7. PERMUTATIONS AND COMBINATIONS

Fundamental principle of counting

Multiplication principle (Fundamental Principle of Counting)

Suppose an event E can occur in m different ways and associated with each way of occurring of E, another event F can occur in n different ways, then the total number of occurrence of the two events in the given order is $m \times n$.

Addition principle

If an event E can occur in m ways and another event F can occur in n ways, and suppose that both can not occur together, then E or F can occur in $m + n$ ways

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places □ □ □ □ by the 4 letters, Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals = $5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals = $5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals = $20 + 60 + 120 + 120 = 320$.

Example 5 How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that
(i) repetition of the digits is allowed?
(ii) repetition of the digits is not allowed?

Solution

(i) There will be as many ways as there are ways of filling 3 vacant places in succession by the given five digits. In this case, repetition of digits is allowed. Therefore, the units place can be filled in by any of the given five digits. Similarly, tens and hundreds digits can be filled in by any of the given five digits. Thus, by the multiplication principle, the number of ways in which three-digit numbers can be formed from the given digits is $5 \times 5 \times 5 = 125$

(ii) In this case, repetition of digits is not allowed. Here, if units place is filled in first, then it can be filled by any of the given five digits.
Therefore, the number of ways of filling the units place of the three-digit number is 5. Then, the tens place can be filled with any of the remaining four digits and the hundreds place can be filled with any of the remaining three digits. Thus, by the multiplication principle, the number of ways in which three-digit numbers can be formed without repeating the given digits is $5 \times 4 \times 3 = 60$

Example 6

How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

Solution

There will be as many ways as there are ways of filling 3 vacant places in succession by the given six digits In this case, the units place can be filled by 2 or 4 or 6 only i.e., the units place can be filled in 3 ways.
The tens place can be filled by any of the 6 digits in 6 different ways and also the hundreds place can be filled by any of the 6 digits in 6 different ways, as the digits can be repeated. Therefore, by multiplication principle, the required number of three digit even numbers is $3 \times 6 \times 6 = 108$

Example 7

How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

Solution

There are as many codes as there are ways of filling 4 vacant places in succession by the first 10 letters of the English alphabet, keeping in mind that the repetition of letters is not allowed.
The first place can be filled in 10 different ways by any of the first 10 letters of the English alphabet following which, the second place can be filled in by any of the remaining letters in 9 different ways. The third place can be filled in by any of the remaining 8 letters in 8 different ways and the fourth place can be filled in by any of the remaining 7 letters in 7 different ways.
Therefore, by multiplication principle, the required numbers of ways in which 4 vacant places can be filled is $10 \times 9 \times 8 \times 7 = 5040$

Hence, 5040 four-letter codes can be formed using the first 10 letters of the English alphabet, if no letter is repeated.

Example 8

How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

Solution

It is given that the 5-digit telephone numbers always start with 67. Therefore, there will be as many phone numbers as there are ways of filling 3 vacant places 6,7, _, _, _ by the digits 0 – 9, keeping in mind that the digits cannot be repeated.
The units place can be filled by any of the digits from 0 – 9, except digits 6 and 7. Therefore, the units place can be filled in 8 different ways following which, the tens place can be filled in by any of the remaining 7 digits in 7 different ways, and the hundreds place can be filled in by any of the remaining 6 digits in 6 different ways.
Therefore, by multiplication principle, the required number of ways in which 5-digit telephone numbers can be constructed is $8 \times 7 \times 6 = 336$

Example 9

A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

Solution

When a coin is tossed once, the number of outcomes is 2 (Head and tail) i.e., in each throw, the number of ways of showing a different face is 2.
Thus, by multiplication principle, the required number of possible outcomes is $2 \times 2 \times 2 = 8$

Example 10

Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

Solution

Each signal requires the use of 2 flags.
There will be as many flags as there are ways of filling in 2 vacant places in succession by the given 5 flags of different colours.
The upper vacant place can be filled in 5 different ways by any one of the 5 flags following which, the lower vacant place can be filled in 4 different ways by any one of the remaining 4 different flags.
Thus, by multiplication principle, the number of different signals that can be generated is $5 \times 4 = 20$.

