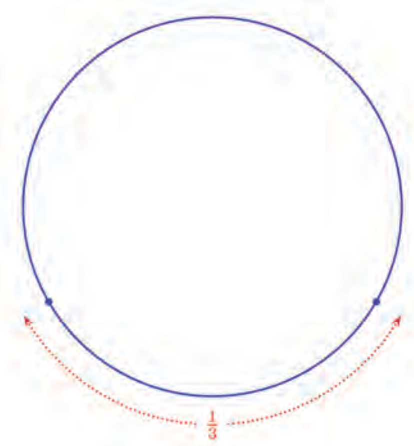
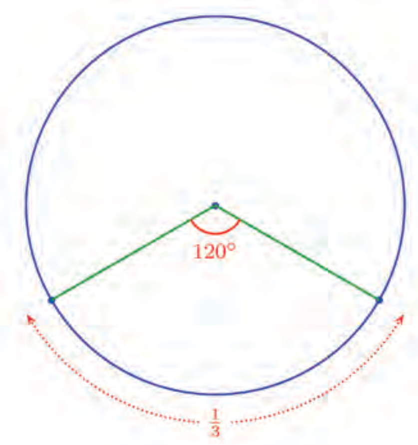
2 CIRCLES AND ANGLES

Parts of a circle

How do we cut out $\frac{1}{3}$ of a circular ring ?

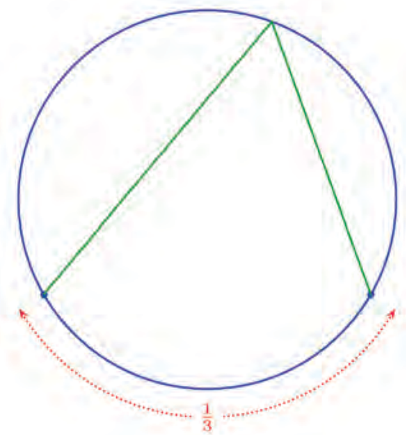


As we saw in the section **Length and angle** of the chapter **Parts of Circles** in the Class 9 textbook , the central angle of this arc is $\frac{1}{3} \times 360° = 120°$.

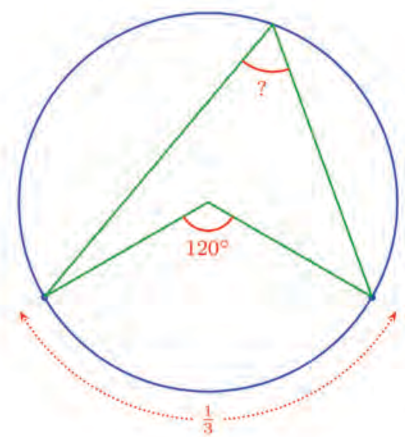


So, one method is to first locate the centre of the circle using any of the techniques discussed in the section **Chords** of the chapter **Circles** in the Class 8 textbook, and then draw an angle of 120° at the centre to mark an arc which is $\frac{1}{3}$ of the circle.

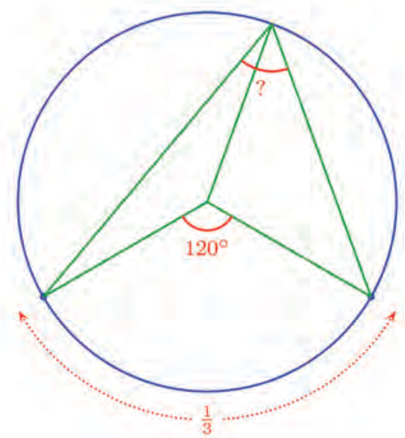
Can we mark such an arc without locating the centre ? In other words, is there a way to mark such an arc by drawing two lines from some point of the circle itself ?



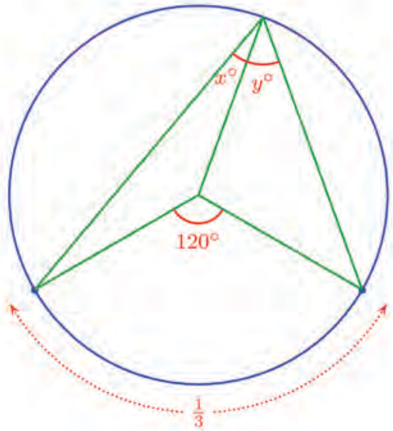
Let's make this question more precise : what angle should we choose in the picture above, so that the angle at the centre is 120° ?

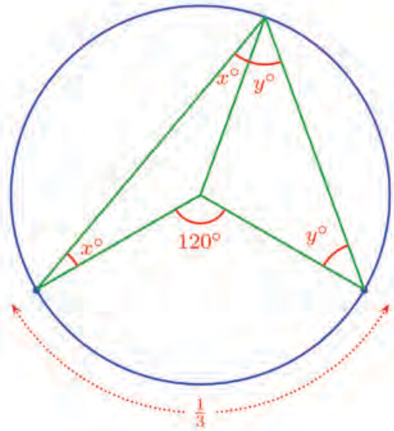


To compute this angle , we join the centre of the circle to the corner of this angle and split it into two parts :

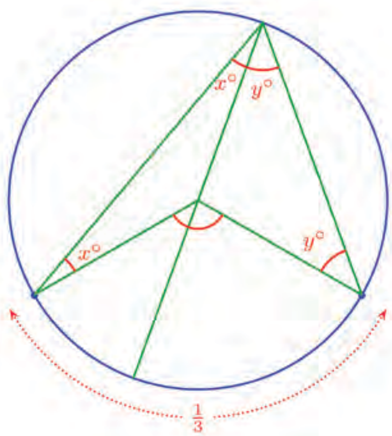


Now there are two triangles in the picture. In both, two of the sides are radii of the circle, and so are of equal length. In other words, both are isosceles triangles. So, if we take the parts of the top angle as $x°$ and $y°$, then they are also the bottom angles of the triangles :

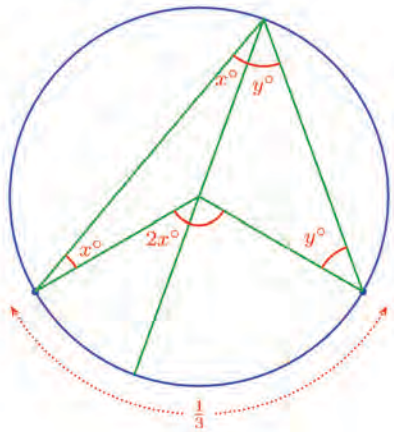


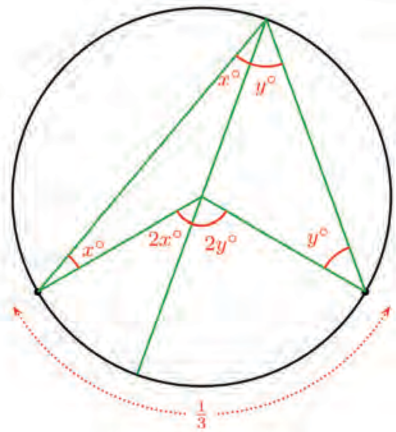


Now if we extend the common side of these triangles, then the central angle also is split into two :



These parts of the central angles are the outer angles at the third vertices of the triangles on the left and right ; and so they are equal to the sum of the inner angles at the other two vertices (the section **Outer angles** of the chapter **Polygons**, in the Class 8 textbook).





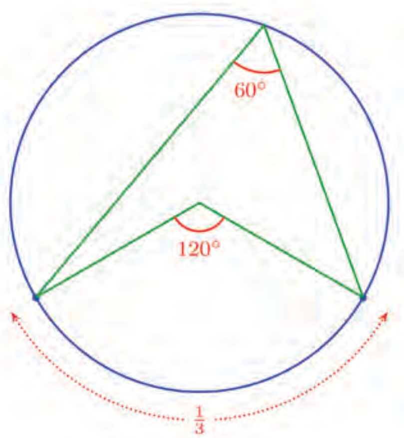
Since the central angle is 120°, we have

$$2x + 2y = 120$$

and from this, we get

$$x + y = 60$$

This means , the angle made by the ends of the arc, at the point on the circle is 60°.



Make an angle slider a. Draw a circle centred at a point A and mark a point B on it. Select the Angle with Given Size tool and click on B and then on A. In the dialogue window, give a as Angle. We get a point B' on the circle. Join AB and AB', and mark the central angle. Make sure that the central angle is less than 180°. Mark the point C on the larger arc and join CB and CB', and mark the angle at C. What is the relation between the angles at A and C ? Change the position of C. What if we change the value of a, using the slider ?

