LIMITS AND DERIVATIVES

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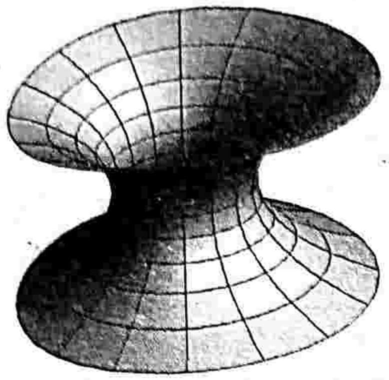

"What you should learn"

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13.1 INTRODUCTION

Calculus is a branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change. This chapter is an introduction to calculus. We also introduce the concept of derivative and process of differentiation.

GOTTFRIED WILHELM LEIBNIZ

(July 1, 1646–November 14, 1716) was a German mathematician. He developed the infinitesimal calculus independently of Isaac Newton. Leibniz's mathematical notation has been widely used ever since it was published.

13. 2 INTUITIVE IDEA OF DERIVATIVES

We know that the displacement s of a freely falling stone at time t is given by the equation $s = 4.9t^2$. (s in metre and t in sec). Clearly s is a function of t. The following table gives the distance travelled by stone at various times.

t	0	1	1.8	2	2.5	2.7	2.8	2.9	2.95	3	3.05	3.1	3.2	4
s	0	4.9	15.876	19.6	30.625	35.721	38.416	41.209	42.642	44.1	45.58	47.08	50.17	78.4

Let us try to find the velocity at 3 seconds. We cannot calculate it directly. Hence we consider the average velocity in different time intervals (Starting from t and ending in 3).

$$\text{Average velocity in } [t_1, t_2] = \frac{\text{distance travelled between } t_1 \text{ and } t_2}{\text{time interval } t_2 - t_1}$$

$$\text{Average velocity in } [0, 3] = \frac{44.1-0}{3-0} = 14.7$$

$$\text{Average velocity in } [2, 3] = \frac{44.1-19.6}{3-2} = 24.5$$

Similarly we can find the average velocity for different time intervals. The following table shows the average velocity in the interval between 3 and t seconds.

Interval	[0, 3]	[2, 3]	[2.7, 3]	[2.9, 3]	[2.95, 3]	[3, 3.05]	[3, 3.1]	[3, 4]
Average velocity	14.7	24.5	27.93	28.91	29.16	29.64	29.89	34.3

From the table we observe that as the time approaches from 0 to 3 sec, average velocity is increasing and when the time is approaching from 4 to 3 the average velocity is decreasing. Also we observe that the average velocity at 3 sec is a value between 29.16 and 29.64. This method can be used to find the average (instantaneous) velocity at any time.

We say that the derivative of the function $s = 4.9t^2$ at $t = 3$ is between 29.64 and 29.16.

$$\therefore \text{ The average velocity at 3 seconds } = \frac{29.64+29.16}{2} = 29.4 \; m/s$$

Consider a regular polygon inscribed in a circle of given radius.

