Illustration 5

Consider the function $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\left(\dfrac{1}{x}\right), & x > 0 \end{cases}$

The graph is given in fig. 13.7. From the graph we observe that when x approaches to zero from the left of zero, the value of $f(x)$ remains zero.

$$\lim_{x \to 0-} f(x) = 0$$

When x approaches to zero from the right of zero the graph oscillates between – 1 and 1.

i.e., the value of $f(x)$ is not approaching to a finite number. Therefore $\lim_{x \to 0+} f(x)$ does not exist.

Also $f(0) = 0$

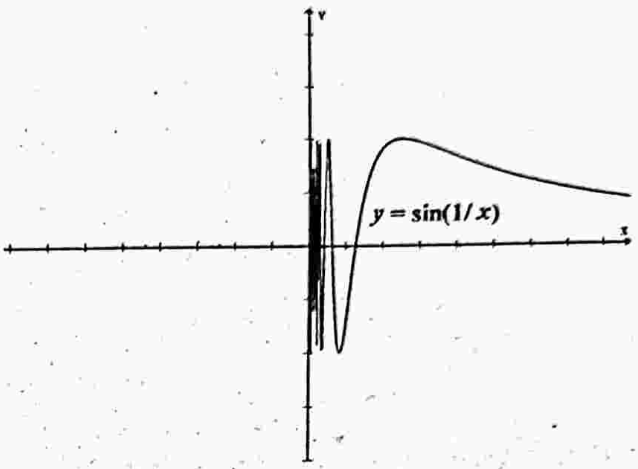

Fig. 13.7

Illustration 6

Consider the function $f(x) = \begin{cases} \dfrac{1}{|x|}, & x < 0 \\ x, & x \ge 0 \end{cases}$

From the graph (fig. 13.8), we observe that when x approaches to zero through the left of zero the value of $f(x)$ never approaches to a finite number. So $\lim_{x \to 0-} f(x)$ does not exist

$\lim_{x \to 0+} f(x) = 0$. Also $f(0) = 0$.

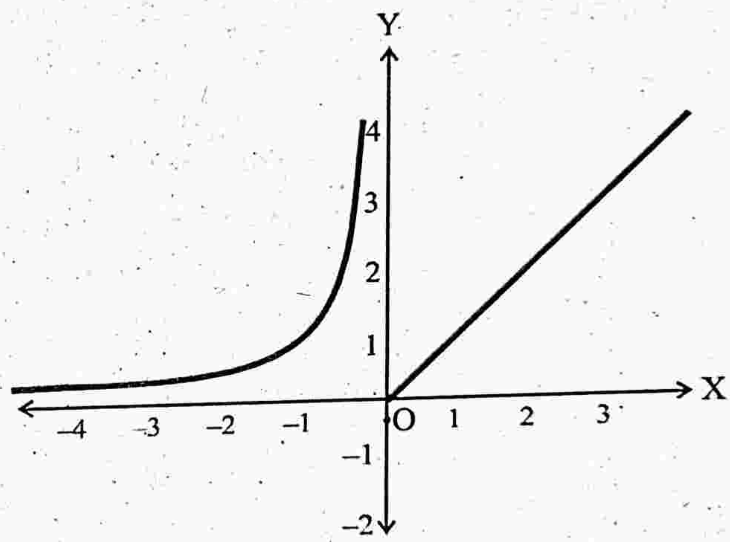

Fig. 13.8

Illustration 7

Consider the function $f(x) = \dfrac{1}{x}$ $x \neq 0$.

Let us draw the graph of the function.

x	– 3	– 2	– 1	1	2	3	0.9	0.6	0.5	0.4	0.3	– 0.9	–0.6	–0.5	–0.4	–0.3
$y = \dfrac{1}{x}$	–0.33	–0.50	– 1	1	0.50	0.33	1.1	1.67	2	2.5	3.3	– 1.1	–1.67	– 2	–2.5	–3.3

From the graph we observe that when x approaches to zero from the left of zero, the value of $f(x)$ never approaches to a finite number. Therefore the left hand limit of f at $x = 0$ does not exist.

i.e., $\lim_{x \to 0-} f(x)$ does not exist.