Combinations On many occasions we are not interested in arranging but only in selecting r objects from given n objects. A combination is a selection of some or all of a number of different objects where the order of selection is immaterial. The number of selections of r objects from the given n objects is denoted by ${}^nC_r$, and is given by ${}^n\mathbf{C}_r = \dfrac{n!}{(n-r)!r!}$

Remarks

1. Use permutations if a problem calls for the number of arrangements of objects and different orders are to be counted.
2. Use combinations if a problem calls for the number of ways of selecting objects and the order of selection is not to be counted.

Some important results

Let n and r be positive integers such that $r \le n$. Then

(i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$

Example 17 If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.

Solution We have ${}^nC_9 = {}^nC_8$

i.e., $$\frac{n!}{9!(n-9)!} = \frac{n!}{(n-8)!\,8!}$$

or $$\frac{1}{9} = \frac{1}{n-8} \quad \text{or} \quad n - 8 = 9 \quad \text{or} \quad n = 17$$

Therefore $${}^nC_{17} = {}^{17}C_{17} = 1.$$

Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = ${}^5C_3 = \dfrac{5!}{3!\,2!} = \dfrac{4\times5}{2} = 10$.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways. Therefore, the required number of committees

$$= {}^2C_1 \times {}^3C_2 = \frac{2!}{1!\,1!} \times \frac{3!}{2!\,1!} = 6.$$

Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,
(ii) four cards belong to four different suits,
(iii) are face cards,
(iv) two are red cards and two are black cards,
(v) cards are of the same colour?

Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways = $^{52}C_4 = \dfrac{52!}{4!\ 48!} = \dfrac{49\times50\times51\times52}{2\times3\times4}$

$= 270725$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are $^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are $^{13}C_4$ ways of choosing 4 clubs, $^{13}C_4$ ways of choosing 4 spades and $^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$.

$= 4\times\dfrac{13!}{4!\ 9!} = 2860$

(ii) There are13 cards in each suit.

Therefore, there are $^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, $^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, $^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, $^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in $^{12}C_4$ ways. Therefore, the required number of ways = $\dfrac{12!}{4!\ 8!} = 495$.

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2 \times \frac{26!}{4!\ 22!} = 29900.$$

Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ?

Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in ${}^7C_5$ ways. Therefore, the required

number of ways $= {}^7C_5 = \frac{7!}{5!\ 2!} = \frac{6 \times 7}{2} = 21$

(ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of

(a) 1 boy and 4 girls (b) 2 boys and 3 girls

(c) 3 boys and 2 girls (d) 4 boys and 1 girl.

1 boy and 4 girls can be selected in ${}^7C_1 \times {}^4C_4$ ways.

2 boys and 3 girls can be selected in ${}^7C_2 \times {}^4C_3$ ways.

3 boys and 2 girls can be selected in ${}^7C_3 \times {}^4C_2$ ways.

4 boys and 1 girl can be selected in ${}^7C_4 \times {}^4C_1$ ways.

Therefore, the required number of ways

$$= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$$

$$= 7 + 84 + 210 + 140 = 441$$

(iii) Since, the team has to consist of at least 3 girls, the team can consist of (a) 3 girls and 2 boys, or (b) 4 girls and 1 boy.

Note that the team cannot have all 5 girls, because, the group has only 4 girls.

3 girls and 2 boys can be selected in ${}^4C_3 \times {}^7C_2$ ways.

4 girls and 1 boy can be selected in ${}^4C_4 \times {}^7C_1$ ways.

Therefore, the required number of ways

$$= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 84 + 7 = 91$$

PYQ & EXPECTED QUESTIONS

1. If ${}^nC_8 = {}^nC_2$, find ${}^nC_2$.
2. Determine n if
   (i) ${}^{2n}C_3 : {}^nC_3 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^nC_3 = 11 : 1$
3. How many chords can be drawn through 21 points on a circle?
4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?
5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.
6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.
7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?
8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.
9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Q) A group consists of 4 girls and 7 boys. In how many ways, can a team of 5 members can be selected if the team should have at least 3 girls?

Q) How many chords can be drawn through 12 points on a circle?

Q) Write the value of ${}^7C_5$

Q) ${}^{29}C_{29}$ = ____________

Q) ${}^nC_{n-1}$ = ____________

a. n-1 b. n c.0 d.1

Q)

a) In how many ways can cricket of 11 players be selected from 15 players?

b) A bag contains 5 white, 6 red and 4 blue balls. Determine the number of ways in which 2 white, 3 red and 2 blue balls can be selected.

Q) How many words, with or without meaning, can be formed using all the letters of the word CHEMISTRY, using each letter exactly once? How many of them start with C and end with Y?