Example 43

Consider the line joining the points P(–4, 1) and Q(0, 5).

i. Write the coordinates of the midpoint of PQ.

ii. Find the equation of the line passing through the midpoint of PQ and parallel to the line $3x - 4y + 2 = 0$. **(September 2013)**

Solution

i. Midpoint of PQ $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4+0}{2}, \frac{1+5}{2}\right) = (-2, 3)$

ii. Equation of the line parallel to the given line is $3x - 4y + K = 0$.

Since it passes through (–2, 3) we get $3(-2) - 4(3) + K = 0$ $\therefore K = 18$

Equation of the line is $3x - 4y + 18 = 0$

Example 44

i. Find the point of intersection of the lines $2x + y - 3 = 0$, $3x - y - 2 = 0$

ii. Find the equation of the line passing through the above point of intersection and parallel to the line $x + y + 1 = 0$. **(March 2012)**

Solution

i. $2x + y = 3$ ....(1)

$3x - y = 2$ ....(2)

$(1) + (2) \rightarrow 5x = 5$ $\therefore x = 1$

From (1), we get $y = 1$

$\therefore$ Point of intersection is (1,1)

ii. Equation of the line parallel to $x + y + 1 = 0$ is $x + y + k = 0$

Since this line passes through (1,1)

we get $1 + 1 + k = 0$

$\therefore k = -2$

Equation of the line is $x + y - 2 = 0$

Example 45

Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x - intercept 3. **(NCERT)**

Solution

Equation of the perpendicular line is $-7x - y + K = 0$

Since it passes through (3, 0) we get $-7(3) - 0 + K = 0$. $\therefore K = 21$

Equation of the required line is $-7x - y + 21 = 0$ or $7x + y - 21 = 0$

Example 46

Find the equation of a line perpendicular to the line $x - 2y + 3 = 0$ and passing through the point (1, –2). **(NCERT, March 2013)**

Solution

Equation of a line perpendicular to the line

$x - 2y + 3 = 0$ is $2x + y + k = 0$ ......(1)

Since (1) passes through (1, –2),

we get $2(1) + (-2) + k = 0 \quad \Rightarrow k = 0$

$\therefore$ The required line is $2x + y = 0$

Example 47

Consider the straight line $3x + 4y + 8 = 0$

i. What is the slope of the line which is perpendicular to the given line?

ii. If the perpendicular line passes through (2,3), form its equation.

iii. Find the foot of the perpendicular drawn from (2,3) to the given line. **(March 2011)**

Solution

i. Slope of the given line, $m = \dfrac{-\text{Coefficient of } x}{\text{Coefficient of } y} = \dfrac{-3}{4}$

Slope of the line perpendicular to the given line $= \dfrac{-1}{m} = \dfrac{4}{3}$

ii. Equation of the perpendicular line passing through (2,3) is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{4}{3}(x - 2)$$

$$3y - 9 = 4x - 8$$

$$4x - 3y + 1 = 0$$

STUDY TIP

- Equation of the line parallel to the line $ax + by + c = 0$ and passing through (x_1, y_1) is $a(x - x_1) + b(y - y_1) = 0$
- Equation of the line perpendicular to the line $ax + by + c = 0$ and passing through (x_1, y_1) is $b(x - x_1) - a(y - y_1) = 0$

Another Method (ii)

The equation of a line perpendicular to $ax + by + c = 0$ and passing through (x_1, y_1) is $b(x - x_1) - a(y - y_1) = 0$:

Hence the equation of the line perpendicular to $3x + 4y + 8 = 0$ and passing through (2,3) is

$4(x - 2) - 3(y - 3) = 0$

$4x - 3y + 1 = 0$

iii. The foot of any point on the perpendicular line to the line $3x + 4y + 8 = 0$ is the point of intersection of the line $3x + 4y + 8 = 0$ and its perpendicular.

