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Pairs of Equations

Mental math and algebra

Let's start with a problem:

There are 100 beads, black and white, in a box; 10 more black than white. How many black and how many white ?

This can be thought out in different ways.

If we keep the extra 10 black beads apart for the time being, there are 90 in the box, with black and white equal; which means 45 each. Now putting back the 10 black removed, there are 55 black and 45 white.

We can use a little bit of algebra (Remember the lesson, **Equations** in Class 8?)

Taking the number of black beads as x, the number of white is $x - 10$. Since there are 100 altogether,

$$x + (x - 10) = 100$$

We can extract x from this:

$$2x - 10 = 100$$
$$2x = 110$$
$$x = 55$$

Thus we get the number of black beads as 55; and subtracting 10 from this, we can get the number of white as 45.

Let's look at another one:

The price of a table and chair is 11000 rupees. The price of the table and four chairs is 14000 rupees. What's the price of each ?

First see if you can work this out in your head. The price of a table and four chairs is 3000 rupees more, because of three extra chairs, right? So, this extra 3000 is the price of 3 chairs; which means the price of a chair is 1000 rupees. And this gives the price of the table as 10000 rupees.

Instead of thinking like this, we can just start by taking the price of a chair as x rupees. A little thinking shows, the price of the table is then $11000 - x$ rupees. So the price of the table and four chairs is

$$(11000 - x) + 4x = 11000 + 3x$$

This is given to be 14000 rupees. That is,

$$11000 + 3x = 14000$$

And we can get x from this:

$$11000 + 3x = 14000$$
$$3x = 3000$$
$$x = 1000$$

Thus we get the price of a chair as 1000 rupees and then the price of the table as $11000 - 1000 = 10000$ rupees.

There's another way. Let's take the price of the chair as x and the price of the table as y. Then what is given in the problem can be written as two equations:

$$x + y = 11000$$
$$4x + y = 14000$$

Now the relation between the numbers x and y given in the first equation, we can write like this:

$$y = 11000 - x$$

So, in the second equation, we can use $11000 - x$ in the place of y:

$$4x + (11000 - x) = 14000$$

That is,

$$11000 + 3x = 14000$$

This is the same old equation we got by taking only the price of a chair as x rupees, isn't it? So, we can compute the prices as before

Another problem:

> Of two numbers, the larger one is 5 times the smaller one; and subtracting the smaller from the larger gives 32. What are the numbers?

Can you do it in your head?

One number is five times the other; so what can we say instead of subtracting the smaller from the larger?

Subtracting the smaller number from five times itself, right?

Which means four times the number

This is 32, we are told. Thus 4 times the smaller number is 32, and so the number itself is 8.

Now we can compute the larger number as 5 times 8, which is 40.

How about using algebra?

Let's start by taking the smaller number as x, so that the larger is $5x$.

Subtracting the smaller from the larger, $5x - x = 4x$

Since the difference is 32, we have $4x = 32$

This gives $x = 8$

So, the smaller number is 8 and the larger, $5 \times 8 = 40$

What if we start by taking the smaller number as x and the larger y ?

We can write the information given in the problem as two equations:

$$y = 5x$$

$$y - x = 32$$

In the second equation, we can use $5x$ in the place of y, right?

$$5x - x = 32$$

That is,

$$4x = 32$$

From this we get $x = 8$ and then $y = 5 \times 8 = 40$.

One more problem:

A fraction simplified after adding 1 to its numerator becomes $\frac{1}{2}$. If instead, 1 is added to the denominator and then simplified, it becomes $\frac{1}{3}$. What is the fraction ?

Can you do it in your head?

Even after taking only the numerator or denominator as x, we can't get far. Let's denote the numerator as x and denominator as y, so that the fraction is $\frac{x}{y}$.

Now let's write each bit of information given, as an equation:

$$\frac{x+1}{y} = \frac{1}{2}$$

$$\frac{x}{y+1} = \frac{1}{3}$$

What does the first equation tell us ?

The fraction $\frac{x+1}{y}$ is another form of $\frac{1}{2}$

In all the various forms of $\frac{1}{2}$ the denominator is twice the numerator, isn't it?

So,

$$y = 2(x + 1)$$

What about the second equation, $\frac{x}{y+1} = \frac{1}{3}$?

