3

Arithmetic Sequences and Algebra

Algebraic form

We have noted that a number sequence is an arrangement of numbers according to a definite rule. How do we state this rule?

For example, how do we describe the sequence 2, 4, 6, ... of even numbers?

The term at every position is twice the number giving the position:

Position	1	2	3	...
Term	$2 \times 1 = 2$	$2 \times 2 = 4$	$2 \times 3 = 6$	...

What about the sequence 1, 3, 5, ... of odd numbers?

The term at every position is one less than twice the number giving the position:

Position	1	2	3	...
Term	$(2 \times 1) - 1 = 1$	$(2 \times 2) - 1 = 3$	$(2 \times 3) - 1 = 5$	...

We can use algebra to concisely state these.

Thus we can describe the first sequence like this:

The number in the n^{th} position is $2n$

And the second one?

The number in the n^{th} position is $2n - 1$

In writing the terms of a general sequence as letters, it is not possible to use a different letter for each term. So, we use a single letter together with the position number to denote each term of a sequence. For example, we can write x_1 for the first term, x_2 for the second term, x_3 for the third term and so on. We can use different letters like x or y to denote different sequences

Using this notation, the two sequences above can be written like this:

The sequence 2, 4, 6, ... of even numbers: $x_n = 2n$

The sequence 1, 3, 5, ... of odd numbers: $y_n = 2n - 1$

This representation of the relation between the positions and terms of a sequence can be called the algebraic form of the sequence.

Now let's see how we can write the algebraic form of any arithmetic sequence

For example, consider the sequence

$$12, 23, 34, \ldots$$

What is the next term of this sequence? And the term after that?

Instead of computing term by term like this, we can directly find the term at any position. How do we compute the 10th term of this sequence?

(i) The 1st term is 12

(ii) The common difference is 11

(iii) The change in position from the 1st term to the 10th term is $10 - 1 = 9$

(iv) The 10th term is $12 + (9 \times 11) = 111$

We can compute the term at any position like this.

In general how do we write the n^{th} term?

(i) The change in position from the 1st term to the n^{th} term is $n - 1$

(ii) The n^{th} term is $12 + ((n - 1) \times 11) = 12 + 11(n - 1)$

In this, we can rewrite $11(n - 1)$ as

$$11(n - 1) = 11n - 11$$

This gives

$$12 + 11(n - 1) = 12 + (11n - 11) = 11n + 1$$

Thus the algebraic form of this sequence is

$$x_n = 11n + 1$$

In this if we take n as 1, 2, 3, ... we get the terms in these positions as $x_1, x_2, x_3, \ldots$

$$x_1 = (11 \times 1) + 1 = 12$$
$$x_2 = (11 \times 2) + 1 = 23$$
$$x_3 = (11 \times 3) + 1 = 34$$

....

General forms

We have noted that the algebraic form of the sequence 1, 3, 5, ... of odd numbers is $x_n = 2n - 1$. Here the number n indicates the position of each term. Thus when n is taken as 1, 2, 3, ... we get the first terms as 1, second term as 3 and so on for all odd numbers.

In the lesson Algebra of the Class 7 textbook, the general form of an odd number is given as $2n + 1$ ($n = 0, 1, 2, \ldots$). In this n denotes the quotient on dividing the odd number by 2. So when we take $n = 0, 1, 2, \ldots$ we get the numbers with quotient 0, 1, 2, ... and remainder 1 on division by 2; that is the odd numbers 1, 3, 5, ...

Let's look at another example: the arithmetic sequence starting with $\frac{1}{4}$ and proceeding by successively adding $\frac{1}{2}$ is

$$\frac{1}{4}, \frac{3}{4}, 1\frac{1}{4}, 1\frac{3}{4}, \ldots$$

How do we compute its n[th] term?

(i) 1[st] term is $\frac{1}{4}$

(ii) Common difference is $\frac{1}{2}$

(iii) The position change from the 1[st] term to the n^{th} term is $n - 1$

(iv) n^{th} term is $\frac{1}{4} + (n-1) \times \frac{1}{2}$

$$\frac{1}{4} + (n-1) \times \frac{1}{2} = \frac{1}{4} + \left(n \times \frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}n - \frac{1}{4}$$

The algebraic form of the sequence is

$$x_n = \frac{1}{2}n - \frac{1}{4}$$

In the first example, the n^{th} term is got by multiplying n by 11 and adding 1

In the second example, the n^{th} term is got by multiplying n by $\frac{1}{2}$ and subtracting $\frac{1}{4}$, which means adding $-\frac{1}{4}$

Is the n^{th} term of every arithmetic sequence got by multiplying n by a fixed number and adding a fixed number ?

