XI PHYSICS - CHAPTER 7

SYSTEMS OF PARTICLES AND ROTATIONAL MOTION
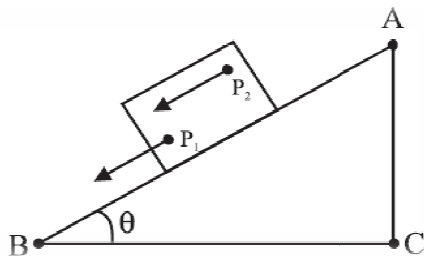
(Prepared By Ayyappan C. HSST Physics, GHSS Udma, Kasaragod, Mob: 9961985448)

RIGID BODY

- A rigid body is a body with a perfectly definite and unchanging shape.
- The distances between different pairs of such a body do not change
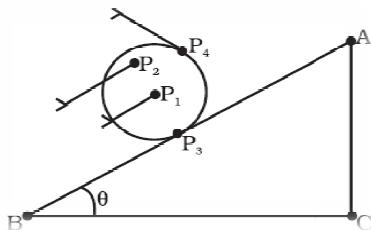
MOTIONS OF A RIGID BODY

PURE TRANSLATION

- In pure translational motion **at any instant of time every particle of the body** has the **same velocity**.

TRANSLATION AND ROTATION

- Consider the rolling motion of a solid metallic or wooden cylinder down the inclined plane .

- *Points P1, P2, P3 and P4 have different velocities (shown by arrows) at any instant of time.*
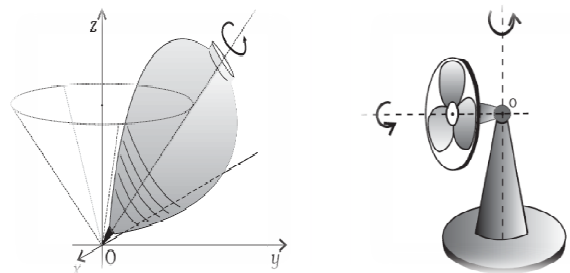
- The line along which the body is fixed is termed as its axis of rotation.

ROTATION ABOUT A FIXED AXIS

- In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.
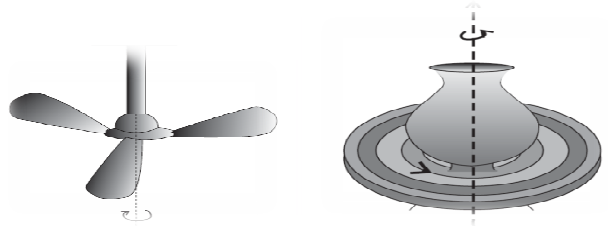
## ROTATION ABOUT AN AXIS NOT FIXED

- The axis of a spinning top moves around the vertical through its point of contact with the ground, sweeping out a cone.

- The movement of the axis of the top around the vertical is termed **precession.**

- The point of contact of the top with ground is fixed.

- The axis of rotation of the top at any instant passes through the point of contact.

- **The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation.**
- **The motion of a rigid body which is pivoted or fixed in some way is rotation.**

CENTRE OF MASS

- **Centre of mass is the point at which the entire mass of the body can be assumed to be concentrated.**
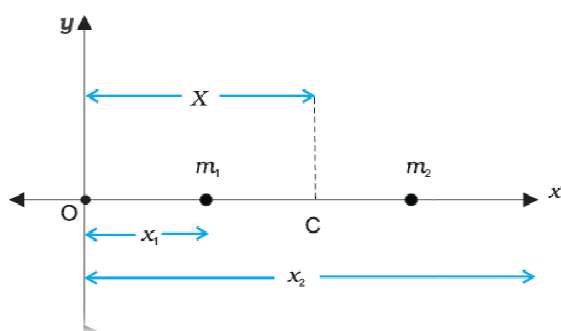
- For two particle system as shown in fig. the centre of mass is given by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

- If the two particles have the same mass *,* *then*

$$X = \frac{m x_1 + m x_2}{2m} = \frac{x_1 + x_2}{2}$$

- If we have *n particle*

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum m_i x_i}{\sum m_i}$$

- The centre of mass C of the system of the three particles is defined and located by the coordinates (*X, Y)* given by

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

- For the particles of equal mass *m*

$$X = \frac{m(x_1 + x_2 + x_3)}{3m} = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{m(y_1 + y_2 + y_3)}{3m} = \frac{y_1 + y_2 + y_3}{3}$$

- Thus, for three particles of equal mass, the centre of mass coincides with the **centroid of the triangle** formed by the particles.