As before, this gives,

$$y + 1 = 3x$$

The first equation says, the numbers y and $2(x + 1)$ are equal. So, we can write $2(x + 1)$ instead of y in the second equation.

This gives

$$2(x + 1) + 1 = 3x$$

That is,

$$2x + 2 + 1 = 3x$$

$$2x + 3 = 3x$$

From this, we get $x = 3$. Then from the first equation, we get $y = 2 \times (3 + 1) = 2 \times 4 = 8$

Thus the fraction in question is $\frac{3}{8}$

Now try these problems in any way you like: as mental math, as an equation with a single letter or as a pair of equations with two letters:

(1) Priya bought a bag and a pair of slippers for 1100 rupees. The bag costs 300 rupees more than the slippers. What is the price of the slippers? And the price of the bag?

(2) The sum of two numbers is 26 and their difference is 4. What are the numbers ?

(3) The perimeter of a rectangle is 40 centimetres, and one side is 8 centimetres longer than the other. Calculate the lengths of the sides.

(4) A wire three and a half metres long is to be cut into two pieces, with one piece bent into a square and the other into an equilateral triangle. The lengths of the sides of both must be the same. How should the wire be cut ?

(5) In a class, there are 4 more girls than boys. On a day when only 8 boys were absent, the number of girls was twice the number of boys. How many boys and girls are there in the class ?

(6) A fraction simplified after adding 1 to its numerator becomes $\frac{1}{3}$. If instead, 1 is added to the denominator and then simplified, it becomes $\frac{1}{4}$. What is the fraction?

(7) A person invested 100000 rupees in two schemes, with interest rates 7% and 6%. After one year, they got 6750 rupees as interest from both these together. How much did they invest in each scheme?

(8) An object starts with a speed of u m/s and travels along a straight line. If the speed increases at the rate of a m/s every second, the speed at time t seconds is $u + at$. The speed at one second is 5 m/s and at five seconds, 13 m/s. What is the rate at which speed is increasing ? What was the starting speed?

Two equations

See this problem:

The price of 2 pens and 3 notebooks is 110 rupees. The price of 2 pens and 5 notebooks is 170 rupees. What is the price of a pen? And a notebook?

Think as we did in the table and chair problem. How did the cost increase from 110 rupees to 170 rupees ? Because of two more notebooks, right?

So, the 60 rupees increase is the price of 2 extra notebooks; and this means the price of a notebook is 30 rupees. Now to get the price of two pens from the first statement, we need only subtract the price of three note books from 110

This gives the price of two pens as 110 – 90 = 20 rupees and the price of one pen as 10 rupees.

Information and solutions

If we are only told that a box contains ten beads, black and white, we cannot definitely say how many of each. It can be just one black and nine white, two black and eight white and so on. If we are also told that there are two more black than white, we can fix the numbers as six black and four white.

In the language of algebra, there are many pairs of numbers x and y satisfying the single equation $x + y = 10$. But there is only one pair satisfying both the equations

$$x + y = 10$$
$$x - y = 2$$

and these are $x = 6, y = 4$

Now let's see how we do this by taking the price of a pen as x rupees, the price of a notebook as y rupees and converting the pieces of information given into algebraic equations:

Price of 2 pens and 3 notebooks is 110 rupees	$2x + 3y = 110$
Price of 2 pens and 5 notebooks is 170 rupees	$2x + 5y = 170$
Increase is the price of 2 notebooks	$(2x + 5y) - (2x + 3y) = 2y$
Increase is 60 rupees	$170 - 110 = 60$
Price of two notebooks is 60 rupees	$2y = 60$
Price of a notebook is 30 rupees	$y = 30$
Price of 2 pens is 90 rupees subtracted from 110 rupees	$2x = 110 - (3 \times 30) = 20$
Price of a pen is 10 rupees	$x = 10$