When x approaches to zero from the right of zero, the value of $f(x)$ never approaches to a finite number

i.e., $\lim_{x \to 0+} f(x)$ does not exist.

Also $f(0)$ does not exist.

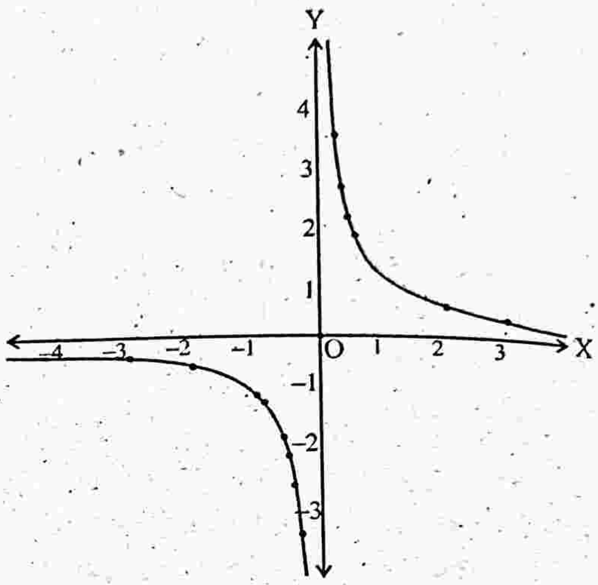

Fig. 13.9

From the above illustrations we observe the following

i. Both left hand and right hand limits exist and are equal (Illustration 1, 2)

ii. Both left hand and right hand limits exist and are not equal (Illustration 3, 4)

iii. Left hand limit exists and right hand limit does not exist (Illustration 5)

iv. Left hand limit does not exist and right hand limit exists (Illustration 6)

v. Both left hand right hand limits do not exist (Illustration 7).

If both the left hand and right hand limit exist and are equal, the common value is defined as the ***limit of the function.***

Left hand limit of f at $x = a$

When x approaches a from the left of a and the value of the function $f(x)$ approaches to a finite number l, then the finite number l is the left hand limit of f at $x = a$

i.e., $\lim_{x \to a-} f(x) = l$

Right hand limit of f at $x = a$

When x approaches 'a' from the right of a and the value of the function $f(x)$, approaches to a finite number l', then the finite number l' is the right hand limit of f at $x = a$

i.e., $\lim_{x \to a+} f(x) = l'$

Existence of limit of f at $x = a$

If the left hand and right hand limits at $x = a$ exist and $l = l'$, then the limit of f at $x = a$ exists and is equal to l. i.e., $\lim_{x \to a} f(x) = l$

The value of a function and the limit of a function at $x = a$

If $\lim_{x \to a} f(x)$ and $f(a)$ exist, then there are two possibilities.

i. $\lim_{x \to a} f(x) = f(a)$ ii. $\lim_{x \to a} f(x) \neq f(a)$

13.3.1 Algebra of limits

Illustration 8

i. Consider the function $f(x) = 1$.
The graph of the function is given in Fig. 13.10.

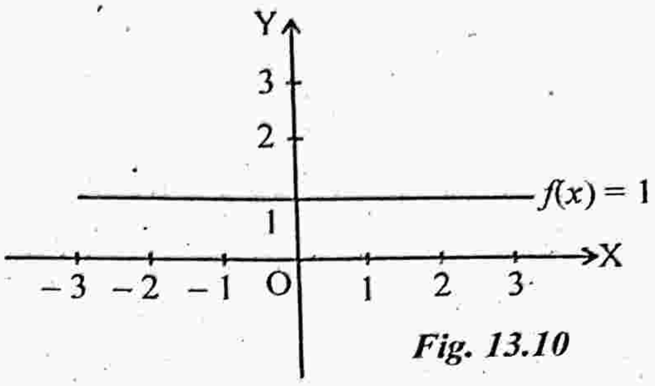

Fig. 13.10

From the graph we get $\lim_{x\to3} f(x) = \lim_{x\to3}(1) = 1$

Limit of a constant is the constant itself

ii. Consider the function $f(x) = x$
The graph of the function is given in Fig. 13.11