- In general the centre of mass of a system distributed in space is at (*X, Y, Z*).

$$X = \frac{\sum m_i x_i}{M}$$

$$Y = \frac{\sum m_i y_i}{M}$$

$$\text{and} \quad Z = \frac{\sum m_i z_i}{M}$$

- In terms of position vectors

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M}$$

- The centre of mass of **homogeneous bodies of regular shapes like rings, discs, spheres, rods** lie at their **geometric centers**.

MOTION OF CENTRE OF MASS

- From Newton's second law,

$$M\mathbf{A} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$$

- Thus the **total mass of a system of particles times the acceleration of its centre of mass is the vector sum of all the forces acting on the system of particles.**

- Only the external forces contribute to the equation

$$M\mathbf{A} = \mathbf{F}_{ext}$$

- **The centre of mass of a system of particles moves as if all the mass of the system was**

concentrated at the centre of mass and all the external forces were applied at that point.

Illustration- projectile explodes into fragments

- Consider a projectile, following the usual parabolic trajectory, explodes into fragments midway in air.

- The forces leading to the explosion are internal forces. They contribute nothing to the motion of the centre of mass.

- The total external force, namely, the force of gravity acting on the body, is the same before and after the explosion.

- The centre of mass under the influence of the external force continues, therefore, along the same parabolic trajectory as it would have followed if there were no explosion.
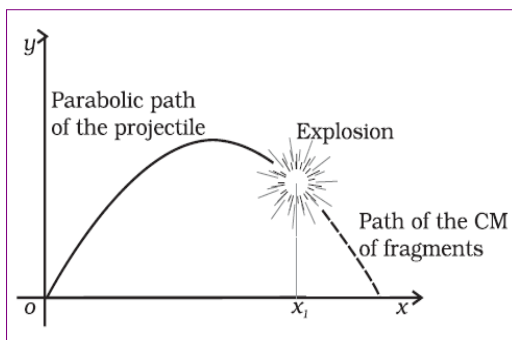
LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

- The linear momentum of a particle is defined as **p= *mv.***

- Newton's second law form for a single particle is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

- For the system of *n particles, the linear* momentum of the system is defined to be the vector sum of all individual particles of the system.
- Therefore

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n$$
$$= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n$$

- Thus, the total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass*.*

$$\mathbf{P} = M\mathbf{V}$$

- Differentiating above equation with respect to time,

$$\frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{V}}{dt} = M\mathbf{A}$$

- Or

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{ext}$$

- This is the statement of Newton's second law extended to a system of particles.

- When the total external force acting on a system of particles is zero, the total linear momentum of the system is constant.

$$\frac{d\mathbf{P}}{dt} = 0 \quad \text{or} \quad \mathbf{P} = \text{Constant}$$

- This is the law of conservation of the total linear momentum of a system of particles.

- This also means that **when the total external force on the system is zero the velocity of the centre of mass remains constant.**

**ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY**

- The average angular velocity of the particle over the interval Δt is **Δθ /Δt.**

- The instantaneous angular velocity

ω = dθ/dt.

- The magnitude of linear velocity *v of a particle moving in a circle is* related to the angular velocity of the particle ω by the simple relation

$$\upsilon = \omega r$$

- At any given instant this relation *applies to all particles of the* rigid body.
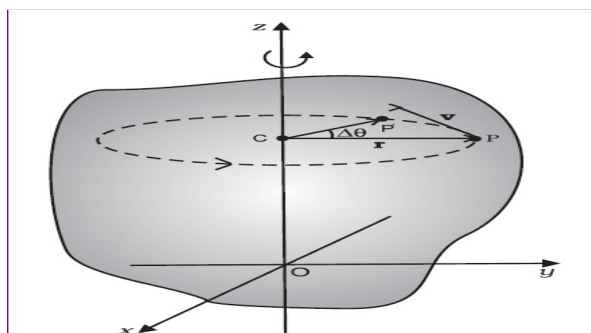
- The angular velocity is a vector quantity.