Let's take the first term of a general arithmetic sequence as f and the common difference as d. Then the n^{th} term of the sequence is got by adding $(n - 1)$ times the common difference d to the first term f.

Thus, the n^{th} term is

$$f + (n - 1)d = dn + (f - d)$$

This means the n^{th} term is got by multiplying d by n and adding $(f - d)$

So, what can we say in general?

In any arithmetic sequence, the term in any position is got by multiplying the position number by a fixed number and then adding a fixed number. In other words, the algebraic form of any arithmetic sequence is

$$x_n = an + b$$

> The algebraic form of any arithmetic sequence is
>
> $$x_n = an + b$$
>
> where ***a*** and ***b*** are specific numbers.

In this, if we take as n, the natural numbers 1, 2, 3, ... we get the terms of the sequence as

$$a + b, 2a + b, 3a + b, \ldots$$

In other words,

> The terms of any arithmetic sequence are got by multiplying the natural numbers, 1, 2, 3, ... by a fixed number and adding a fixed number.

In such an arithmetic sequence, the common difference is the fixed number used to multiply the natural numbers.

Using this idea, the algebraic form of an arithmetic sequence can be easily found. For example, consider the arithmetic sequence got by starting with $\frac{1}{2}$ and successively adding $\frac{1}{3}$:

$$\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \ldots$$

Since the common difference is $\frac{1}{3}$, the terms of this sequence are got by multiplying the natural numbers by $\frac{1}{3}$ and adding another fixed number.

What is the number added?

The first term is this number added to $\frac{1}{3}$ itself

The first term is $\frac{1}{2}$. What number is to be added to $\frac{1}{3}$ to get $\frac{1}{2}$?

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Thus the terms of this arithmetic sequence are got by multiplying the natural numbers by $\frac{1}{3}$ and then adding $\frac{1}{6}$.

That is,

$$x_n = \frac{1}{3}n + \frac{1}{6}$$

From the algebraic form of an arithmetic sequence, we can gain much information about the sequence. For example, first note that the algebraic form of the sequence in this example can be written

$$x_n = \frac{n}{3} + \frac{1}{6} = \frac{2n + 1}{6}$$

Now $2n + 1$ is an odd number, whatever natural number we take as n; and the denominator 6 is an even number. So, in none of the fractions $\frac{2n + 1}{6}$ is the numerator a multiple of the denominator. Thus none of the terms of this sequence are natural numbers

The language of the law

We have noted that to find the terms of a sequence, the law of formation should be specified. And we have seen some examples of how such rules can be algebraically specified.

But not all sequences can be algebraically described. For example, no algebraic formula to find the n^{th} prime number has been discovered; that is, no formula to directly compute the number at a specified position in the sequence 2, 3, 5, 7, 11, 13, ... of primes is known.

Again, there is no algebraic formula to compute the digit at a specified location in the sequence 3, 1, 4, 1, 5, 9, ... got from the decimal representation of π. In such cases, we can only specify the rule of forming the terms in ordinary language.

Next look at the sequence got by starting with $\frac{1}{2}$ and repeatedly adding $\frac{1}{4}$.

The algebraic form of this sequence can be found as

$$x_n = \frac{n}{4} + \frac{1}{4}$$

This we can write as

$$x_n = \frac{n+1}{4}$$

In this, if we take n as 3, 7, 11, ... we get

$$x_3 = 1$$

$$x_7 = 2$$

$$x_{11} = 3$$

....

In other words, all natural numbers are terms of this sequence

(1) Find the algebraic form of the arithmetic sequences given below:

(i) 1, 6, 11, 16, ... (ii) 2, 7, 12, 17, ... (iii) 21, 32, 43, 54, ...

(iv) 19, 28, 37, ... (v) $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$ (vi) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \ldots$

(2) The terms of some arithmetic sequences in two specified positions are given below. Find the algebraic form of each:

(i) 1st term 5, 10th term 23 (ii) 1st term 5, 7th term 23 (iii) 5th term 10, 10th term 5 (iv) 8th term 2, 12th term 8

(3) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{1}{6}$ contains all natural numbers.

(4) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{2}{3}$ contains all odd numbers, but no even numbers.

(5) Prove that in the arithmetic sequence 4, 7, 10, ... the squares of all terms are also terms of the sequence.

(6) Prove that the arithmetic sequence 5, 8, 11, ... does not contain any perfect square.