The point of intersection is obtained by solving

$3x + 4y + 8 = 0$ and

$4x - 3y + 1 = 0$

Solving, we get $x = \frac{-28}{25}$ and $y = \frac{-29}{25}$

$\therefore$ Foot of the perpendicular from (2,3) to the line $3x + 4y + 8 = 0$ is $\left(\frac{-28}{25}, \frac{-29}{25}\right)$

Example 48

Consider the points A(–2, –3) and B(1, 6).

i. Find the equation of the line passing through A and B.

ii. Find the equation of the line passing through (2, 1) and perpendicular to AB.

iii. Find the foot of the above perpendicular to AB. **(June 2008)**

Solution

i. Equation of AB is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

i.e., $\frac{y - ^{-}3}{6 - ^{-}3} = \frac{x - ^{-}2}{1 - ^{-}2}$ or $\frac{y + 3}{9} = \frac{x + 2}{3}$

or $y + 3 = 3(x + 2)$ or $y - 3x - 3 = 0$ or $-3x + y - 3 = 0$

ii. The equation of a line passing through (x_1, y_1) and perpendicular to $ax + by + c = 0$ is $b(x - x_1) - a(y - y_1) = 0$

$\therefore$ Equation of the line passing through (2,1) and perpendicular to $-3x + y - 3 = 0$ is

$1(x - 2) + 3(y - 1) = 0$

$x + 3y - 5 = 0$

iii. The foot of any point on the perpendicular to line AB is the point of their intersection.

$\therefore$ Solving $-3x + y - 3 = 0$ and $x + 3y - 5 = 0$, we get

$x = \frac{-2}{5}$ and $y = \frac{9}{5}$

Hence the foot of the perpendicular to AB is $\left(\frac{-2}{5}, \frac{9}{5}\right)$

Example 49

Consider the points A(2, 2) and B(5, 3).

i. Find the slope of the line through the points A and B.

ii. Find the equation of the line passing through the points A and B.

iii. Find the image of the point (1, 2) in the line through A and B. **(August 2009, NCERT)**

Solution

i. Slope of AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{5-2} = \frac{1}{3}$

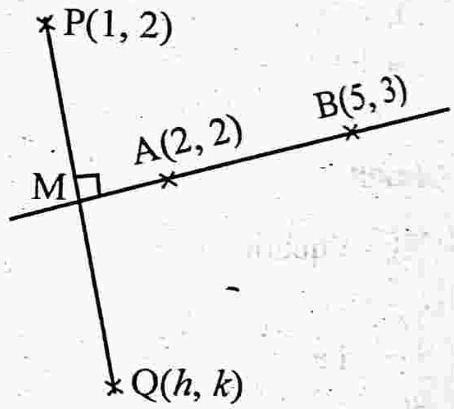
ii. Since the slope of line AB is $\frac{1}{3}$ and (2, 2) is a point on AB

$\therefore$ Equation of the line AB is $y - 2 = \frac{1}{3}(x-2)$

i.e., $3y - 6 = x - 2 \quad \Rightarrow 3y - x - 4 = 0$ or $x - 3y + 4 = 0$

iii. Let P(1, 2) be the given point and Q(h, k) be the image of P w.r.t. the line AB. Let PQ intersect AB at M.

Hence PQ $\perp$ AB and M is the midpoint of PQ.



Slope of AB $= \frac{1}{3}$ *from (i)*

$\therefore$ Slope of PQ $= \frac{-1}{\text{Slope of AB}} = \frac{-1}{\left(\frac{1}{3}\right)} = -3$

Again slope of PQ $= \frac{k-2}{h-1}$

Hence we get $\frac{k-2}{h-1} = -3$ or $3h + k - 5 = 0$ .... (1)

Coordinate of M = Midpoint of PQ $= \left(\frac{h+1}{2}, \frac{k+2}{2}\right)$

The equation of the line AB is $x - 3y + 4 = 0$ *from (ii)*

Since M is a point on the line AB

we get $\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0$

$h + 1 - 3k - 6 + 8 = 0$

or $h - 3k + 3 = 0$ .... (2)

Solving (1) and (2), we get $h = \frac{6}{5}$, $k = \frac{7}{5}$

$\therefore$ The image of (1, 2) w.r.t. line AB $= \left(\frac{6}{5}, \frac{7}{5}\right)$

10.5 DISTANCE OF A POINT FROM A LINE

Let $Ax + By + C = 0$ be the equation of the line l intersecting the x - axis at S and y - axis at R. The x - intercept of the line is $\frac{-C}{A}$ and y - intercept of the line is $\frac{-C}{B}$.

$\therefore$ S is the point $\left(\frac{-C}{A}, 0\right)$ and R is the point $\left(0, \frac{-C}{B}\right)$.

Let $P(x_1, y_1)$ be a point in the xy-plane. Draw PN $\perp$ RS.