- For rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction in which a right handed screw would advance, if the head of the screw is rotated with the body.
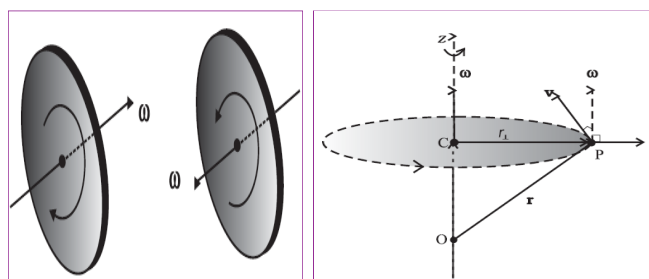
- In general $\mathbf{v} = \omega \times \mathbf{r}$
- For rotation about a fixed axis, the direction of the vector ω does not change with time. Its magnitude may change from instant to instant.
- For the more general rotation, both the magnitude and the direction of ω may change from instant to instant.

Angular acceleration

- Angular acceleration α is the time rate of change of angular velocity.
- Thus,

$$\alpha = \frac{d\omega}{dt}$$

- If the axis of rotation is fixed, the direction of ω and hence, that of α is fixed.

Moment of force (Torque)

- The **rotational analogue of force** is moment of force or torque.
- If a force acts on a single particle at a point P whose position with respect to the origin O is given by the position vector r , the moment of the force acting on the particle with respect to the origin O is defined as the vector product

$$\tau = r \times F$$

- The moment of force (or torque) is a vector quantity.
- The symbol τ stands for the Greek letter tau.
- The magnitude of τ is

$$\tau = rF\ sin\theta$$

- Moment of force has dimensions same as those of work or energy.

$$M\ L^2\ T^{-2}$$

- Moment of a force is a vector, while work is a scalar.
- The SI unit of moment of force is **Newton-metre** (Nm).

Angular momentum of a particle

- Angular momentum is the **rotational analogue of linear momentum**.
- The angular momentum l of the particle with respect to the origin O is defined to be

$$l = r \times p$$

- The magnitude of the angular momentum vector is

$$l = r\ p\ sin\theta$$

Relation Between Angular Momentum and Torque

- Differentiating **l = r × p** with respect to time,

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$

$$\frac{d}{dt}(r \times p) = (r \times \frac{dp}{dt}) + (\frac{dr}{dt} \times p)$$

$$\frac{d}{dt}(r \times p) = (r \times F) + (v \times (mv))$$

Here F= (dp/dt) and p= mv

$$\frac{d}{dt}(r \times p) = (r \times F) + (0)$$, since (**v x v**) = 0

- **Thus**

$$\frac{dl}{dt} = (r \times F)$$

- Therefore

$$\frac{d\mathbf{l}}{dt} = \boldsymbol{\tau}$$

- Thus, the time rate of change of the angular momentum of a particle is equal to the torque acting on it.

Torque and angular momentum for a system of particles

- For a system of *n particles,*

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \ldots + \mathbf{l}_n = \sum_{i=1}^{n} \mathbf{l}_i$$

- *Or*

$$\mathbf{L} = \sum_i \mathbf{l}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_{ext}$$

Conservation of Angular Momentum

- If the external torque acting on a system is zero, then

$$\frac{d\mathbf{L}}{dt} = 0$$
$$\mathbf{L} = \text{constant.}$$

- Thus **if total external torque on a system is zero the angular momentum is conserved**.

EQUILIBRIUM OF A RIGID BODY

- A rigid body is said to be in mechanical equilibrium, if both its linear momentum and angular momentum are not changing with time.

Translational equilibrium

- **The total force, i.e. the vector sum of the forces, on the rigid body is zero; ie linear momentum is conserved.**

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0}$$
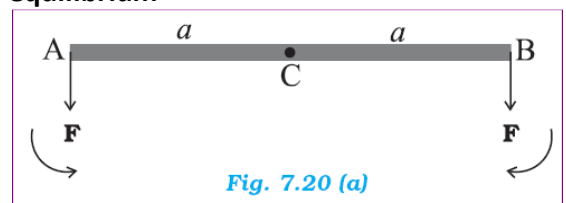
Rotational equilibrium

- **The total torque, i.e. the vector sum of the torques on the rigid body is zero, ie, angular momentum is conserved.**

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \ldots + \boldsymbol{\tau}_n = \sum_{i=1}^{n} \boldsymbol{\tau}_i = \mathbf{0}$$