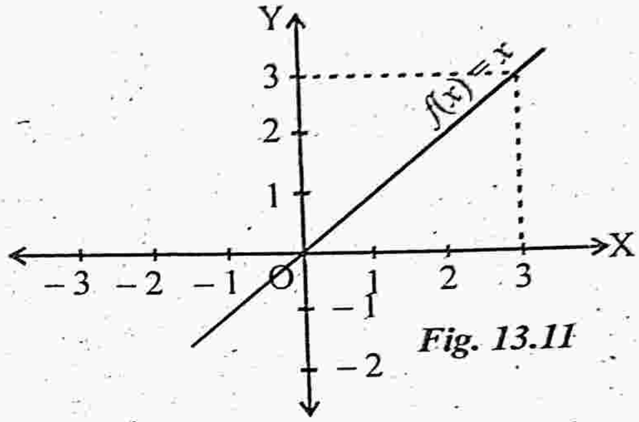

Fig. 13.11

From the graph, we get $\lim_{x\to3} f(x) = \lim_{x\to3}(x) = 3$

$\lim_{x\to a}(x) = a$

iii. Consider the function $f(x) = x + 1$
The graph of the function is given in Fig. 13.3.

From the graph we get $\lim_{x\to3} f(x) = \lim_{x\to3}(x+1) = 4$

From (i) and (ii), we get $\lim_{x\to3}(x) + \lim_{x\to3}(1) = 3 + 1 = 4$

Thus $\lim_{x\to3}(x+1) = \lim_{x\to3}(x) + \lim_{x\to3}(1)$

iv. Consider the function $f(x) = x - 1$.
The graph of the function is given in Fig. 13.12.

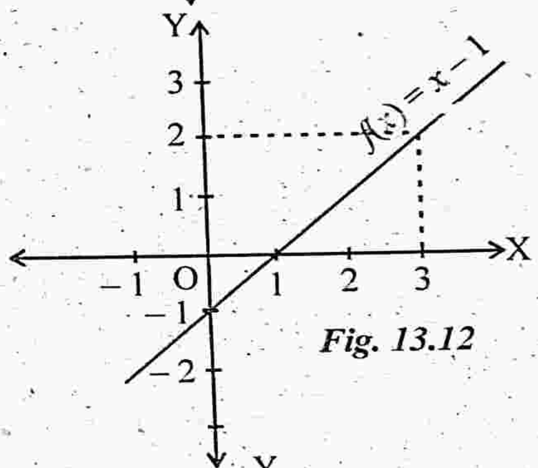

Fig. 13.12

From the graph we get $\lim_{x\to3} f(x) = \lim_{x\to3}(x-1) = 2$

From (i) and (ii), we get $\lim_{x\to3}(x) - \lim_{x\to3}(1) = 3 - 1 = 2$

Thus $\lim_{x\to3}(x-1) = \lim_{x\to3}(x) - \lim_{x\to3}(1)$

v. Consider the function $f(x) = 3x$
The graph of the function is given in Fig. 13.13