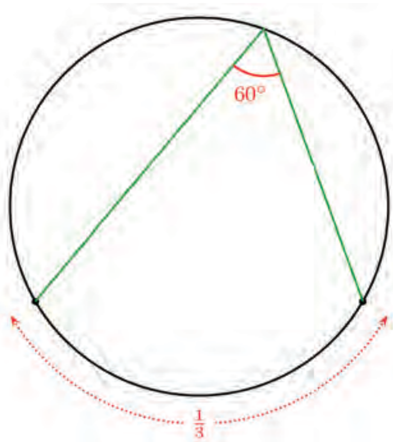
What do we see here ?

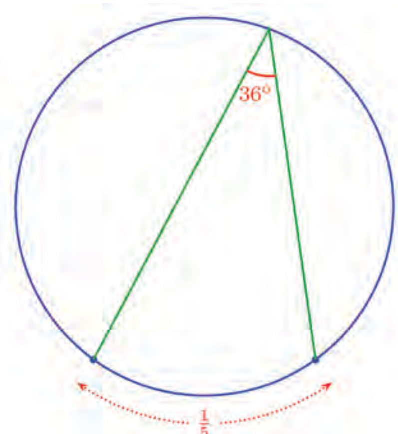
To mark $\frac{1}{3}$ of a circle , we need only make a 60° angle at a point on the circle, instead of a 120° angle at the centre.

Compute what angle we should mark at a point on the circle, to mark off $\frac{1}{5}$ of the circle.

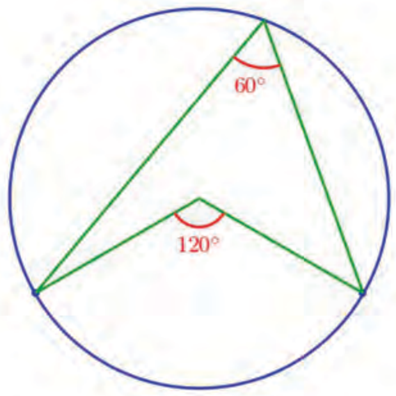
Arcs and angles

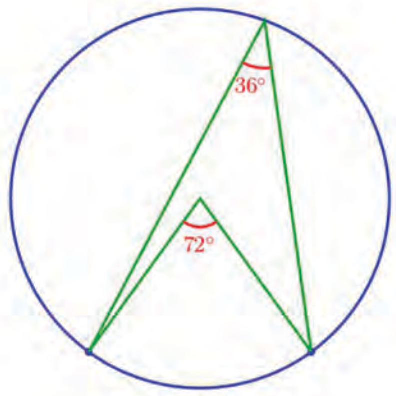
We have seen that if we mark a 60° angle at a point on a circle, we get an arc which is $\frac{1}{3}$ of the circle; and if we mark a 36° angle at a point on a circle, we get an arc which is $\frac{1}{5}$ of the circle





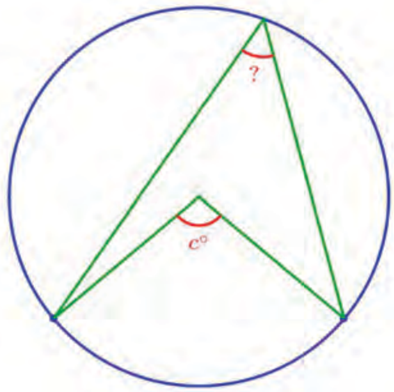
We can state this using central angles of arcs, instead of parts of a circle :





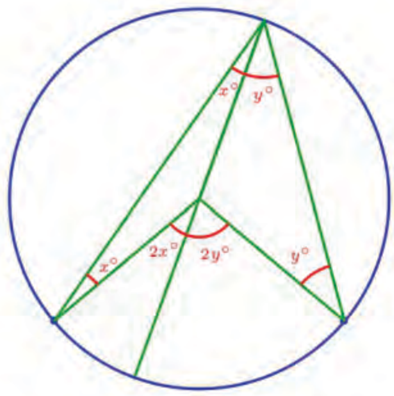
Now here's a question : if we join the ends of an arc to a point on the circle, would we get half the central angle ?

Let's take $c°$ as the central angle of an arc :



As before , let's join the centre of the circle and the point on the circle and take the parts into which this cuts the top angle as $x°$ and $y°$.

We then extend this line to cut the central angle also. We can then mark the angles as below:

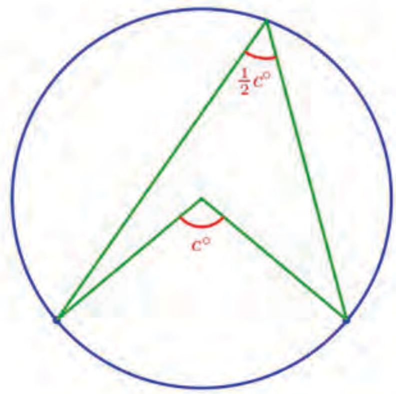


We took the central angle as $c°$. So, we have

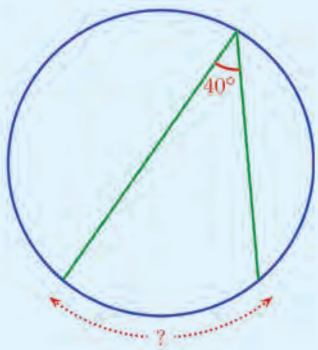
$$2x + 2y = c$$

and from this we get

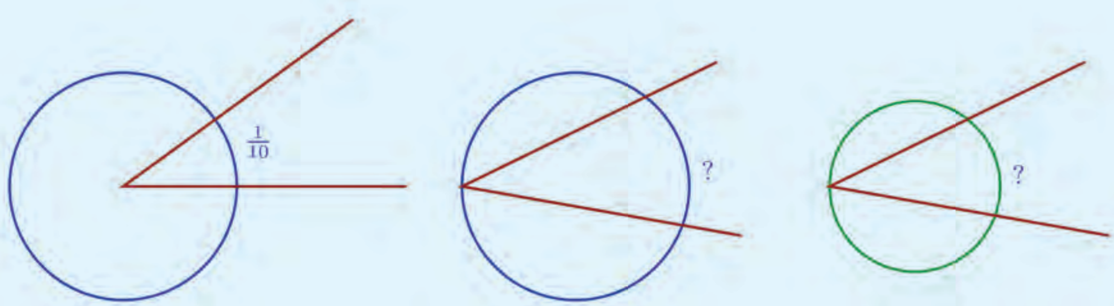
$$x + y = \frac{1}{2}c$$



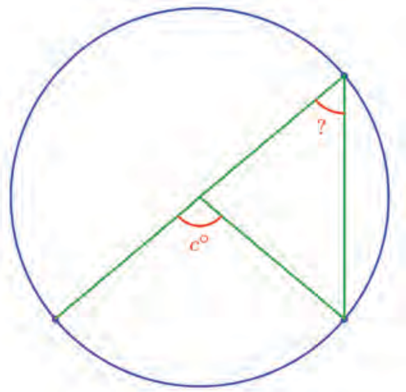
(1) What fraction of the circle is the arc marked in the picture below ?



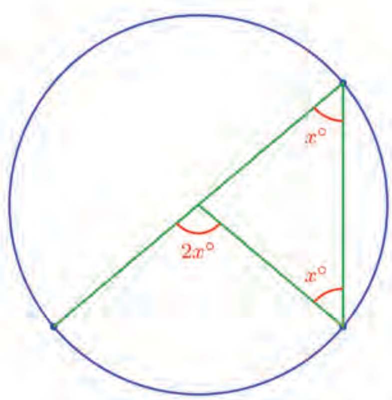
(2) When the corner of a bent wire was placed at the centre of a circle, $\frac{1}{10}$ of the circle was contained within it. If the corner of this wire is placed on a point of this circle as in the second picture, what fraction of the circle would it contain ? What if it is placed at a point on another circle as in the third picture ?

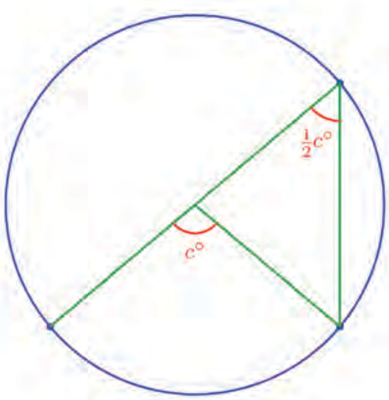


But then , there's another question : would we get half the central angle , if the ends of an arc are joined to any point on the circle ? What if it's like this ?