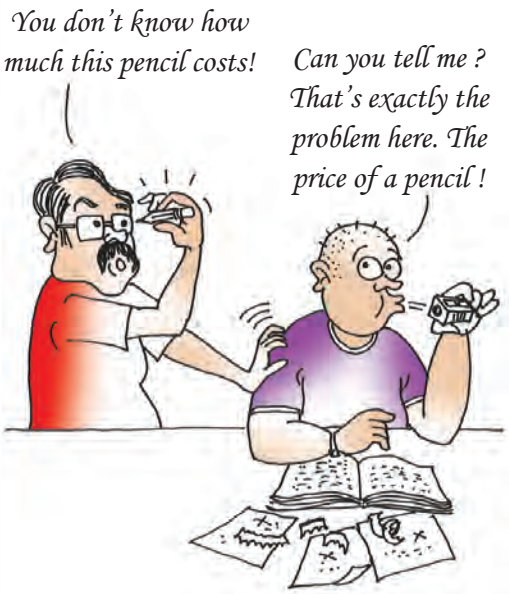
Let's do a slightly different problem:

The price of 3 pencils and 4 pens is 66 rupees; and the price of 6 pencils and 3 pens is 72 rupees. What is the price of each ?

Let's first see if this can be done in head. Here the increase in price is not due to the increase in the number of just one thing, as in the first problem. So, things are not that easy.

If the number of pencils or pens was the same in both occasions, we could've done it as before. How about making it the same ?

Let's write the prices like this:



Pencil	Pen	Price
3	4	66
6	3	72

The number of pencils is 3 in the first line and 6 in the second. Can we make it 6 in the first line also? How about 6 pencils and 8 pens?

	Pencil	Pen	Price
	3	4	66
×2	6	3	72
	6	8	132

From the second line to the third, how much did the price increase ? Why?

The 60 rupees increase is the price of 5 pens only, right ?

So, the price of a pen is 12 rupees.

Now from the first line, we can compute the price of 3 pencils as 66 – 48 = 18 and then the price of a pencil as 6 rupees

Now let's put this all in algebra. Taking the price of a pencil as x rupees, that of a pen as y rupees, we can write the information given in the problem and the method of solution like this:

Price of 3 pencils and 4 pens is 66 rupees.

$$3x + 4y = 66$$

Price of 6 pencils and 3 pens is 72 rupees

$$6x + 3y = 72$$

Price of 6 pencils and 8 pens is 132 rupees

$$6x + 8y = 2(3x + 4y) = 132$$

Increase is the price of 5 pens

$$(6x + 8y) - (6x + 3y) = 5y$$

Increase is 60 rupees

$$132 - 72 = 60$$

Price of 5 pens is 60 rupees

$$5y = 60$$

Price of a pen is 12 rupees

$$y = 12$$

Price of 3 pencils is price of 4 pens subtracted from 66 rupees

$$3x = 66 - (4 \times 12) = 18$$

Price of a pencil is 6 rupees

$$x = 6$$

We can write all this in a compressed form. Let's write the pieces of information given in the problem as equations and number them:

$$3x + 4y = 66 \qquad (1)$$

$$6x + 3y = 72 \qquad (2)$$

Nothing new

Ramu bought a pencil and a pen for 7 rupees. Aju bought 4 pencils and 4 pens for 28 rupees. They tried to work out the price of each using these pieces of information.

Taking the price of a pencil as x rupees. they wrote the price of a pen as $7 - x$, using Ramu's purchase. Then made an equation of Aju's purchase:

$$4x + 4(7 - x) = 28$$

And what did they get on simplification?

$$28 = 28$$

What if they had taken the price of a pencil as x rupees and the price of a pen as y rupees?

$$x + y = 7$$

$$4x + 4y = 28$$

If we write the second equation as

$$4(x + y) = 28$$

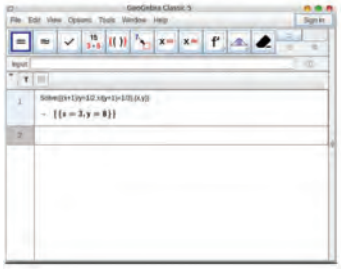
we again get

$$x + y = 7$$

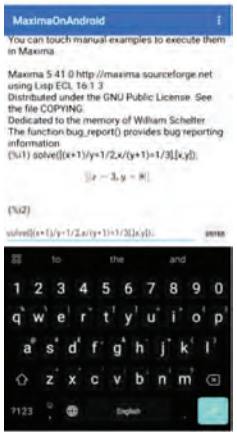
Thus in this problem, there's actually just one piece of information connecting the prices, though they're stated somewhat differently. And we can't compute the prices using this alone.