From the graph, we get $\lim_{x\to3} f(x) = \lim_{x\to3}(3x) = 9$

From (i) and (ii), we get $\lim_{x\to3}(3) . \lim_{x\to3}(x) = 3 \times 3 = 9$

Thus $\lim_{x\to3}(3x) = 3\lim_{x\to3}(x)$

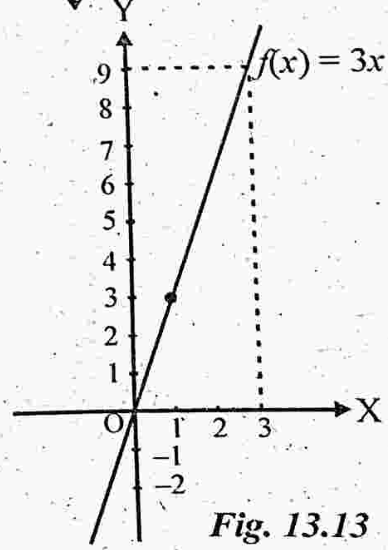

Fig. 13.13

vi. Consider the function $f(x) = x^2 + x$.
The graph of the function is given in Fig. 13.14

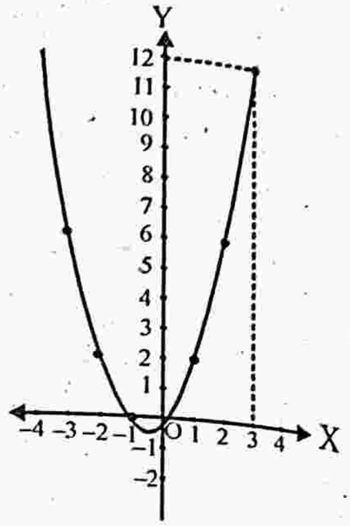

Fig. 13.14

From the graph we get $\lim_{x\to 3} f(x) = \lim_{x\to 3}(x^2 + x) = 12$

from (i) and (ii), we get $\lim_{x\to 3} x . \lim_{x\to 3}(x+1) = 3 \times 4 = 12$

Thus $\lim_{x\to 3} x(x+1) = \lim_{x\to 3}(x) . \lim_{x\to 3}(x+1)$

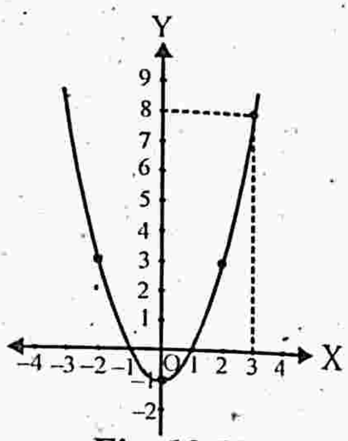

Fig. 13.15

vii. Consider the function $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

The graph of the function is given Fig. 13.4

From the graph we get $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^2-1}{(x-1)} = 4$

The graph of the function $x^2 - 1$ is given in Fig. 13.15.

From Fig 13.12 and Fig. 13.15 we get $\dfrac{\lim_{x\to 3}(x^2-1)}{\lim_{x\to 3}(x-1)} = \dfrac{8}{2} = 4$

Thus $\lim_{x\to 3}\left(\dfrac{x^2-1}{x-1}\right) = \dfrac{\lim_{x\to 3}(x^2-1)}{\lim_{x\to 3}(x-1)}$, since $\lim_{x\to 3}(x-1) \neq 0$

From Illustration 8 (iii to vii), we arrive at the following theorem.

Theorem 1 (Statements only)

Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

i. Limit of sum of two functions is sum of the limits of the functions,

i.e., $\lim_{x \to a}[f(x)+g(x)] = \lim_{x \to a} f(x)+\lim_{x \to a} g(x)$

ii. Limit of difference of two functions is difference of the limits of the functions,

i.e., $\lim_{x \to a}[f(x)-g(x)] = \lim_{x \to a} f(x)-\lim_{x \to a} g(x)$

iii. Limit of scalar product of a function is the scalar product of the limit of the function,

i.e., If c is a constant, then $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$.

iv. Limit of product of two functions is product of the limits of the functions,

i.e., $\lim_{x \to a}[f(x).g(x)] = \lim_{x \to a} f(x) \,.\, \lim_{x \to a} g(x)$

v. Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e., $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

13.3.2 Limits of polynomials and rational functions

Limit of polynomial functions

We have $\lim_{x \to a}(x) = a$

$\therefore \lim_{x \to a}(x^2) = \lim_{x \to a}(x).\lim_{x \to a}(x) = a \,.\, a = a^2$

$\lim_{x \to a}(x^3) = \lim_{x \to a}(x^2).\lim_{x \to a}(x) = a^2 \,.\, a = a^3$

In a similar way, we get $\lim_{x \to a}(x^n) = a^n$

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$, $a_n \neq 0$ and a_is are real numbers, be a polynomial function.