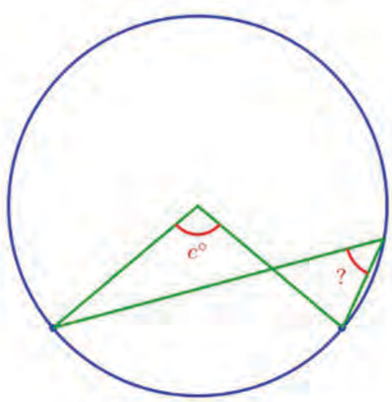


Here the point on the circle is the other end of a diameter through one end of the arc. If we take the angle which the arc makes at this point as $x°$, then we can compute the other angles of the figure:

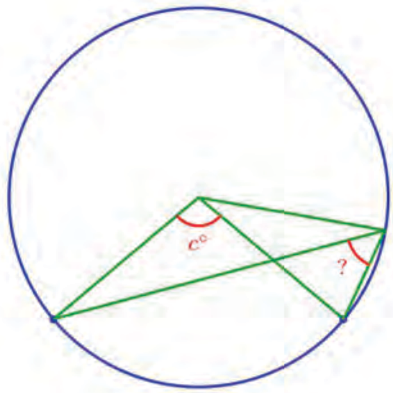




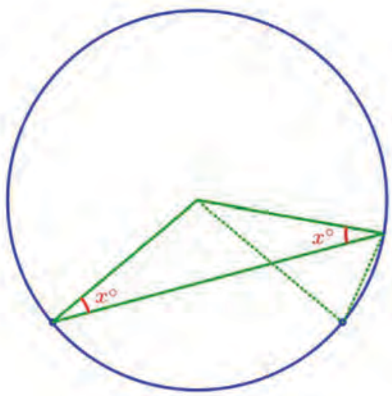
What if we choose a point still lower on the circle as in the picture below ?

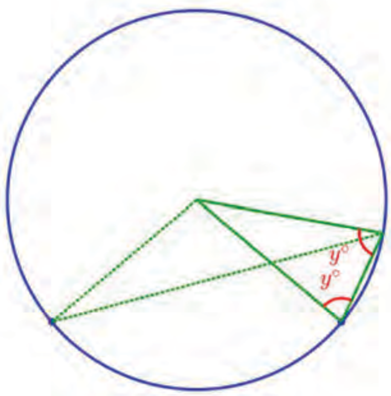


As before , we join this point to the centre of the circle:

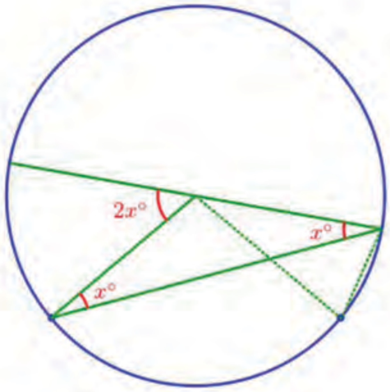


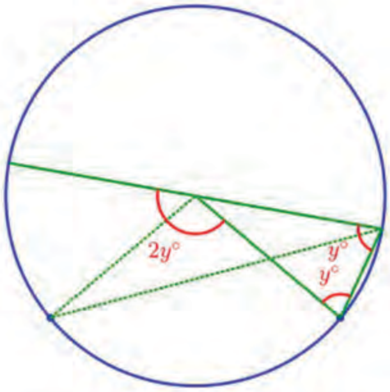
And take the equal angles in one isosceles triangle as $x°$ and in the other as $y°$:



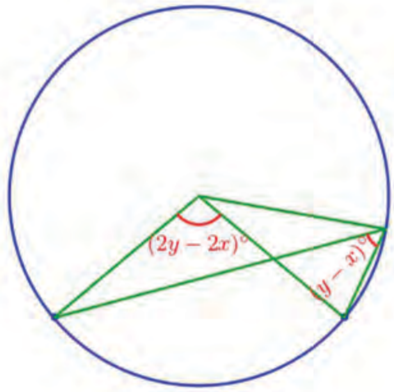


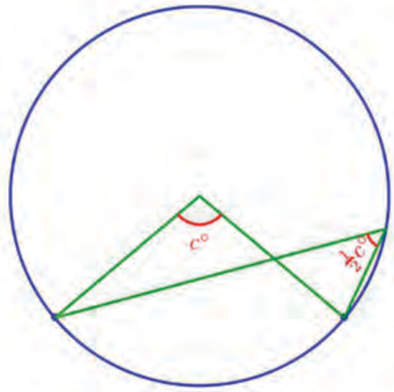
Again we extend the common side of these triangles , and mark the outer angles:



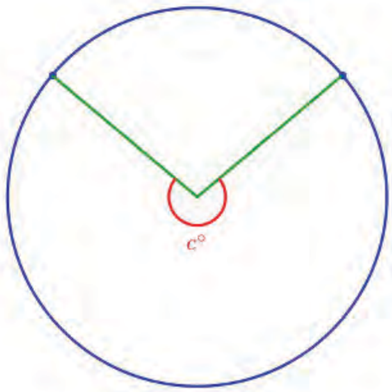


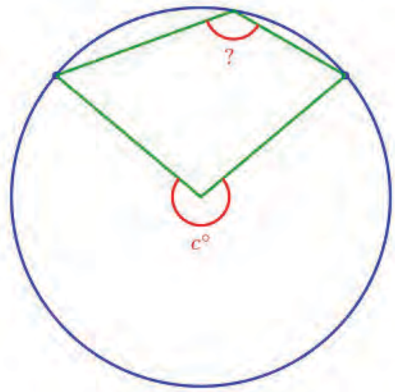
Now we can compute the central angle of the arc and the angle made by its ends at the point on the circle:



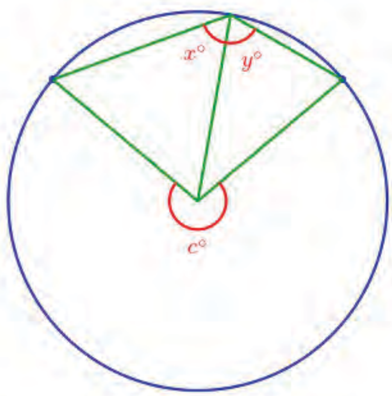


In all the discussion so far, we considered only arcs with central angle less than 180°. What if the central angle is greater than 180° ?

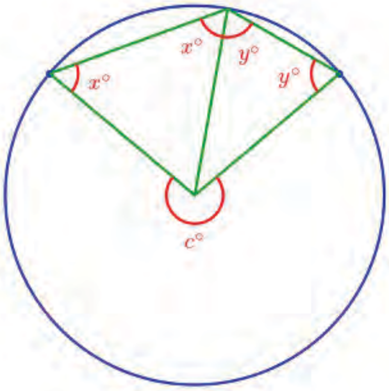




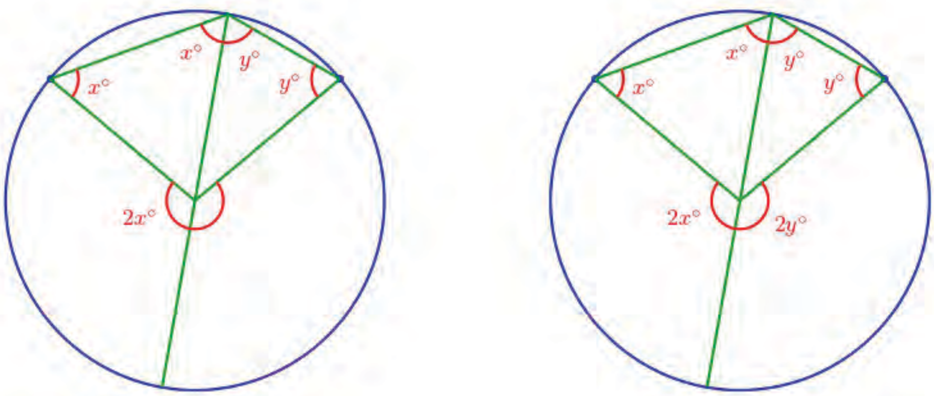
As before, let's join the centre of the circle and the point on the circle ; and take as x° and y° the parts of the angles into which this line splits the angle made at the point on the circle :



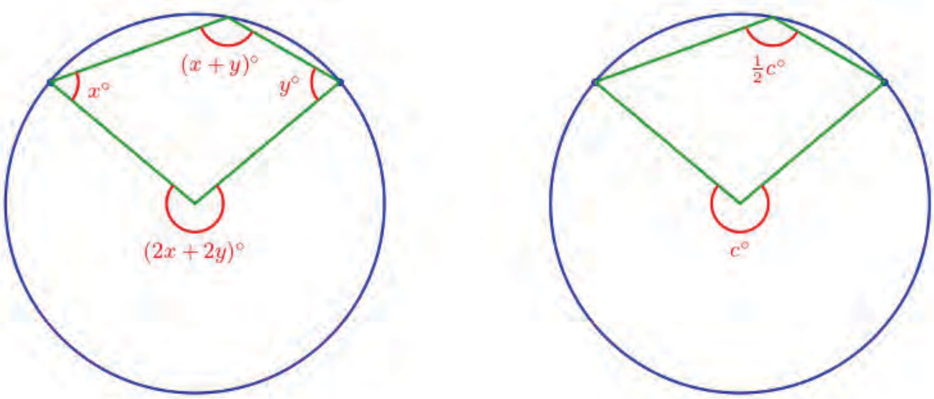
Since the two triangles in the circle are isosceles triangles, the left and right angles are also $x°$ and $y°$:



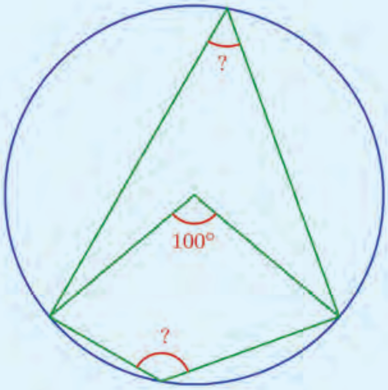
Again, as before, let's extend this line and split the central angle into outer angles of the triangles :

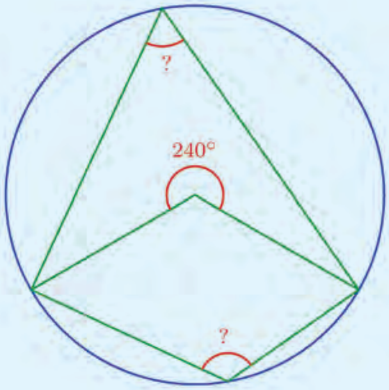


From this we can calculate the central angle of the arc and the angle made by its endpoints at the point on the circle :



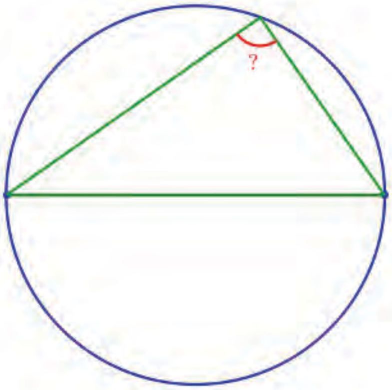
In each picture below, the central angle of an arc of a circle is shown :



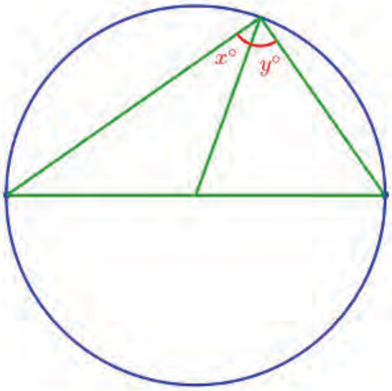


In each, calculate the angles which the arc makes at the other two points

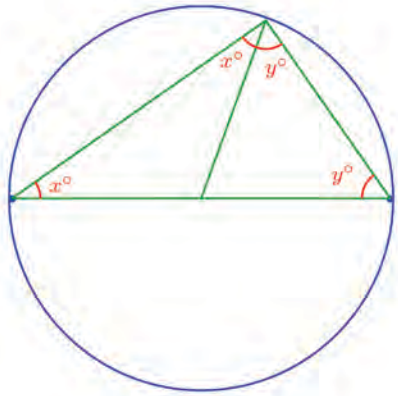
Now what if the central angle is 180° ? That is if the arc is a semicircle ?

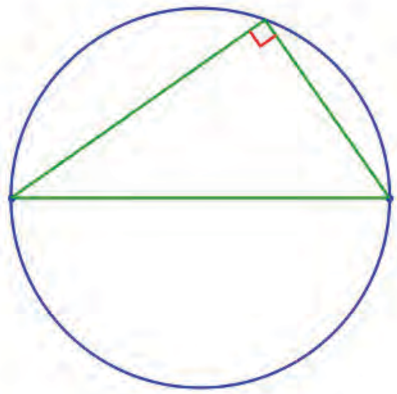


Let's again join the centre of the circle to the point on it, and take as $x°$ and $y°$, the angles into which this line cuts the angle at the point on the circle :



The bottom angles of the isosceles triangles are also $x°$ and $y°$





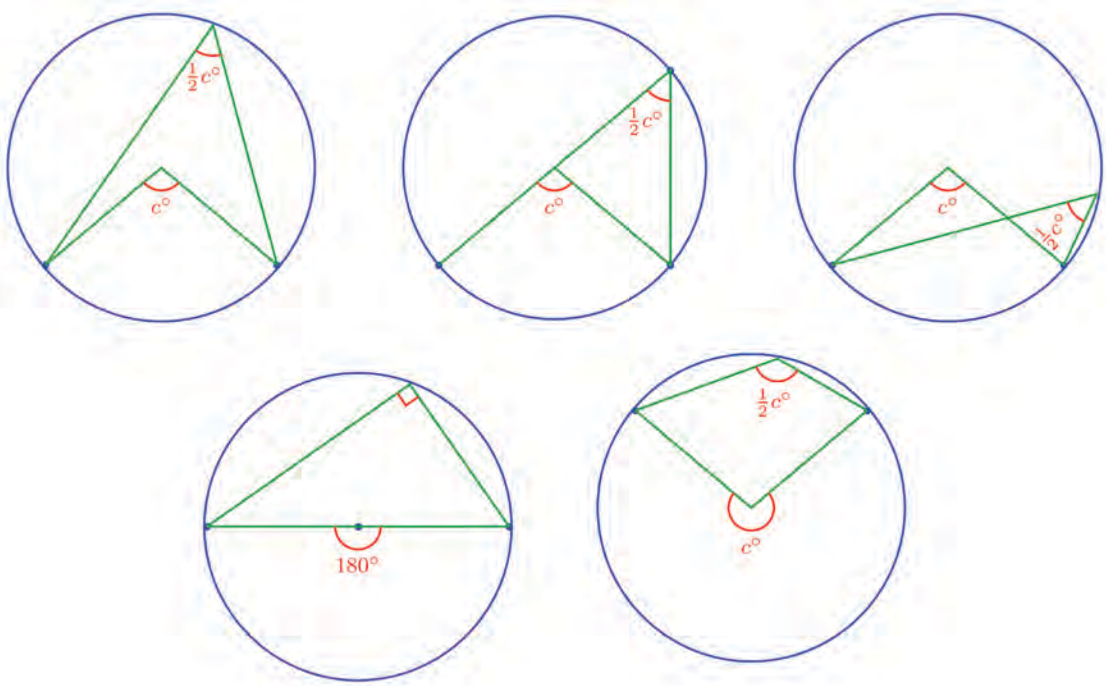
Since the sum of the angles of the large triangle in the circle is 180°, we have

$$(x + y) + x + y = 180$$

and from this we get $x + y = 90$

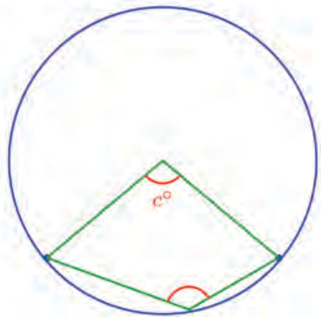
What did we do in all these discussions ?

We found out the relation between the central angle of an arc of a circle and the angle which the ends of the arc make with a point on the circle :



Note that in all these, we join the ends of the arc to a point on the circle, *outside* the arc.

We can readily see that if we join the ends of an arc to a point *within* the arc, then we won't get half the central angle :



So all that we have seen so far can be combined to a general principle :

If the ends of an arc of a circle are joined to a point on the circle, which is not a point on the arc itself, then the angle so made is half the central angle of the arc

The case of a semicircle is worth special mention :

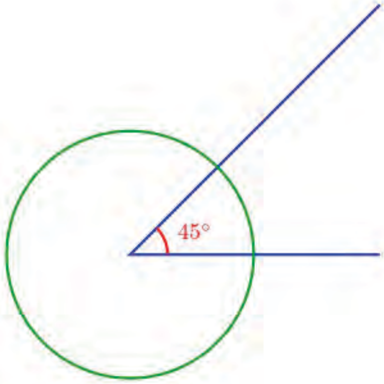
If the ends of a semicircle are joined to another point on the circle, the angle made is a right angle

This can be shortened like this:

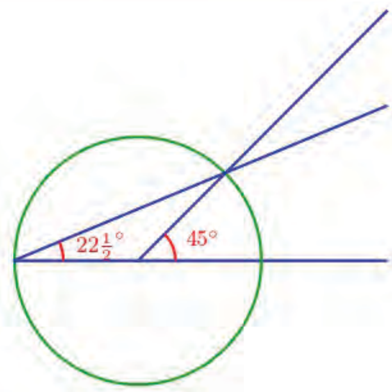
The angle in a semicircle is a right angle