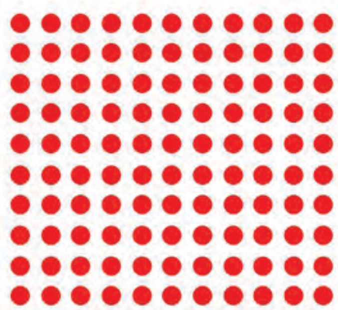
In the sequence of all even numbers, the square of each term is also a term of the sequence. How about other powers?

What about the sequence of odd numbers?

Are there other arithmetic sequences in which the powers of every term is again a term of the sequence?

Sums

See this picture:



How many dots are there?

No need to count one by one. There are 10 rows of dots with 11 dots in each, giving a total of

$$10 \times 11 = 110$$

How many dots in this triangle?

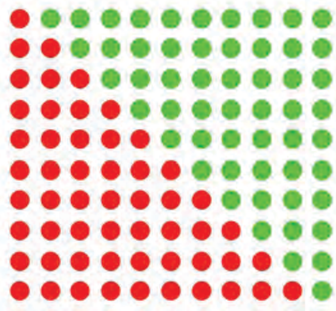
We can count one by one:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

Is there any easier way for this also?

For that, we make another triangle like this:

Turn it upside down and join with the first triangle:



This rectangle has $10 \times 11 = 110$ dots, as seen earlier.

So how many in each triangle?

Half of 110, which is 55

What we did with picture can be done with numbers also

We first write

$$s = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Adding in the other direction,

$$s = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Let's add the numbers in the same position in the two equations:

$$2s = 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 = 110$$

So,

$$s = 110 \div 2 = 55$$

In the same way, we can find the sum of natural numbers from 1 to 100:

$$\begin{aligned} s &= 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\ s &= 100 + 99 + 98 + \cdots + 3 + 2 + 1 \end{aligned}$$

Adding numbers in the same position,

$$\begin{aligned} 2s &= \overbrace{101 + 101 + 101 + \cdots + 101 + 101 + 101 + 101}^{100 \text{ times}} \\ &= 100 \times 101 \end{aligned}$$

A math tale

Haven't you heard about the mathematician, Gauss? It is said that he showed extraordinary mathematical ability right from an early age. There is a tale about it.

This happened when Gauss was just ten and in school. His teacher asked the children to add all numbers from 1 to 100, just to keep them quiet. Young Gauss did it in a flash, explaining his answer like this: 1 and 100 make 101, so does 2 and 99 and so on. 50 pairs of numbers, each with sum 101 make $50 \times 101 = 5050$

From this we get

$$s = \frac{1}{2} \times 100 \times 101 = 5050$$

We can find the sum up to any number instead of 100 in the same way. Thus

The sum of any number of consecutive natural numbers starting from one is the half the product of the last number and the next

In the language of algebra,

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$$

We can use this to find the sum of other arithmetic sequences. For example, consider the sequence 2, 4, 6, ..., 100 of even numbers.

We can write

$$2 + 4 + 6 + \cdots + 100 = 2(1 + 2 + 3 + \ldots +50)$$

In this,

$$1 + 2 + 3 + \cdots + 50 = \frac{1}{2} \times 50 \times 51$$

as seen before

So,

$$2 + 4 + 6 + \cdots + 100 = 2 \times \frac{1}{2} \times 50 \times 51 = 2550$$

In general, the first n even numbers are

$$2, 4, 6, \ldots, 2n$$

and their sum is

$$2 + 4 + 6 + \cdots + 2n = 2 \times \frac{1}{2} \times n(n + 1) = n(n + 1)$$

Now the even numbers 2, 4, 6, ... are multiples of 2; what if we take multiples of 3 instead?

The first n terms of this sequence are

$$3, 6, 9, \ldots, 3n$$

and their sum is

$$\begin{aligned} 3 + 6 + 9 + \cdots + 3n &= 3(1 + 2 + 3 + \cdots + n) \\ &= 3 \times \frac{1}{2}n(n+1) \\ &= \frac{3}{2}n(n + 1) \end{aligned}$$

Can't you find the sums of multiples of 4 and multiples of 5 like this? Try:

Let's now look at the sum of the first n odd numbers

First let's write the sequence of odd numbers in algebraic form

This is the arithmetic sequence which starts at 1 and proceeds by repeatedly adding 2.

So if we write x_n for the n^{th} odd number,

$$x_n = 1 + ((n-1) \times 2) = 2n - 1$$

We get the sequence of odd numbers by taking 1, 2, 3, ... as n.