$\lim_{x \to a} f(x) = \lim_{x \to a} [a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n]$

$= \lim_{x \to a}(a_0) + a_1 \lim_{x \to a}(x) + a_2 \lim_{x \to a}(x^2) + \ldots\ldots\ldots + a_n \lim_{x \to a}(x^n)$

$= a_0 + a_1a + a_2a^2 + \ldots\ldots\ldots + a_na^n = f(a)$

Example 1

Find $\lim_{x\to 3} x(x+1)$ (March 2015)

Solution

$$\lim_{x\to 3} x(x+1) = \lim_{x\to 3} x \cdot \lim_{x\to 3}(x+1)$$
$$= 3 \times 4 = 12$$

Example 2

Find the limits

i. $\lim_{x\to 1}(x^3 - x^2 + 1)$ **(NCERT)**

ii. $\lim_{x\to -1}(1 + x + x^2 + ... + x^{10})$ **(NCERT)**

Solution

i. $\lim_{x\to 1}(x^3 - x^2 + 1) = (1)^3 - (1)^2 + 1 = 1$

ii. $\lim_{x\to -1}(1 + x + x^2 + ... + x^{10}) = 1 + (-1) + (-1)^2 + ,...... (-1)^{10}$

$$= 1 - 1 + 1 - 1 + ... + 1 = 1$$

Limit of rational functions

Consider the rational function $f(x) = \frac{g(x)}{h(x)} = \frac{x^2 - 4x}{x - 2}$

Here $f(3) = -3$ and $\lim_{x\to 3} g(x) = \lim_{x\to 3}(x^2 - 4x) = 3^2 - 4(3) = -3$ $\therefore g(3) = -3$

Also $\lim_{x\to 3} h(x) = \lim_{x\to 3}(x - 2) = 3 - 2 = 1$ $\therefore h(3) = 1$

$$\lim_{x\to 3} f(x) = \frac{\lim_{x\to 3} g(x)}{\lim_{x\to 3} h(x)} = \frac{g(3)}{h(3)} = \frac{-3}{1} = -3 = f(3)$$

Hence we can conclude that, if $f(x) = \frac{g(x)}{h(x)}$ and $h(a) \neq 0$, then $\lim_{x\to a} f(x) = \frac{g(a)}{h(a)} = f(a)$

Consider $f(x) = \frac{g(x)}{h(x)} = \frac{x^2 - 3x + 2}{x^2 - 1}$

$\lim_{x\to 1} g(x) = \lim_{x\to 1}(x^2 - 3x + 2) = g(1) = 0$ and $\lim_{x\to 1} h(x) = \lim_{x\to 1}(x^2 - 1) = h(1) = 0$

$f(1) = \frac{g(1)}{h(1)}$ does not exist

Hence we cannot evaluate $\lim_{x\to 1} f(x)$ directly. We observe that $g(x)$ and $h(x)$ have $(x - 1)$ as common factor

$\therefore f(x) = \dfrac{x^2-3x+2}{x^2-1} = \dfrac{(x-1)(x-2)}{(x-1)(x+1)} = \dfrac{x-2}{x+1}$

$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\left(\dfrac{x-2}{x+1}\right) = -\dfrac{1}{2}$

Consider $f(x) = \dfrac{g(x)}{h(x)} = \dfrac{x-2}{x^3-2x^2-4x+8}$ Here $g(2) = 0$ and $h(2) = 0$

$f(2) = \dfrac{g(2)}{h(2)}$ does not exist. Hence we cannot evaluate $\lim\limits_{x\to 2} f(x)$ directly.