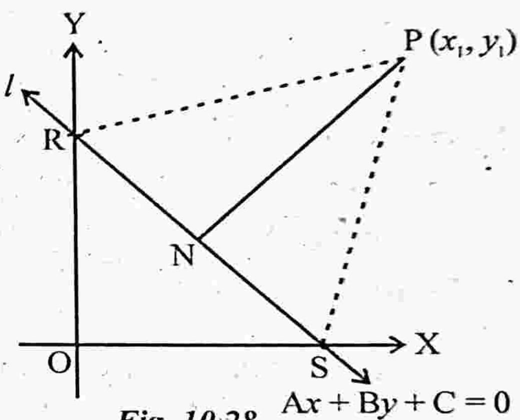


Fig. 10.28

$$\text{Area of } \Delta PSR = \frac{1}{2}\left|x_1\left(0+\frac{C}{B}\right)+\left(\frac{-C}{A}\right)\left(\frac{-C}{B}-y_1\right)+0(y_1-0)\right|$$

$$= \frac{1}{2}\left|\frac{Cx_1}{B}+\frac{Cy_1}{A}+\frac{C^2}{AB}\right| = \left|\frac{C}{2AB}(Ax_1+By_1+C)\right| = \left|\frac{C}{2AB}\right|\left|Ax_1+By_1+C\right| \quad \text{............. (1)}$$

$$\text{Area of } \Delta PSR = \frac{1}{2}\text{ RS. PN} = \frac{1}{2}\sqrt{\frac{C^2}{A^2}+\frac{C^2}{B^2}}\text{ . PN} = \left|\frac{C}{2AB}\right|\sqrt{A^2+B^2}\text{ . PN} \quad \text{............. (2)}$$

From (1) and (2) we get,

$$\left|\frac{C}{2AB}\right|\left|Ax_1+By_1+C\right| = \left|\frac{C}{2AB}\right|\sqrt{A^2+B^2}\text{ PN}$$

$$\therefore \text{PN} = \frac{\left|Ax_1+By_1+C\right|}{\sqrt{A^2+B^2}}$$

Example 50

Find the distance of the point (3, –3) from the line $3x - 4y - 26 = 0$ **(March 2009)**

Solution

Distance of the point (x_1, y_1) from the line $Ax_1 + By_1 + C = 0$ is $\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

Distance of the point (3, –3) from the line $3x - 4y - 26 = 0$

$$= \frac{|3(3) - 4(-3) - 26|}{\sqrt{(3)^2 + (4)^2}} = \frac{|-5|}{\sqrt{25}} = \frac{5}{5} = 1 \text{ unit}$$

Example 51

Find the distance of the line $3x - 4y + 2 = 0$ from the origin. **(March 2015)**

Solution

Perpendicular distance from origin to line

$$= \frac{|c|}{\sqrt{A^2 + B^2}} = \frac{|2|}{\sqrt{3^2 + (-4)^2}} = \frac{2}{5}$$

STUDY TIP

The length of the perpendicular from the origin to the line $Ax + By + C = 0$ is $\frac{|C|}{\sqrt{A^2 + B^2}}$

Example 52

Consider the line $3x - 4y + 2 = 0$ and the point (2, – 3)

i. Find the distance of the point from the line.

ii. Find the image of the point about the line. **(October 2011)**

Solution

i. Distance of the point (2, – 3) from the line $3x - 4y + 2 = 0$ is $\left|\frac{3(2) - 4(-3) + 2}{\sqrt{3^2 + (-4)^2}}\right|$

$$= \left|\frac{6 + 12 + 2}{\sqrt{9 + 16}}\right| = \frac{20}{5} = 4$$

ii. Let P (2, – 3) be the given point and Q (*h*, *k*) be the image of P with respect to the line $3x - 4y + 2 = 0$

Slope of the given line is $\frac{3}{4}$

Slope of PQ = $\frac{k + 3}{h - 2}$

Since PQ is perpendicular to the given line, the product of their slopes = – 1

i.e, $\frac{3}{4} \times \frac{(k + 3)}{(h - 2)} = -1 \quad \Rightarrow \quad 3k + 9 = -4h + 8$

$4h + 3k = -1$ ....(1)

Let M be the midpoint of PQ.