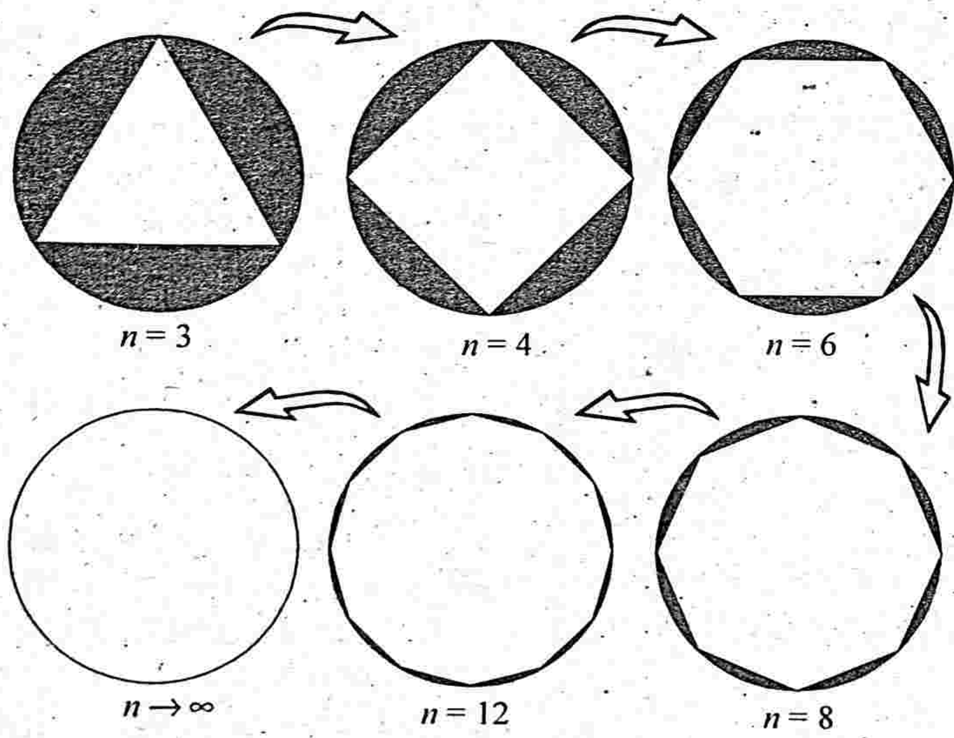

Fig 13.1

From the above figures, we observe that the area of the polygon cannot be greater than the area of the circle. When n (number of sides of the polygon) increases, the area of the shaded portion decreases. The difference between the area of the circle and the area of the polygon can be made as small as we please by increasing the number of sides of the polygon. Thus the area of the polygon of n sides is equal to the area of the circle as $n \to \infty$.

13.3 LIMITS

Meaning of x approaches to 'a' in a real number line

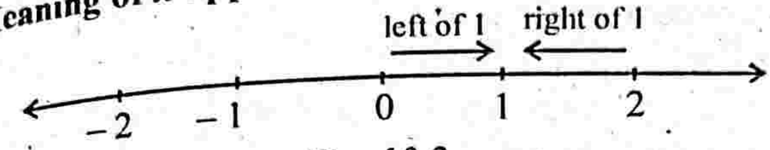

Fig. 13.2

Let x be a variable point in the real number line. x approaches to 1 in the number line means that x can assume all values from the left of 1 and from the right of 1. This is denoted by $x \to 1$ (read as x tends to 1). Here x is not assuming the value 1.

x approaches to 1 from the left side of 1 is denoted by $x \to 1-$ (read as x tends to 1 –).

x approaches to 1 from the right side of 1 is denoted by $x \to 1+$ (read as x tends to 1 +).

In general $x \to a$ means that x can assume all values less than a and greater than a excluding the value a.

$x \to a$ means both $x \to a-$ and $x \to a+$

Limit of a function

Illustration 1

Consider the function $f(x) = x + 1$. Let us draw the graph of the given function

x	– 2	– 1	0	1	2
$f(x)$	– 1	0	1	2	3

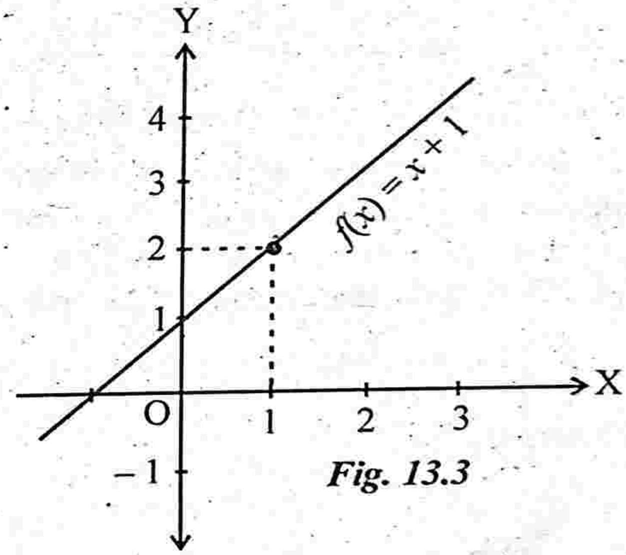

Fig. 13.3

From the graph, we observe that when x approaches to 1 from the left of 1, the value of $f(x)$ approaches to 2. i.e., when $x \to 1-$, $f(x) \to 2$. We define this value as the **left hand limit** of f at $x = 1$, denoted by $\lim_{x \to 1-} f(x) = 2$.

Also we observe that when x approaches to 1 from the right of 1, the value of $f(x)$ approaches to 2 i.e., when $x \to 1+$, $f(x) \to 2$. We define this value as the **right hand limit** of f at $x = 1$, denoted by $\lim_{x \to 1+} f(x) = 2$. Also $f(1) = 2$.