Let's look at some applications of these results

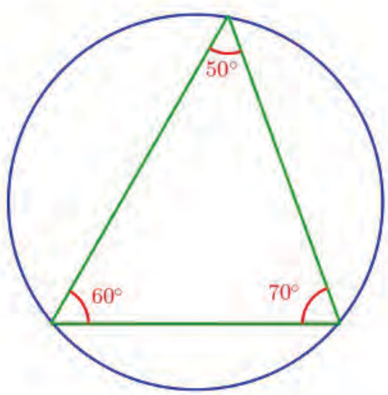
We can use the first result to draw half a given angle . See this picture :



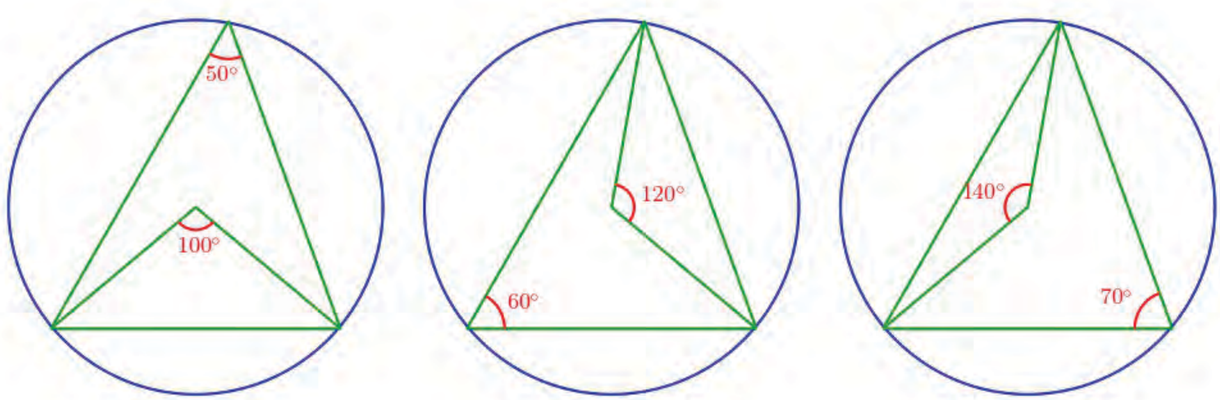
The circle is drawn with centre at the vertex of the angle . Now if we extend the bottom line of the angle to meet the circle, and join this point to the point where the top line of the angle intersects the circle , we have half the angle



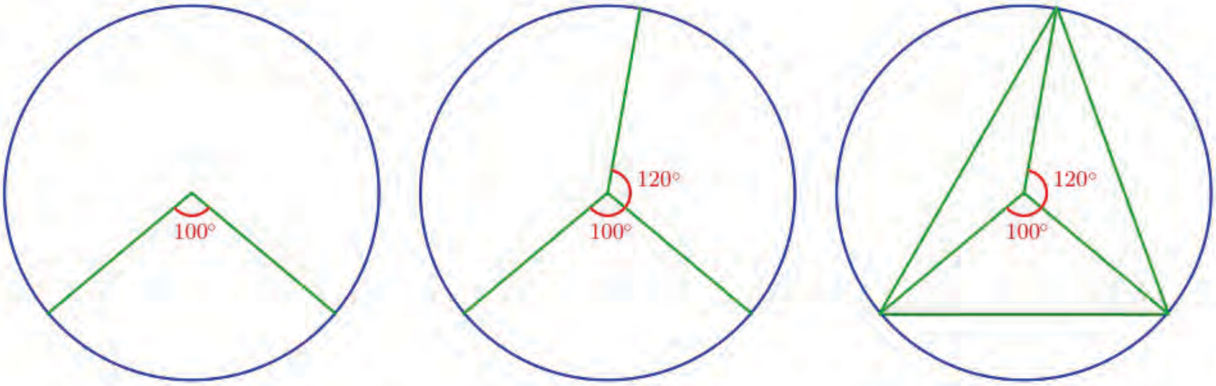
We can also use the result to draw a triangle with specified angles within a circle. For example, let's see how we can draw a triangle of angles 50°, 60° and 70° within a circle.



The vertices of the triangle split the circle into three arcs. What are their central angles ?

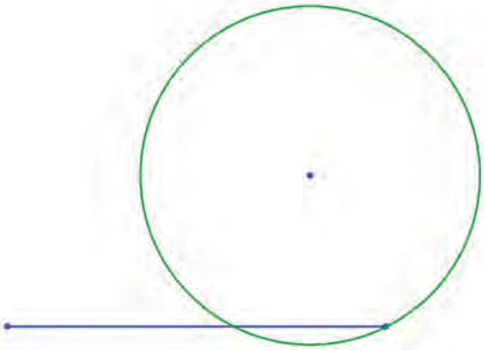


So to draw the triangle , we need only draw two of these angles at the centre :

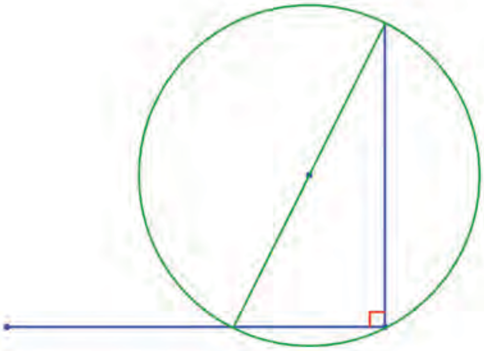


Next we use the result that the angle in a semicircle is a right angle, to draw a perpendicular to a line. Suppose we want to draw the perpendicular to the line below through its right endpoint :

For that , we first draw a circle which passes through this point and intersects the line at another point:



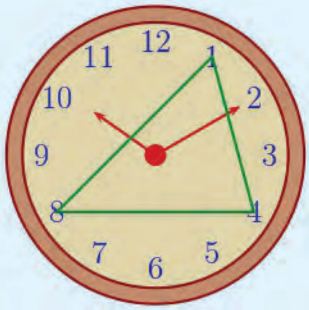
Next draw the diameter through this point and join its other end to the end of the first line:



Since the angle in a semicircle is a right angle , this line is perpendicular to the first line

Remember another method of drawing a perpendicular to a line using ruler and compass, seen in Class 8 ? (The section **Bisectors of a line** in the lesson **Bisectors**).

(1) A triangle is drawn joining the numbers 1, 4 and 8 on a clock face:



(i) Calculate the angles of this triangle.

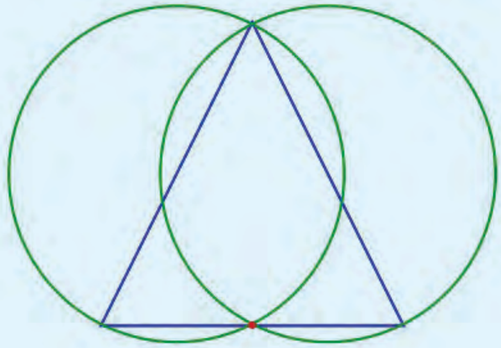
(ii) How many equilateral triangles can we make by joining the numbers on a clock face ?

(2) Draw an equilateral triangle with circumradius 3.5 centimetres.

(3) Draw a triangle with circumradius 3 centimetres and two of the angles $32\frac{1}{2}^{\circ}$ and $37\frac{1}{2}^{\circ}$.

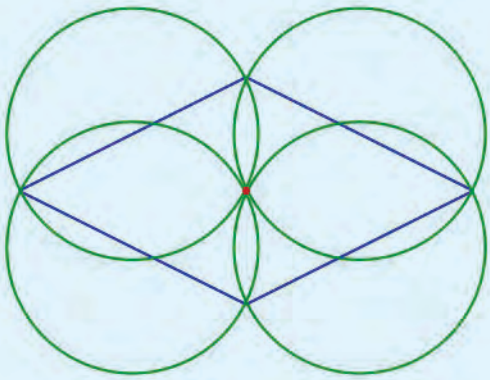
(4) In the picture , a semicircle is drawn with a line as diameter and a smaller semicircle with half this line as diameter . Prove that a line joining the point where the semicircles meet, to any point on the larger semicircle is bisected by the smaller semicircle:

(5) Prove that circle drawn on the equal sides of an isosceles triangle as diameters pass through the midpoint of the third side:

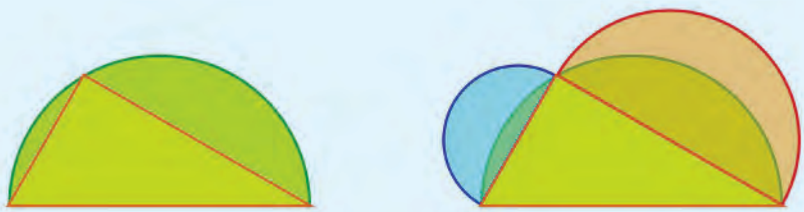


(Hint : Consider the circles one by one).