Computer algebra

Computers can be used not only to do complicated numerical computations, but also to solve algebraic problems. Such software are collectively called Computer Algebra System (CAS) or Symbolic Algebra System. Two prominent free software in this category are Sage Math and Maxima. GeoGebra also has some CAS capabilities:



GeoGebra and Maxima are available in Android phones also:



We can use the CAS in GeoGebra to solve a pair of equations. For example, to solve the equations, $5x + 2y = 20$, $2x + 3y = 19$ open the CAS window by

View → CAS and type

Solve ({$5x + 2y = 20$, $2x + 3y = 19$}, {x, y})

Equation (1) says the number $3x + 4y$ is 66. So, twice that is 132:

$$6x + 8y = 132 \quad (3)$$

Now we can use Equations (2) and (3) to write

$$(6x + 8y) - (6x + 3y) = 132 - 72$$

Simplified, this gives

$$5y = 60$$

and from this, we get $y = 12$

Now putting $y = 12$ in Equation (1), we can compute x:

$$3x + (4 \times 12) = 66$$

$$3x = 66 - 48 = 18$$

$$x = 6$$

Now try these:

(1) The price of 4 pens and 3 pencils is 66 rupees. The price of 7 pens and 3 pencils is 111 rupees. What is the price of a pen? The price of a pencil ?

(2) The perimeter of a rectangle is 26 centimetres. Another rectangle with twice the length and thrice the breadth has perimeter 62 centimetres. What are the length and breadth of the first rectangle ?

Let's do another problem:

When a small vessel was filled and emptied five times and a big one two times into a bucket, it contains 20 litres. Instead when this was done twice with the small vessel and thrice with the big, it contained only 19 litres. How much can each vessel hold ?

Let's take the capacity of the small vessel as x litres and the capacity of the big one as y litres, to convert the given pieces of information into equations:

$$5x + 2y = 20 \qquad (1)$$
$$2x + 3y = 19 \qquad (2)$$

As in the last problem, to change $5x$ to $2x$ in Equation (1), we'll have to multiply by $\frac{2}{5}$; on the other hand, to change $2x$ to $5x$ in Equation (2), we'll have to multiply by $\frac{5}{2}$. We can find the solutions either way.

Is there a way to avoid fractions?

We can multiply the equations by any numbers. How about multiplying the first equation by one number and multiplying the second by a different number to make the multiplier of x the same in both?

$$(1) \times 2 : 10x + 4y = 40 \qquad (3)$$
$$(2) \times 5 : 10x + 15y = 95 \qquad (4)$$

Now we can subtract (3) from (4) to get

$$11y = 55$$

which gives

$$y = 5$$

And then we can use this in (1) to find x:

$$5x + 10 = 20$$
$$5x = 10$$
$$x = 2$$

Thus we find that the small vessel can hold 2 litres and the big vessel 5litres.

Only numbers

Remember how we did the pencil-pen problem by writing down just the numbers without using letters? Other problems can also be done like this.

For example, we can start doing the vessel problem by first writing like this:

Small	Large	Water
5	2	20
2	3	19

We write next steps like this:

	5	2	20
×2	2	3	19
	10	4	40
	10	15	95

×5

Now we look at only the last two lines and continue like this:

	10	4	40
−	10	15	95
	0	11	55
÷ 11	0	1	5

Thus we get the capacity of the large vessel as 5 litres. Now the capacity of the small vessel can be calculated in the head to be 2 litres. Long before algebra as we now use was developed in Europe, ancient Chinese mathematicians did problems of this kind in this manner.

(1) The price of two kilograms of orange and three kilograms of apple is 520 rupees. The price of three kilograms of orange and two kilograms of apple is 480 rupees. What is the price of each ?

(2) A wire one metre long is cut into two pieces, one of which is bent into a square and the other into an equilateral triangle. Three times the side of the square and two times the side of the equilateral triangle makes 71 centimetres. What are the lengths of the pieces ?

(3) Four years ago, Rahim's age was three times the age of Ramu. After two years, this would become two times. What are their ages now ?

Let's look at another problem:

Four times a number added to three times another number gives 43. Two times the second number subtracted from three times the first gives 11. What are the numbers ?

We can start with taking the first number as x and the second number as y.