$\therefore$ M is $\left(\frac{h+2}{2}, \frac{k-3}{2}\right)$

M lies on the line $3x - 4y + 2 = 0$

i.e., $3\left(\frac{h+2}{2}\right) - 4\left(\frac{k-3}{2}\right) + 2 = 0$

$3h + 6 - 4k + 12 + 4 = 0$

$3h - 4k = -22$ ....(2)

Solving (1) and (2), we get $h = \frac{-14}{5}$, $k = \frac{17}{5}$

$\therefore$ The image of $(2, -3)$ with respect to the line $3x - 4y + 2 = 0$ is $\left(\frac{-14}{5}, \frac{17}{5}\right)$

10.5.1 Distance between two parallel lines

We know that $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ are the equations of two parallel lines.

Let $P(x_1, y_1)$ be a point on the line $Ax + By + C_1 = 0$

$\therefore Ax_1 + By_1 + C_1 = 0$

$Ax_1 + By_1 = -C_1$ .... (1)

Draw PM perpendicular to the line $Ax + By + C_2 = 0$

PM is the perpendicular distance from $P(x_1, y_1)$ to the line $Ax + By + C_2 = 0$.

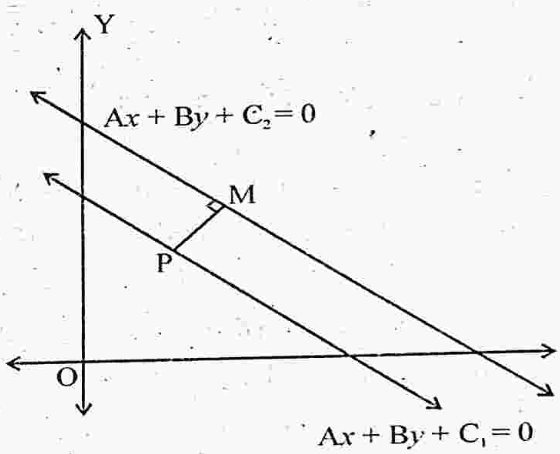


Fig 10.29

$$\therefore PM = \frac{|Ax_1 + By_1 + C_2|}{\sqrt{A^2 + B^2}} = \frac{|-C_1 + C_2|}{\sqrt{A^2 + B^2}} \quad \textit{from (1)}$$

$$\therefore PM = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

But PM is the distance between the parallel lines.

The distance between the lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is $\frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$

Example 53

Find the distance between the parallel lines

$3x - 4y + 5 = 0$ and $3x - 4y + 7 = 0$ **(NCERT)**

Solution

The given equations are $3x - 4y + 5 = 0$ ——— (1) and $3x - 4y + 7 = 0$ ——— (2)

Here $C_1 = 5$ and $C_2 = 7$, $A = 3$, $B = -4$

$$\text{Distance } d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}} = \frac{|7-5|}{\sqrt{(3)^2 + (-4)^2}} = \frac{|2|}{\sqrt{9+16}} = \frac{2}{5}$$

Example 54

i. Reduce the equation $3x + 4y - 12 = 0$ into intercept form.

ii. Find the distance of the above line from the origin.

iii. Find the distance of the above line from the line $6x + 8y - 18 = 0$. **(March 2010)**

Solution

i. The equation of the line is $3x + 4y - 12 = 0$

i.e., $3x + 4y = 12$

Dividing by 12, we get $\frac{3x}{12} + \frac{4y}{12} = \frac{12}{12}$ or $\frac{x}{4} + \frac{y}{3} = 1$, which is in intercept form.

ii. Distance of the line $3x + 4y - 12 = 0$ from the origin $= \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|-12|}{\sqrt{3^2 + 4^2}} = \frac{12}{5}$.

iii. The lines $3x + 4y - 12 = 0$ and $6x + 8y - 18 = 0$ are parallel, since the ratio of the x-coordinates is equal to the ratio of the y-coordinates.

Let $3x + 4y - 12 = 0$ .... (1)

$6x + 8y - 18 = 0$ .... (2)

Multiplying (1) by 2, we get

$6x + 8y - 24 = 0$ .... (3)

Distance between lines (1) and (2) = Distance between lines (2) and (3)

$$= \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}} = \frac{|-24 - {}^-18|}{\sqrt{6^2 + 8^2}} = \frac{|-6|}{\sqrt{100}} = \frac{6}{10} = \frac{3}{5}$$