Partial equilibrium

- A body may be in partial equilibrium, i.e., it may be in translational equilibrium and not in rotational equilibrium, or it may be in rotational equilibrium and not in translational equilibrium.
- When two parallel forces both equal in magnitude are applied perpendicular to alight rod , **the system will be in rotational equilibrium, and not in translational equilibrium**


Fig. 7.20 (a)

- When two forces are applied perpendicular in two opposite directions, the body is in translational equilibrium; but not in rotational equilibrium.
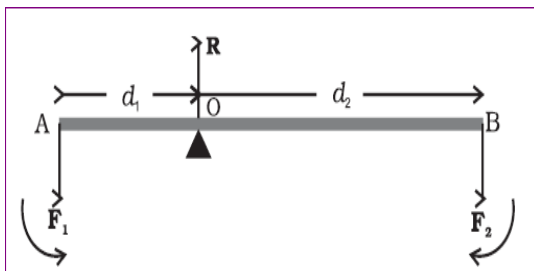

Fig. 7.20 (b)
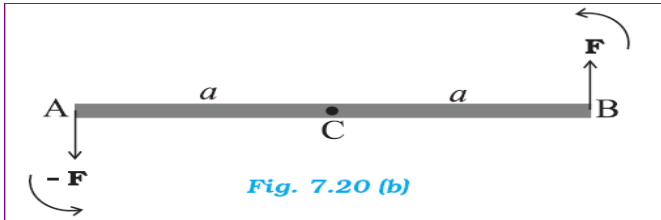
Couple

- A pair of equal and opposite forces with different lines of action is known as a **couple.**

- **A couple produces rotation without translation**.

- When we open the lid of a bottle by turning it, our fingers are applying a **couple** to the lid.
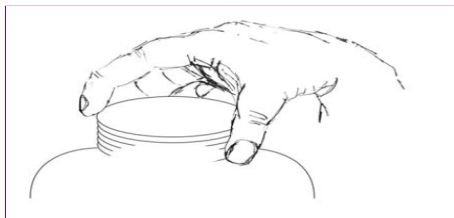
Principle of moments

- For a lever at equilibrium the moment on the left = moment on the right

load arm × load = effort arm × effort

$$d_1 F_1 = d_2 F_2$$

- The above equation expresses the **_principle of moments for a lever._**

- A l lever is a light rod pivoted at a point along its length. This point is called the **fulcrum.**

- A seesaw on the children's playground is a typical example of a lever.

- Normally the a**nticlockwise (clockwise) moments are taken to be positive (negative).**

- In the case of the lever force *F1 is usually* some weight to be lifted. It is called the *load* and its distance from the fulcrum *d1 is* called the *load arm.*

- Force *F2 is the **effort*** applied to lift the load; distance *d2 of the effort from the* fulcrum is the *effort arm.*

- The ratio **F1/F2** *is called **the Mechanical Advantage (M.A.);***

$$M.A. = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

- If the effort arm *d2 is larger than the load* arm, the mechanical advantage is greater than one.

- Mechanical advantage greater than one means that a small effort can be used to lift a large load.

Centre of gravity

- The CG of a body is the point where the **total gravitational torque on the body is zero.**
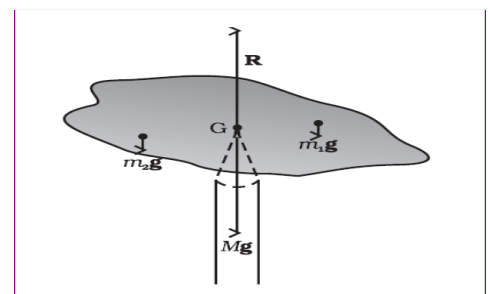
MOMENT OF INERTIA

- Moment of inertia is the rotational analogue of mass of a body.

- The moment of inertia *I* given by

$$I = \sum_{i=1}^{n} m_i r_i^2$$

- It is independent of the magnitude of the angular velocity.

The moment of inertia of a rigid body depends on :

- the mass of the body,

- its shape and size

- distribution of mass about the axis of rotation,

- The position and orientation of the axis of rotation.

Rotational kinetic energy

- The kinetic energy in terms of moment of inertia is

- We have kinetic energy

$$K = \frac{1}{2} m v^2$$

- The rotational motion the velocity is given by

$$v = r\omega$$

- Thus

$$K = \frac{1}{2} m r^2 \omega^2$$

- Therefore

$$K = \frac{1}{2} I \omega^2$$

where ω is the angular velocity.

