greater than the second ?

(iii) What is the probability of the first digit being less than the second ?

(3) Two dice with faces numbered from 1 to 6 are rolled together. What are the possible sums that can be got ? Which sum has the maximum possibility ?

(4) One box contains 10 slips numbered 1 to 10 and another contains slips numbered with multiples of 5, up to 25. One slip is drawn from each box.

(i) What is the probability of both numbers being odd ?

(ii) What is the probability of the product of the numbers being odd ?

(iii) What is the probability of the sum of the numbers being odd ?