Writing the given information as equations, we get

$$4x + 3y = 43 \quad (1)$$
$$3x - 2y = 11 \quad (2)$$

As before we multiply Equation (1) by 3 and Equation (2) by 4 to make the multipliers of x the same:

$$(1) \times 3 : \ 12x + 9y = 129 \quad (3)$$
$$(2) \times 4 : 12x - 8y = 44 \quad (4)$$

Now subtracting (4) from (3) and simplifying, we get

$$12x + 9y - (12x - 8y) = 129 - 44$$
$$12x + 9y - 12x + 8y = 85$$
$$17y = 85$$
$$y = 5$$

And using $y = 5$ in (1), we can find x:

$$4x + 15 = 43$$
$$4x = 28$$
$$x = 7$$

Thus we see that the first number is 7 and the second, 5

One more problem:

The sum of two numbers is 28 and their difference is 12. What are the numbers ?

Remember doing such a problem before ?

Let's take the larger number as x and smaller number as y and write out the given information as equations:

Sum of the numbers $x + y = 28$

Difference $x - y = 12$

Adding up these equations, we get,

$$(x + y) + (x - y) = 28 + 12$$

That is, $2x = 40$

$$x = 20$$

What about their difference ?

$$(x + y) - (x - y) = 28 - 12$$

$$x + y - x + y = 16$$

That is, $2y = 16$

$$y = 8$$

Thus the larger number is 20 and the smaller one is 8.

We also get a general principle from this problem:

The sum and difference of two numbers added together gives twice the larger number:

$$(x + y) + (x - y) = 2x$$

When the difference is subtracted from the sum, we get twice the smaller number:

$$(x + y) - (x - y) = 2y$$

Math and reality

A rectangle of perimeter 10 metres is to be constructed. The length should be 5.5 metres more than the breadth. What should be the length and breadth?

If we take the breadth as x metres the length should be $x + 5.5$ metres. Since the perimeter is to be 10 metres, we should have

$$x + (x + 5.5) = \frac{10}{2} = 5$$

That is,

$$2x + 5.5 = 5$$

which gives

$$2x = -0.5$$

This is not right. How can the length of the side of a rectangle be a negative number ?

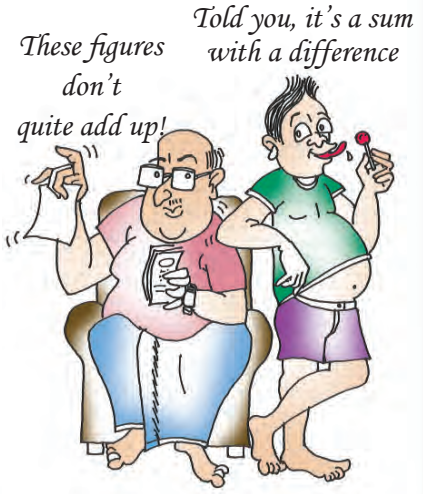
This just means that no rectangle can be made satisfying these conditions

Had we taken length as x and breadth as y, we would have got

$$x + y = 5$$

$$y - x = 5.5$$

And we would have immediately seen that there are no positive numbers satisfying these conditions. (The sum of two positive numbers cannot be less than their difference, right?)



(1) The difference of the two smaller angles of a right triangle is 20°. Calculate all three angles.

(2) When a larger number is divided by a smaller number, the quotient and remainder are both 2. When 5 times the smaller is divided by the larger, the quotient and remainder are still both 2. What are the numbers ?

(3) The sum of the digits of a two digit number is 11. The number got by interchanging the digits is 27 more than the original number. What is the number ?

(4) The price of 17 trophies and 16 medals is 2180 rupees. The price of 16 trophies and 17 medals is 2110 rupees. What is the price of each ?

(5) An object starts with a speed of u m/s and travels along a straight line. If the speed increases at the rate of a m/s every second, the distance travelled in time t seconds is $ut + \frac{1}{2}at^2$. The distance travelled in 2 seconds is 10 metres and the distance travelled in 4 seconds is 28 metres. What was the starting speed ? What is the rate at which speed is increasing ?

(6) A two digit number is equal to 6 times the sum of its digits. The number got by interchanging the digits is 9 more than 4 times the sum of the digits. What is the number ?

(7) 11 added to a number gives twice another number. 20 added to the second number gives twice the first number. What are the numbers ?