(6) Prove that all the circles drawn on the four sides of a rhombus as diameters pass through a common point:



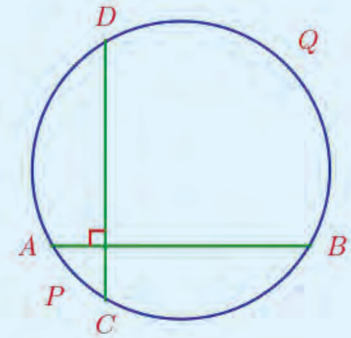
(7) In the picture below, a triangle is drawn joining the ends of the diameter of a circle and another point on the semicircle; and then semicircles on the other sides of the triangle as diameters:



Prove that the sum of the areas of the blue and red crescents in the second picture is equal to the area of the triangle

(Hint : See the last problem of the lesson, **Parts of Circles** of Class 9 textbook).

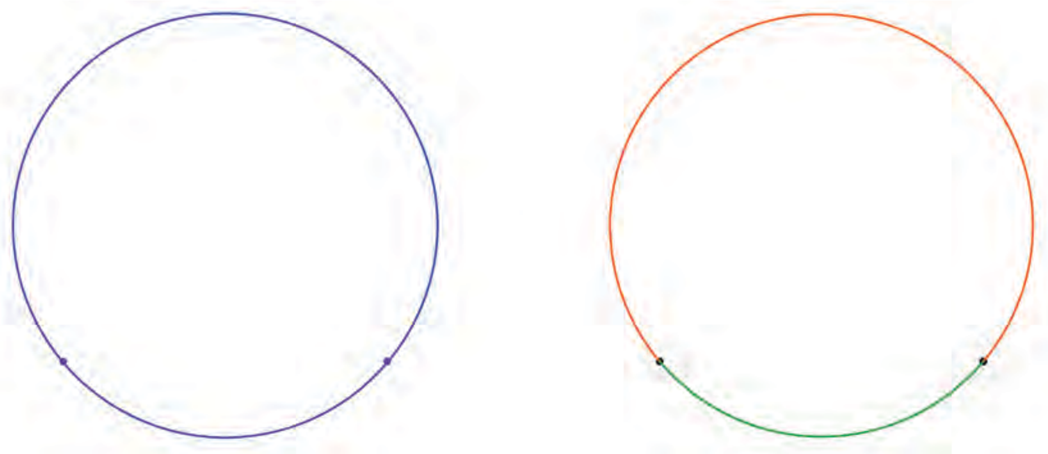
(8) In the picture, AB and CD are perpendicular chords of the circle:



Prove that the arcs APC and BQD joined together make a semicircle.

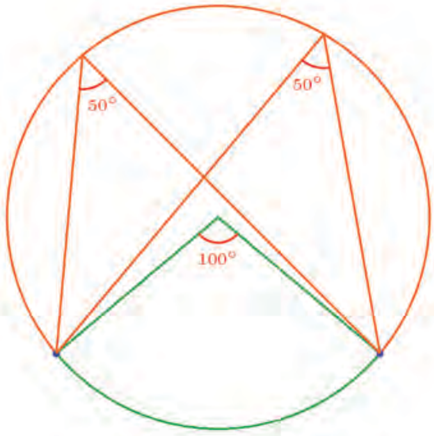
Segments of a circle

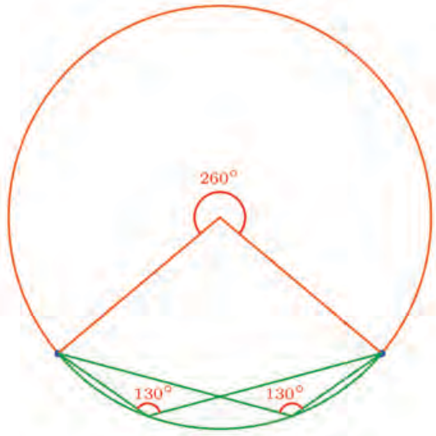
Any two points on a circle splits the circle into two arcs :



Each of these arcs may be called the **alternate arc** of the other . Any point on the circle is either on one of these arcs or its alternate arc . So , the relation between the central angle of an arc and the angle made by joining the ends of the arc to a point on the circle can be stated like this

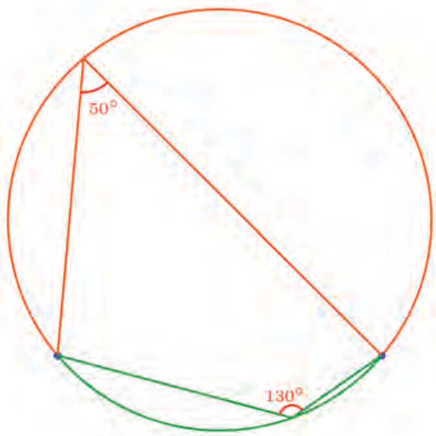
> The angle made by joining the ends of an arc of a circle to any point on the alternate arc is half the central angle of the arc



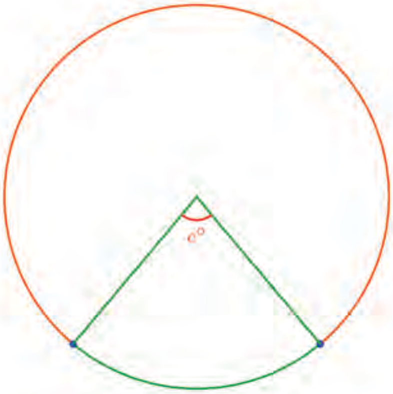


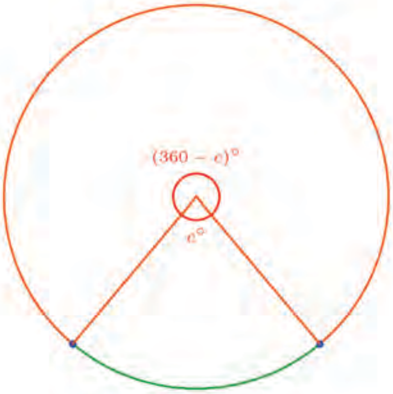
In other words , two points on a circle divide the circle into two arcs. The angles made by joining these points to any point on one of the arcs are all equal.

What is the relation between the angle made by joining these points to points on one arc and its alternate arc ?

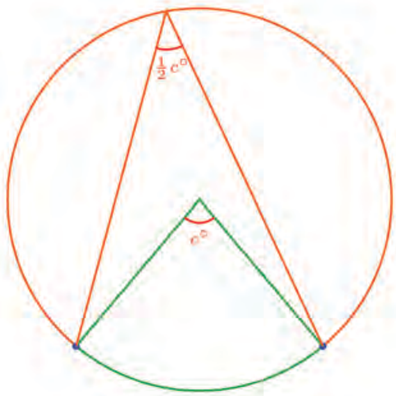


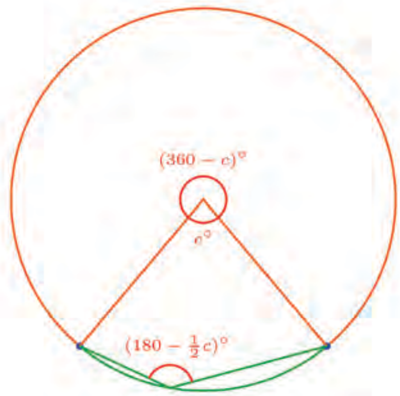
The sum of the central angles of an arc and its alternate arc is 360°, isn't it ?





And the angles made at points on each arc is half its central angle:

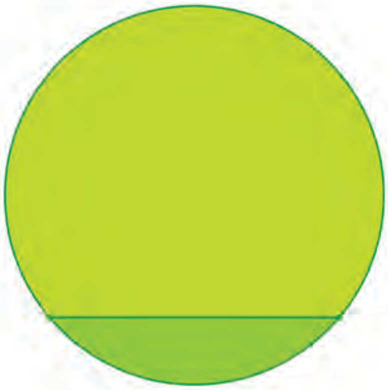




So what can we say ?

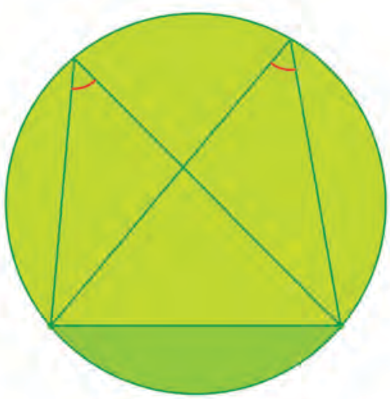
The sum of the angles made by two points on a circle, at points on one of the arcs and the alternate arc, is 180°

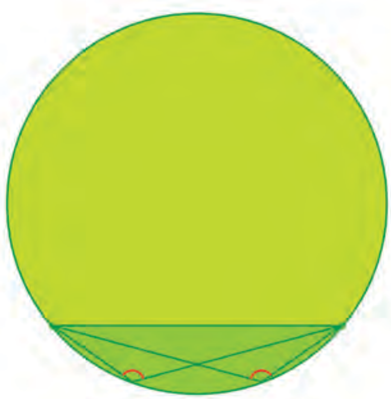
This can put in another way. A chord joining two points on a circle splits the entire circular region into two parts.