Moment of Inertia of a thin Ring

- Consider a thin ring of radius *R and mass M, rotating in its own plane around its* centre with angular velocity ω.

- Each mass element of the ring is at a distance R from the axis, and moves with a speed *Rω.*

- The kinetic energy is therefore,

$$K = \frac{1}{2} M v^2 = \frac{1}{2} M R^2 \omega^2$$
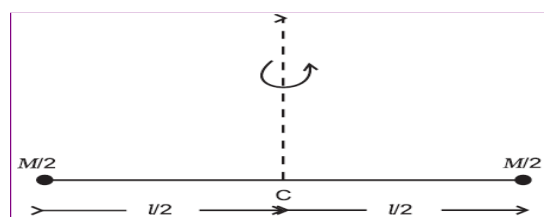
- Therefore comparing the equation with

$$K = \frac{1}{2} I \omega^2$$

- We get $$I = MR^2$$

Moment of Inertia of a rigid Rod

- Consider a rigid massless rod of length with a pair of small masses, rotating about an axis through the centre of mass perpendicular to the rod.

- Each mass *M/2 is at a distance l/2 from the axis.*

- The moment of inertia of the masses is therefore given by

$$(M/2)(l/2)^2 + (M/2)(l/2)^2$$

- Thus, for the pair of masses, rotating about the axis through the centre of mass perpendicular to the rod

$$I = Ml^2 / 4$$

- As the moment of inertia about a given axis of rotation resists a change in its rotational motion, it can be regarded as a measure of **rotational inertia of the body**.

- It is a measure of the way in which different parts of the body are distributed at different distances from the axis.

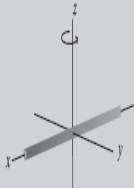## Radius of Gyration

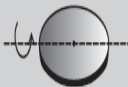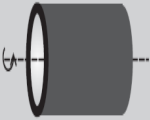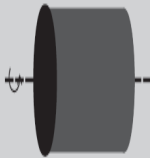- Moment of inertia can be written as

$$I = Mk^2$$

- Here the length *k is a* geometric property of the body and axis of rotation. It is called the **radius of gyration.**

- **The radius of gyration**

$$k = \sqrt{\frac{I}{M}}$$

- **The radius of gyration of a body about an axis** may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

Moment of inertia of different bodies

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius R	Perpendicular to plane, at centre		MR^2
(2)	Thin circular ring, radius R	Diameter		$MR^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$ML^2/12$

(4)	Circular disc, radius R	Perpendicular to disc at centre		$MR^2/2$
(5)	Circular disc, radius R	Diameter		$MR^2/4$
(6)	Hollow cylinder, radius R	Axis of cylinder		MR^2
(7)	Solid cylinder, radius R	Axis of cylinder		$MR^2/2$
(8)	Solid sphere, radius R	Diameter		$2MR^2/5$

Practical uses of moment of inertia

- The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a **flywheel.**

- Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle.

- It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

Theorem of Perpendicular Axes

- **It states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.**
- **Thus**

$$I_z = I_x + I_y$$

- This theorem is applicable to bodies which are planar.

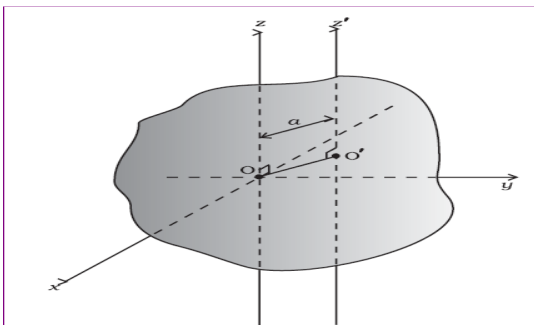- The figure shows a planar body
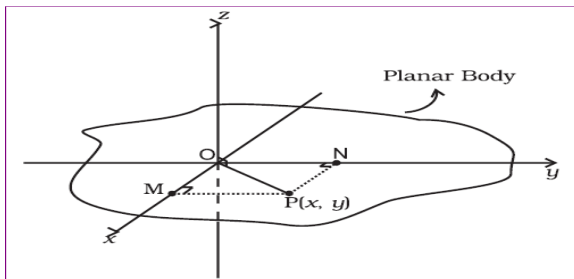
Theorem of parallel axes

- **The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.**

$$I_{z'} = I_z + Ma^2$$

- This theorem is applicable to a body of any shape.

KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

- The kinematical equations of linear motion with uniform (i.e. constant) acceleration are:

$$v = v_0 + at$$
$$x = x_0 + v_0 t + \frac{1}{2} at^2$$
$$v^2 = v_0^2 + 2ax$$

- The kinematic equations for *rotational motion* with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\text{and } \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
3	Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\alpha$
6	Work $dW = F\,ds$	Work $W = \tau\,d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = Fv$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Work done by a torque
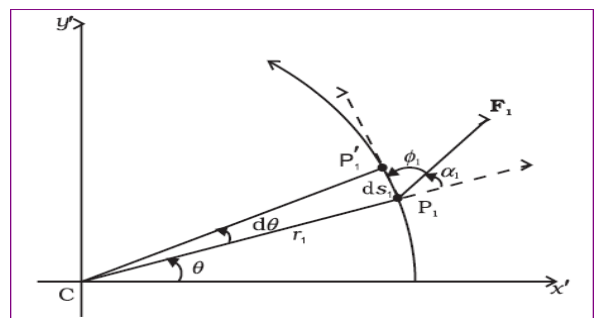
- The work done by the total (external) torque τ which acts on the body rotating about a fixed axis is given by

$$dW = \tau d\theta$$

- The instantaneous power is given by

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$
$$\text{or } P = \tau\omega$$

- Also the torque is
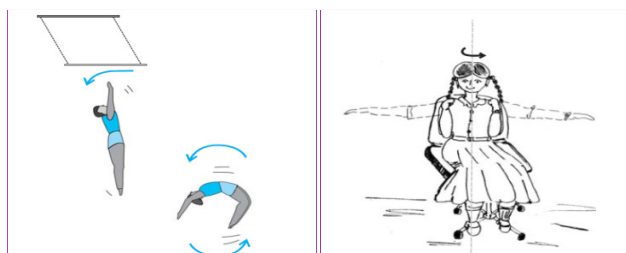
$$\tau = I\alpha$$

Conservation of angular momentum

- The angular momentum is given by

L = I ω

- If the external torque is zero,

$$L_z = I\omega = \text{constant}$$

- The below given diagrams are examples in which conservation of angular momentum takes place ,

- When we stretch hand angular speed is reduced (moment of inertia is increased) and when hand is closed angular speed is increased (moment inertia is decreased)
- A circus acrobat and a diver take advantage of this principle.
- Also, skaters and classical, Indian or western, dancers performing a Pirouette on the toes of one foot display 'mastery' over this principle.

ROLLING MOTION

- Rolling motion is the combination of translation and rotation.

- All wheels used in transportation have rolling motion.
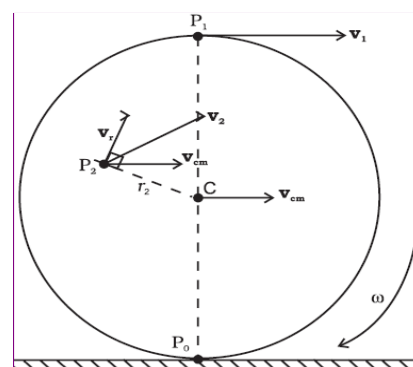
- For the disc the condition for rolling without slipping is

$$v_{cm} = R\omega$$

- This means that the velocity of point P1 at the top of the disc (**v1**) has a magnitude

$$v_{cm} + R\omega \text{ or } 2\ v_{cm}$$

  and is directed parallel to the level surface.

Kinetic Energy of Rolling Motion

- The kinetic energy of a rolling body can be separated into kinetic energy of translation and kinetic energy of rotation.

- Thus

$$K = K' + MV^2/2$$

- Where *K' is the* kinetic energy of rotational motion about the centre of mass of the system of particles

- Kinetic energy can be written as

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}^2$$

- Substituting for I and velocity (**I = mk^2 and ω= v$_{cm}$/R**)

$$K = \frac{1}{2}\frac{mk^2 v_{cm}^2}{R^2} + \frac{1}{2}mv_{cm}^2$$
$$\text{or } K = \frac{1}{2}mv_{cm}^2\left(1 + \frac{k^2}{R^2}\right)$$