> **NOTE**
> Left hand limit of $f(x)$ at $x = a$ is $f(a-)$
> Right hand limit of $f(x)$ at $x = a$ is $f(a+)$

Illustration 2

Consider the function $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \end{cases}$

Let us draw the graph of the given function $f(x) = \dfrac{(x-1)(x+1)}{x-1} = x+1,\ x \neq 1$

We can draw the graph of $f(x) = x + 1$ except at $x = 1$.

x	-1	0	1	2
$f(x)$	0	1	not defined	3

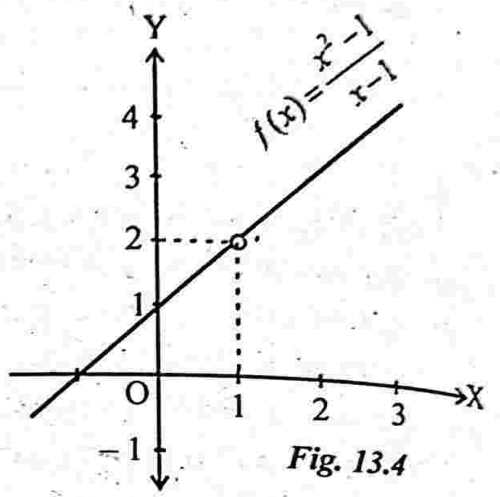

Fig. 13.4

From the graph, we observe that $\lim\limits_{x\to1-} f(x)=2$ and $\lim\limits_{x\to1+} f(x)=2$. Also $f(1)$ does not exist.

Illustration 3

Consider the function $f(x) = \begin{cases} -1, & x<0 \\ 1, & x>0 \end{cases}$

Let us draw the graph of the given function.

x	$x<0$	$x>0$
$f(x)$	-1	1

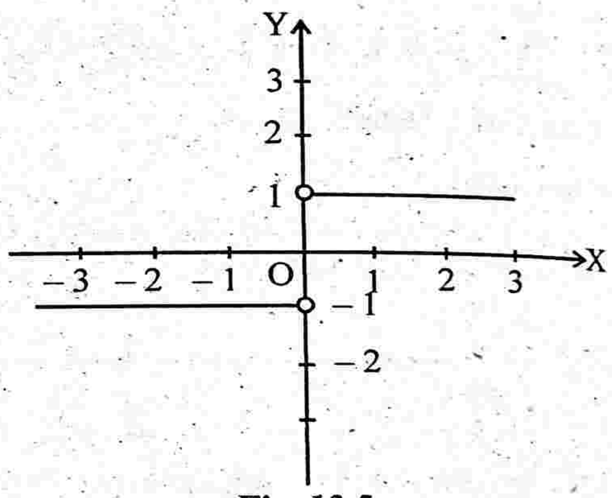

Fig. 13.5

From the graph and table we observe that when x approaches to zero from the left of zero, the value of $f(x)$ approaches to -1. Also when x approaches to zero from the right of zero, the value of $f(x)$ approaches to 1.

$\therefore \lim\limits_{x\to0-} f(x)=-1,\ \lim\limits_{x\to0+} f(x)=1$. Also $f(0)$ does not exist

Illustration 4

Consider the function $f(x) = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$

Let us draw the graph of the given function.

From the graph we observe that $\lim\limits_{x\to0-} f(x)=-1$,

$\lim\limits_{x\to0+} f(x)=1$ Also $f(0)=0$

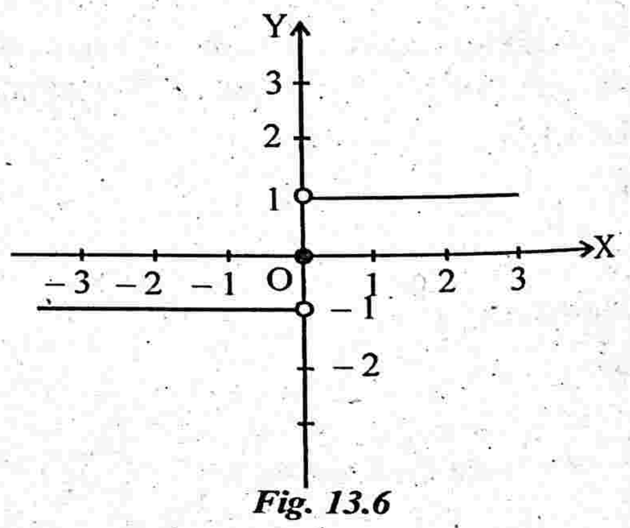

Fig. 13.6