We observe that $g(x)$ and $h(x)$ have $(x-2)$ as common factor

$f(x) = \dfrac{x-2}{x^3-2x^2-4x+8} = \dfrac{(x-2)}{(x-2)(x-2)(x+2)} = \dfrac{1}{(x-2)(x+2)}$

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\dfrac{1}{(x-2)(x+2)}$ does not exist, since $\lim\limits_{x\to 2}(x-2) = 0$

WORKING RULE

To find the limit of a function of the form $f(x) = \dfrac{g(x)}{h(x)}$ at $x = a$

* If $h(a) \neq 0$; $\lim\limits_{x\to a} f(x) = f(a) = \dfrac{g(a)}{h(a)}$
* If $h(a) = 0$ and $g(a) \neq 0$, then $\lim\limits_{x\to a} f(x)$ does not exist.
* If $h(a) = 0$ and $g(a) = 0$, then $f(x) = \dfrac{(x-a)^m g_1(x)}{(x-a)^n h_1(x)}$.
    - If $m > n$, then $\lim\limits_{x\to a} f(x) = 0$
    - If $m = n$, then $\lim\limits_{x\to a} f(x) = \dfrac{g_1(a)}{h_1(a)}$.
    - If $m < n$, then $\lim\limits_{x\to a} f(x)$ does not exist.

Example 3

Evaluate $\lim\limits_{x\to 0}\dfrac{4x+3}{2x-1}$ **(March 2011)**

Solution

$\lim\limits_{x\to 0}\dfrac{4x+3}{2x-1} = \dfrac{4(0)+3}{2(0)-1} = \dfrac{3}{-1} = -3$

Example 4

Evaluate $\lim_{x\to -1} \frac{x^2-5x+6}{x-1}$

Solution

$$\lim_{x\to -1} \frac{x^2-5x+6}{x-1} = \frac{(-1)^2-5(-1)+6}{-1-1} = \frac{1+5+6}{-2} = \frac{12}{-2} = -6$$

Example 5

Evaluate $\lim_{x\to a} \frac{\sqrt{x}+\sqrt{a}}{x+a}$.

Solution

$$\lim_{x\to a} \frac{\sqrt{x}+\sqrt{a}}{x+a} = \frac{\sqrt{a}+\sqrt{a}}{a+a} = \frac{2\sqrt{a}}{2a} = \frac{1}{\sqrt{a}}$$

Example 6

Given $f(x) = \frac{x^3-1}{x+2}$, find $\lim_{x\to 0} f(x)$.

Solution

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^3-1}{x+2} = \frac{(0)^3-1}{0+2} = \frac{-1}{2}$$

Example 7

Evaluate $\lim_{x\to 2} \frac{3x^2-x-10}{x^2-4}$. **(NCERT)**

Solution .

Evaluating the function at $x=2$, we get it of the form $\frac{0}{0}$

$$\therefore \lim_{x\to 2} \frac{3x^2-x-10}{x^2-4} = \lim_{x\to 2} \frac{3x^2-6x+5x-10}{x^2-2^2} = \lim_{x\to 2} \frac{3x(x-2)+5(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \left(\frac{3x+5}{x+2}\right) = \frac{3(2)+5}{2+2} = \frac{11}{4}$$

Example 8

Find $\lim_{x\to -2} \frac{x^2+5x+6}{x^2+3x+2}$

Solution

$$\lim_{x\to -2} \frac{x^2+5x+6}{x^2+3x+2} = \lim_{x\to -2} \frac{(x+2)(x+3)}{(x+2)(x+1)} = \lim_{x\to -2} \frac{x+3}{x+1} = \frac{-2+3}{-2+1} = \frac{1}{-1} = -1$$