13.6 DERIVATIVES

The concept of derivative arose mainly as the result of many centuries of effort spent in drawing tangents to curves and finding the velocities of bodies in non-uniform motion. Derivatives are used widely in science, economics, medicine and computer science to calculate velocity and acceleration, to explain the behaviour of machinery and to estimate the decrease in water level as water is pumped out of a tank.

In this section the concept of limit is applied to find the derivative of a function. Derivative deals with the change in the value of the function and rate of change of the function. An intuitive idea of derivative of a function is discussed in section 13.2.

Derivative at a point

Let $y = f(x)$ be a real valued function and 'a' be any point in its domain. Let 'h' be any real number, then the derivative of 'f' at 'a' is defined by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided this limit exists.

It is denoted by $f'(a)$.

$$\therefore f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 32

Find the derivative of $f(x) = 3$ at $x = 0$ and $x = 3$ **(NCERT)**

Solution

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$\therefore f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{3-3}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{3-3}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Example 33

Find the derivative at $x = 2$ of the function $f(x) = 3x$. **(NCERT)**

Solution

We have, $f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{3(2+h)-3(2)}{h}$

$$= \lim_{h\to 0}\frac{6+3h-6}{h} = \lim_{h\to 0}\frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

$\therefore$ The derivative of the function $3x$ at $x = 2$ is 3.

Example 34

Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also prove that $f'(0) + 3f'(-1) = 0$.

Solution

$f(x) = 2x^2 + 3x - 5$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\left\{\frac{[2(0+h)^2+3(0+h)-5]-[2(0)^2+3(0)-5]}{h}\right\}$$

$$= \lim_{h\to 0}\frac{2h^2+3h-5+5}{h} = \lim_{h\to 0}(2h+3) = 3$$

Example 35

Find the derivative of $\sin x$ at $x = 0$ **(NCERT)**

Solution

Let $f(x) = \sin x$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\sin(0+h)-\sin 0}{h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Geometrical interpretation of derivative

Let $y = f(x)$ be a function and let P $(a, f(a))$ and Q $((a + h), f(a + h))$ be two points close to each other on the graph of this function.

From the figure 13.19, we consider the ΔPQR, in which $\tan\phi$ is the slope of the chord PQ.

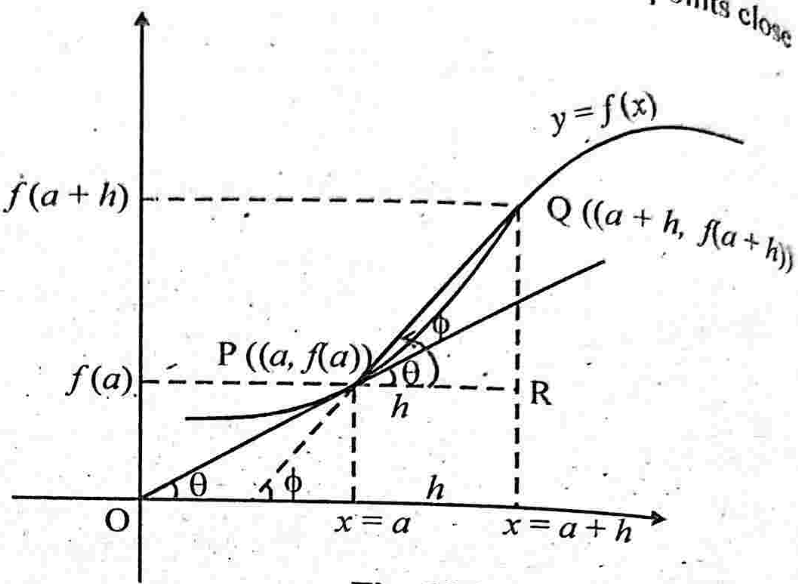

Fig. 13.19

∴ slope of the chord

$$PQ = \frac{QR}{PR} = \frac{f(a+h) - f(a)}{h}$$

In limiting process, as h tends to '0', the point Q tends to P and we write

it as $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim\limits_{Q \to P} \frac{QR}{PR}$.

This means that the chord PQ tends to the tangent at P of the curve $y = f(x)$. Thus this limit turns out to be equal to the slope of the tangent of $f(x)$ at 'a'.

We know that $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Hence $f'(a) = \tan\theta$ [Slope of the tangent to the curve at P]

NOTE

If the tangent is parallel to the x-axis, then the derivative at that point is zero.

Physical Interpretation

Suppose a particle is moving along a line so that we know its position S on that line as a function of time 't', is given by $S = f(t)$.

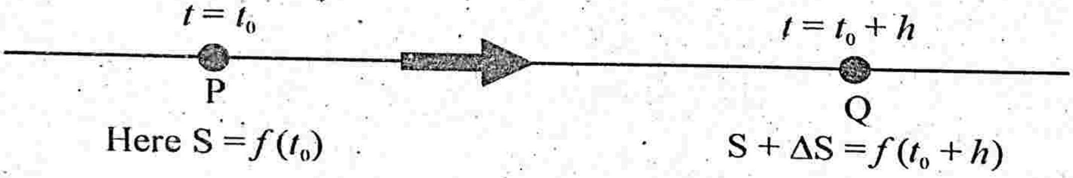

Fig 13.20

The displacement of the particle over the time interval from t_0 to $t_0 + h$ is $\Delta S = f(t_0 + h) - f(t_0)$ and the average rate of change of $f(t)$ on the interval between $t = t_0$ and $t = t_0 + h$ is equal to $\frac{f(t_0 + h) - f(t_0)}{h}$.

In limiting case, as $h \to 0$, we get its instantaneous speed (velocity) at $t = t_0$ is $f'(t_0)$ $\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}$. This limit is the derivative of f with respect to 't' at $t = t_0$.

Derivative of a function

We have discussed the process of finding the derivative of a real valued function at any point in the domain of the function. If we require to find the derivative of the function throughout the domain, then instead of calculating the derivative at any particular point, we consider an arbitrary point 'x' in the domain D of a function 'f'. Derivative of a function at any arbitrary point 'x' is denoted by $f'(x)$.

We can define $f'(x)$ as follows

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, wherever the limit exists

$f'(x)$ is also denoted by $\frac{dy}{dx}$ or $\frac{d[f(x)]}{dx}$ or D ($f(x)$) or y' or y_1.

The derivative of f at $x = a$ is also denoted by $\left.\frac{d}{dx} f(x)\right|_a$ or $\left.\frac{df}{dx}\right|_a$ or $\left(\frac{dy}{dx}\right)_{x=a}$

Differentiation from first principles

The process of finding the derivative of a function by using the definition of derivative is called **differentiation from first principles.**

The process for finding the derivatives of the differentiable functions from first principles is as follows.

WORKING RULE

1. Name the given function as $f(x)$
2. Find $f(x + h)$
3. Evaluate $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$