Such parts of a circle are called **segments** of the circle. Of the two segments into which a chord cuts a circle, each may be called the **alternate segment** of the other.

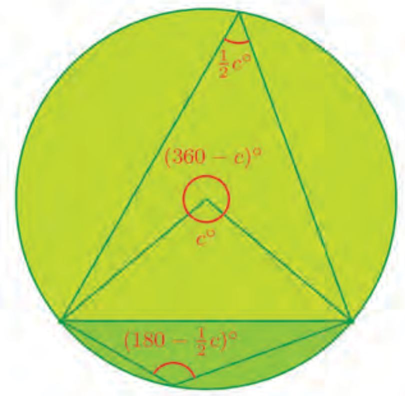
So the result about the angles in a circle can be stated in terms of segments also, instead of arcs.



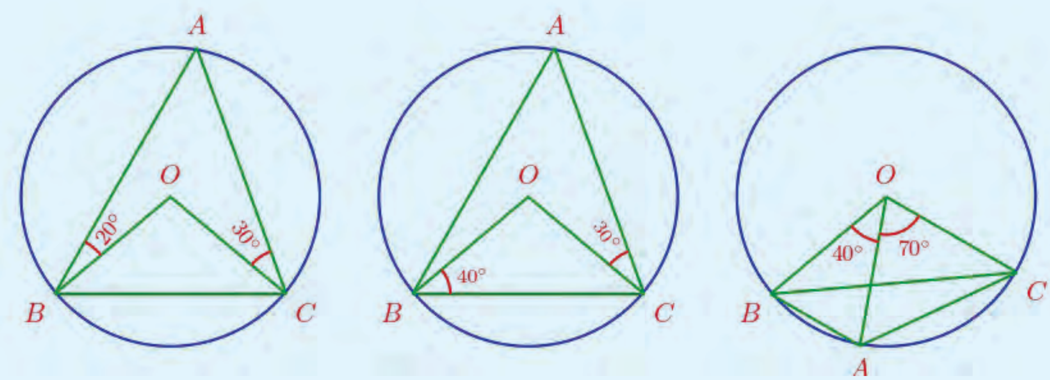


In a circle , angles in the same segment are equal; the sum of the angles in alternate segments is 180°

The relations between the central angles of the two arcs into which two points divide a circle, and the angles in the segments into which the chord joining these points divide the circular region, are shown in the figure below:



(1) In all three pictures below, O is the centre of the circle and A, B, C are points on the circle. In each , calculate all the angles of triangles ABC and OBC:



(2) In each of the problem below , a circle and a chord is to be drawn to split the circle into two parts. The parts must be as specified:

(i) All angles in one part must be 80°

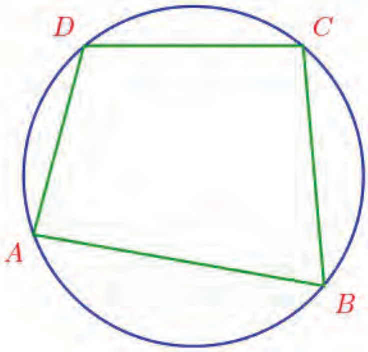
(ii) All angles in one part must be 110°

(iii) All angles in one part must be half the angles in the other part

(iv) All angles in one part must be one and a half times the angles in the other part

Circle and quadrilateral

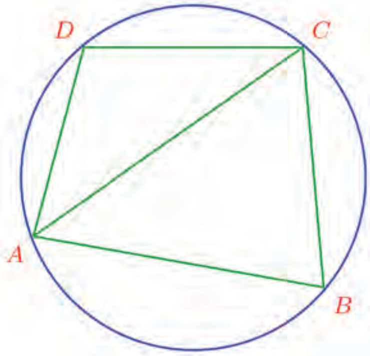
See this picture:

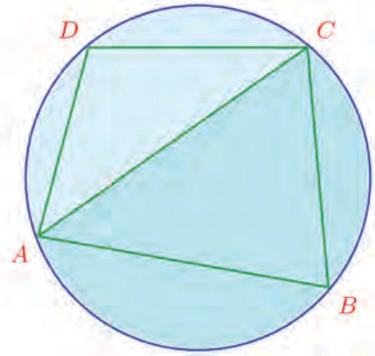


Draw a circle in GeoGebra and draw a quadrilateral with vertices on the circle, using the Polygon tool. Mark all its angles (Just select the Angle tool and click inside the quadrilateral) Check the relation between the opposite angles. Move the vertices on the circle and see what happens

A quadrilateral is drawn joining four points on a circle. Is there any relation between the opposite angles in this quadrilateral ?

If you don't see it, join *AC* and look again:





Now the angles at *B* and *D* are the angles in alternate segments into which the chord *AC* splits the circle. So their sum is 180°

If we join *BD* instead of *AC*, we can similarly see that the sum of the angles at *A* and *C* is also 180°

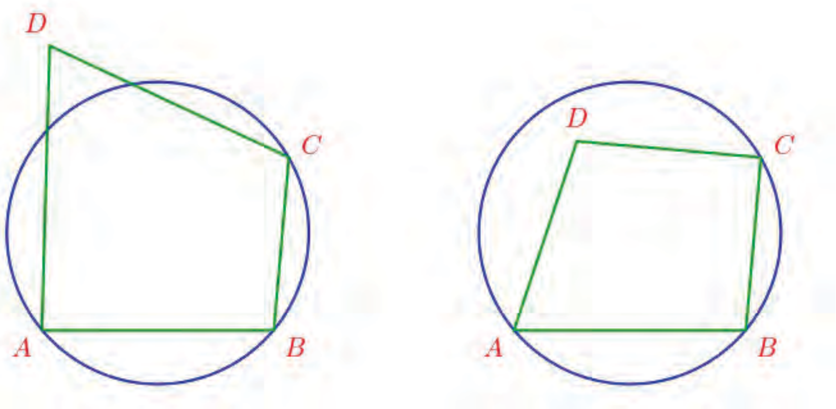
So what can we say in general ?

If all vertices of a quadrilateral are on a circle, then the sum of its opposite angles is 180°

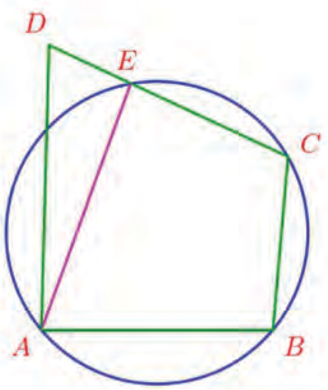
Is the reverse statement true ? That is, if the sum of opposite angles of a quadrilateral is 180°, can we draw a circle through all its vertices ?

We think like this. We can draw a circle through three vertices of any quadrilateral (The section **Lines and circles** of the chapter **Circles** in the Class 8 textbook).

The fourth vertex may be on the circle; or it may be inside or outside the circle. We have seen that if it is on the circle , then the sum of the angles at this vertex and the opposite one is 180°



Look at the first picture, If we join the point where the circle intersects *CD*, and the point *A*, we get a quadrilateral with all four vertices on the circle:



Since *A*, *B*, *C*, *E* are on the circle,

$$\angle B + \angle AEC = 180° \qquad (1)$$

Here $\angle AEC$ is the outer angle at *E* of triangle *AED*. So,

$$\angle AEC = \angle D + \angle EAD$$

This shows that

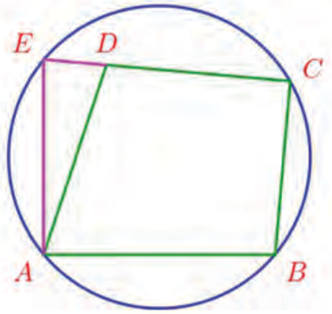
$$\angle D < \angle AEC \qquad (2)$$

Now if we think about the meaning of the algebraic statements marked (1) and (2), we can see that

$$\angle B + \angle D < 180°$$

Thus if the fourth vertex is outside the circle , the sum of the angles at this vertex and the opposite one is less than 180°

Next in the second picture, extend the line *CD* and join the point where it meets the circle, and the point *A*:



Using the Polygon tool, draw a quadrilateral ABCD. Using the Circle through 3 points tool, Draw a circle passing through points A, B, C. Mark the angles of the quadrilateral. Observe ∠D, ∠B. Change the position of D. What is the relationship between these angles when the point D is outside the circle ? When inside the circle ? What if the point D is on the circle ?