So, we can write this sequence as

$$\begin{aligned} x_1 &= (2 \times 1) - 1 \\ x_2 &= (2 \times 2) - 1 \\ &\dots\dots\dots\dots \\ x_n &= (2 \times n) - 1 \end{aligned}$$

If we add these from top to bottom, we get

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= ((2 \times 1) + (2 \times 2) + \cdots + (2 \times n)) - \overbrace{(1 + 1 + \cdots + 1)}^{n \text{ times}} \\ &= 2(1 + 2 + \cdots + n) - n \end{aligned}$$

In this, we have

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

Using this, we get

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 2 \times \frac{1}{2}n(n+1) - n \\ &= n(n+1) - n \\ &= n^2 + n - n = n^2 \end{aligned}$$

Thus the sum of any number of consecutive odd numbers starting with 1 is the square of that number.

We can compute the sum of consecutive terms of any arithmetic sequence like this

Any arithmetic sequence we can write as

$$x_n = an + b$$

To compute the sum of the first n terms of these, we take $n = 1, 2, 3, \ldots$ in this and add

$$\begin{aligned} x_1 &= a + b \\ x_2 &= 2a + b \\ &\dots\dots\dots \\ x_n &= na + b \end{aligned}$$

Adding as before,

$$x_1 + x_2 + \cdots + x_n = (a + 2a + \cdots + na) + \overbrace{(b + b + \cdots + b)}^{n \text{ times}}$$
$$= a(1 + 2 + \cdots + n) + nb$$
$$= \tfrac{1}{2}an(n + 1) + nb$$

For the arithmetic sequence given by

$$x_n = an + b$$

the sum of the first n terms is,

$$x_1 + x_2 + \cdots + x_n = \tfrac{1}{2}an(n + 1) + nb$$

As an example, let's compute the sum of the first 100 terms of the arithmetic sequence

$$1, 4, 7, \ldots$$

The algebraic form of this sequence is

$$x_n = 3n - 2$$

So, the sum of the first 100 terms is

$$3 \times \tfrac{1}{2} \times 100 \times 101 - (2 \times 100) = 14950$$

In general, the sum of the first n terms of this sequence is

In GeoGebra, we can use the sum command to compute the sum of a sequence.

The command

L=Sequence(n^2,n,1,10)

in the input bar gives the list of first ten perfect squares. The command sum (L) gives the sum of these ten numbers. We can directly type sum(n^2,n,1,10). To get the sum of the first 20 terms of the arithmetic sequence 5, 8, 11, ... type

sum(3n+2,n,1,20)

To get the sum of the 10^{th} to the 20^{th} terms of this sequence, type

sum(3n + 2,n,10,20).

$$3 \times \tfrac{1}{2}n(n + 1) - 2n = 3 \times \tfrac{1}{2}n^2 + \tfrac{3}{2}n - 2n = \tfrac{1}{2}(3n^2 - n)$$

The sum of an arithmetic sequence can be computed in another manner. For this, we first rewrite the algebraic form of the sum like this:

$$\tfrac{1}{2}an(n + 1) + bn = \tfrac{1}{2}n(a(n + 1) + 2b)$$
$$= \tfrac{1}{2}n(an + a + 2b)$$
$$= \tfrac{1}{2}n((an + b) + (a + b))$$

In this, $an + b$ is the n^{th} term of the sequence and $a + b$ is the first term

The sum of the first n terms of the arithmetic sequence $x_1, x_2, x_3, \dots$ is

$$x_1 + x_2 + \dots + x_n = \frac{1}{2}n(x_1 + x_n)$$

This can be put in ordinary language

The sum of consecutive terms of an arithmetic sequence is half the product of the sum of the first and last terms by the number of terms

To compute the sum of the first 100 terms of the arithmetic sequence 1, 4, 7, ... using this, first find the 100th term as

$$1 + (99 \times 3) = 298$$

Then we can compute the sum of the first 100 terms as

$$\frac{1}{2} \times 100 \times (1 + 298) = 14950$$

The sum of n terms of all arithmetic sequences have a common algebraic form. To see this, we write the sum as

$$\frac{1}{2}an(n + 1) + bn = \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$$

In this $\frac{1}{2}a$ and $\frac{1}{2}a + b$ are definite numbers associated with the sequence. So, the sum is the sum of the products of n^2 and n by fixed numbers

The algebraic form of the sum of the first n terms of an arithmetic sequence is

$$pn^2 + qn$$

where p and q are specific numbers

For example, it is not difficult to see that the algebraic form of the sequence

$$3, 10, 17, \dots.$$

is

$$7n - 4$$

What is the sum of the first n terms?

$$\frac{1}{2} \times 7 \times n(n + 1) - 4n = \frac{1}{2}(7n^2 + 7n - 8n) = \frac{1}{2}(7n^2 - n)$$

In detail, if we put $n = 1$ in the algebraic expression $\frac{1}{2}(7n^2 - n)$, we get 3, which is the first term of the sequence.