In this we have

$$\angle B + \angle E = 180° \qquad (3)$$

Also from triangle *AED*, we can see that

$$\angle ADC = \angle E + \angle EAD$$

so that

$$\angle ADC > \angle E \qquad (4)$$

From the relations (3) and (4) we get

$$\angle B + \angle ADC > 180°$$

So if the fourth vertex is inside the circle, the sum of the angles at this vertex and the opposite one is greater than 180°

If a circle is drawn through three vertices of a quadrilateral and the fourth vertex is outside this circle, then the sum of the angles at this vertex and the opposite vertex is less than 180°; if the fourth vertex is inside the circle, the sum of these angles is greater than 180°

Now suppose that in a quadrilateral *ABCD*, we have ∠*B* + ∠*D* = 180° and we draw the circle through *A*, *B* and *C*.

Can *D* be outside the circle ? If it is, then the sum of the angles at *B* and *D* would be less than 180°. So, *D* is not outside the circle.

Can *D* be inside the circle ? If it is , then the sum of the angles at *B* and *D* would be greater than 180°. So, *D* is not inside the circle.

Since it is neither outside nor inside the circle, *D* must be on the circle.

Thus we have this result

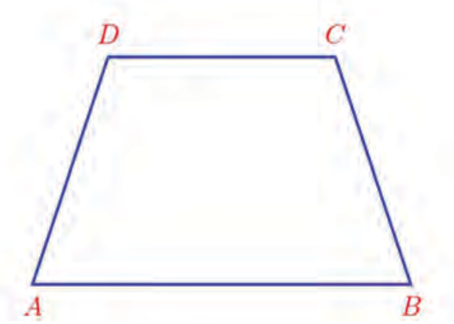
If in a quadrilateral, the sum of opposite angles is 180°, then a circle can be drawn passing through all four of its vertices

A quadrilateral with the property that a circle could be drawn through all four of its vertices, is called a **cyclic quadrilateral.**

What we have seen just now is that a cyclic quadrilateral is a quadrilateral with the sum of opposite angles 180°.

All rectangles are cyclic quadrilaterals, right ?

We can also show that any isosceles trapezium is cyclic. See this picture:



ABCD is an isosceles trapezium. So,

$$\angle A = \angle B$$

Also, since *AB* and *CD* are parallel,

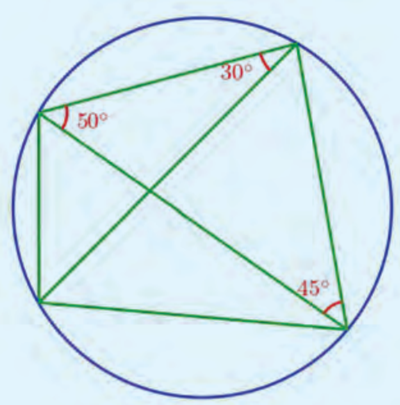
$$\angle A + \angle D = 180°$$

From these two equations, we get

$$\angle B + \angle D = 180°$$

So *ABCD* is a cyclic quadrilateral

(1) Calculate the angles of the quadrilateral shown below and also the angles between its diagonals

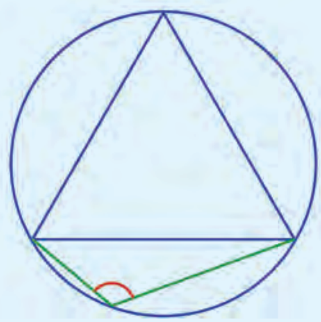


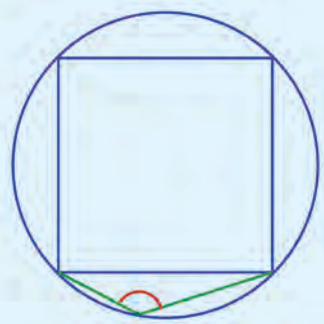
(2) Prove that in a cyclic quadrilateral, the outer angle at any vertex is equal to the inner angle at the opposite vertex

(3) Prove that any parallelogram, which is not a rectangle, is not cyclic.

(4) Prove that a non - isosceles trapezium is not cyclic.

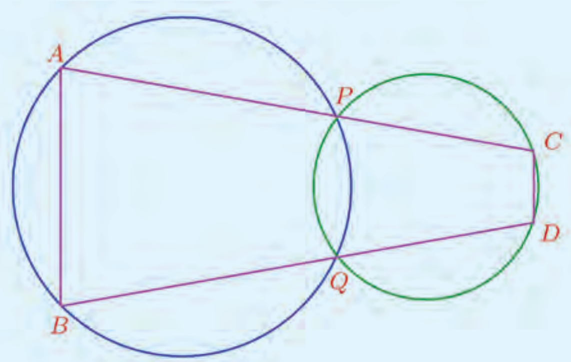
(5) In the first picture below , an equilateral triangle is drawn with vertices on a circle and two of its vertices are joined to a point on the circle. In the second picture, a square is drawn with vertices on a circle and two of its vertices are joined to a point on the circle :



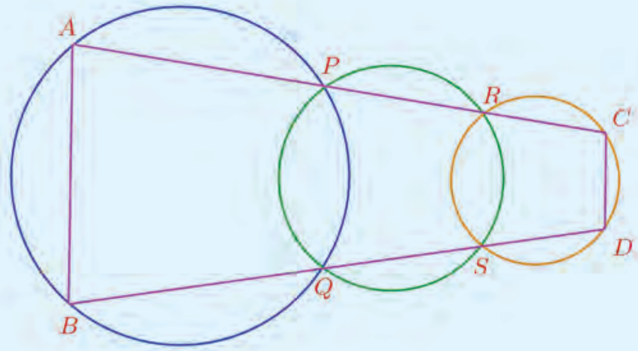


In each picture, calculate the angle marked.

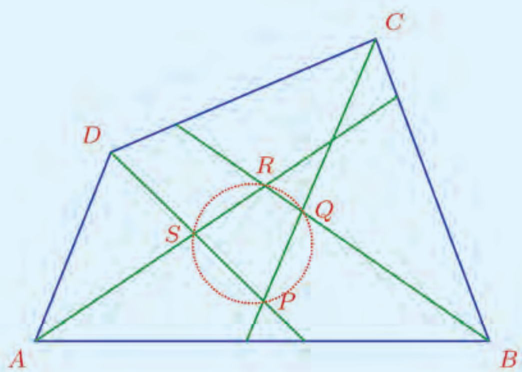
(6) (i) In the picture below , two circles intersect at P and Q. Lines through these points meet the circles at A, B, C, D. The lines AC and BD are not parallel. Prove that if these lines are of equal length, then $ABDC$ is a cyclic quadrilateral.



(ii) In the picture , the circles on the left and right intersect the middle circle at *P*, *Q*, *R*, *S*. Lines joining these meet the left and right circles at *A*, *B*, *C*, *D*. Prove that *ABDC* is a cyclic quadrilateral.



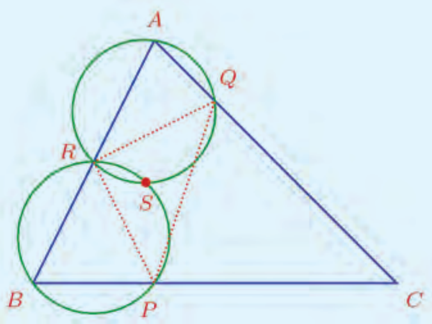
(7) In the picture, the bisectors of the angles of the quadrilateral *ABCD* intersect at *P*, *Q*, *R*, *S*.

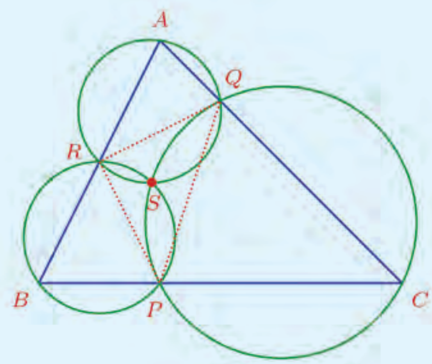


Prove that *PQRS* is a cyclic quadrilateral.

(Hint : Look at the sum of the angles of triangles *PCD* and *RAB*).

(8) In the first picture below, points *P*, *Q*, *R* are marked on the sides *BC*, *CA*, *AB* of triangle *ABC* and circumcircles of triangles *AQR* and *BRP* are drawn. They intersect at the point *S* inside the triangle :





Prove that the circumcircle of triangle *CPQ* also passes through *S*, as in the second picture.

(Hint : In the first figure, join *PS*, *QS* and *RS*. Then find the relations of the angles formed at the point *S* with $\angle A$, $\angle B$ and $\angle C$).