Putting $n = 2$, we get 13, which is the sum of the first two terms

Putting $n = 3$, we get 30, which is the sum of the first three terms and so on

Now a question in the other direction:

> The sum of the first n terms of an arithmetic sequence is
>
> $$3n^2 + n$$
>
> What is the sequence?

How do we do this?

We can think like this:

(i) We get the first term by taking $n = 1$ in the algebraic form of the sum

(ii) $(3 \times 1^2) + 1 = 4$

(iii) We get the sum of the first two terms by taking $n = 2$ in the algebraic form of the sum

(iv) The sum of the first two terms is $(3 \times 2^2) + 2 = 14$

(v) Subtracting the first term from this gives the second term

(vi) Second term is $14 - 4 = 10$

(vii) The sequence is 4, 10, 16, ...

(viii) The algebraic form of the sequence is $x_n = 6n - 2$

If we know the algebraic form of a sequence, we can use the sum function of the Python language to find the sum of any number of consecutive terms. For example, to get the sum of the first 100 perfect squares, type.

```
sum(x**2 for x in range(1,101))
```

at the python prompt.

Sum of squares

We have seen the identity $(x + 1)^2 = x^2 + 2x + 1$. Another such identity is

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$$

From this, we can see that for any number

$$(x + 1)^3 - x^3 = 3x^2 + 3x + 1$$

If we take $x = 1, 2, 3, \ldots, n$ in this we get,

$x = 1,\quad 2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$

$x = 2,\quad 3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1$

....

$x = n - 1,\quad n^3 - (n - 1)^3 = 3(n - 1)^2 + 3(n - 1) + 1$

$x = n,\quad (n + 1)^3 - n^3 = 3n^2 + 3n + 1$

If we add all these, we get

$$2^3 - 1^3 + 3^3 - 2^3 + \ldots + n^3 - (n - 1)^3 + (n + 1)^3 - n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + n$$

$$(n + 1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + n$$

That is,

$$n^3 + 3n^2 + 3n = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + \frac{3}{2}n(n + 1) + n$$

So, we get

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}\left(n^3 + 3n^2 + 3n - \frac{3}{2}n(n + 1) - n\right)$$

Simplifying the right side of this equation, we get

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

We can put it like this. The algebraic form of the arithmetic sequence

$$4, 10, 16, \ldots$$

is

$$x_n = 6n - 2$$

If we calculate the first term, the sum of the first two terms, the sum of the first three terms and so on, we get another sequence

$$4, 14, 30, \ldots$$

And the algebraic form of this sequence

$$y_n = 3n^2 + n$$

(1) Calculate in head the sum of the arithmetic sequences below:

(i) $51 + 52 + 53 + \ldots + 70$

(ii) $1\frac{1}{2} + 2\frac{1}{2} + \ldots + 12\frac{1}{2}$

(iii) $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + \ldots + 12\frac{1}{2}$

(iv) $\frac{1}{101} + \frac{3}{101} + \frac{5}{101} + \ldots + \frac{201}{101}$

(2) Calculate the sum of the first 25 terms of each of the arithmetic sequences below:

(i) 11, 22, 33, ... (ii) 12, 23, 34, ... (iii) 21, 32, 43, ... (iv) 19, 28, 37, ...

(3) Find the sum of all multiples of 9 among three-digit numbers.

(4) The n^{th} term of some arithmetic sequences are given below. Find the sum of the first n terms of each:

(i) $2n + 3$ (ii) $3n + 2$ (iii) $2n - 3$ (iv) $3n - 2$

(5) The sum of the first n terms of some arithmetic sequences are given below. Find the n^{th} term of each:

(i) $n^2 + 2n$ (ii) $2n^2 + n$ (iii) $n^2 - 2n$ (iv) $2n^2 - n$ (v) $n^2 - n$

(6) (i) Calculate the sum of the first 20 natural numbers

(ii) Calculate the sum of the first 20 numbers got by multiplying the natural numbers by 5 and adding 1. Calculate also the sum of the first *n* terms.

(7) How much more is the sum of the first 25 terms of the arithmetic sequence 15, 21, 27, ... than the sum of the first 25 terms of the arithmetic sequence 7, 13, 19, ...?

(8) The 10^{th} term of an arithmetic sequence is 50 and the 21^{st} term is 75. Calculate the sum of the first 30 terms of this sequence.

We know that the sum of any number of consecutive odd numbers starting with 1 is a perfect square

Calculate the sum of the first few consecutive terms of the sequence 4, 12, 20, ...

Are there other arithmetic sequences with the sum of any number of consecutive terms from the first is a perfect square?