9.7 INFINITE GP AND ITS SUM

The geometric progression of the form $a, ar, ar^2, ...$ is called in Infinite G.P.

Consider the G.P. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$

Here $a = 1, r = \frac{1}{2}$ which is less than 1.

$$\therefore \text{Sum to } n \text{ terms, } S_n = \frac{a(1-r^n)}{(1-r)} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)} = 2\left[1-\left(\frac{1}{2}\right)^n\right]$$

The following table shows, the value of $\left(\frac{1}{2}\right)^n$ for larger values of n.

n	1	2	3	10	20
$\left(\frac{1}{2}\right)^n$	0.5	0.25	0.125	0.000965	0.00000095

From the table, we observe that for larger values of n, $\left(\frac{1}{2}\right)^n$ becomes sufficiently small or tends to 0, i.e., when $n \to \infty$, then $\left(\frac{1}{2}\right)^n \to 0$

The above sum becomes sum of an infinite of G.P. as n tends to ∞

$\therefore S_\infty = 2(1-0) = 2$

For a Geometric progression $a, ar, ar^2, \ldots,$ $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$

Then r^n tends to 0 as n tends to ∞

$\therefore$ Sum to infinity is denoted by S_∞ or S

$\therefore S_\infty = \frac{a}{1-r}$

Example 49

Find the sum to infinity : $1, \frac{2}{3}, \frac{4}{9}, \ldots\ldots\ldots$

Solution

This is an infinite GP with $a = 1, r = \frac{2}{3}$

Since $|r| < 1$, $S = \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

Example 50

Evaluate $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots\ldots$

Solution

This is an infinite geometric series with $a = 1, r = \frac{1}{2}$

$\therefore S = \frac{a}{1-r}$, since $|r| < 1$

$S = \frac{1}{1-\frac{1}{2}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

Example 51

Evaluate $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots\ldots\ldots$

Solution

This is an infinite geometric series with $a = 1, r = \frac{-1}{2}$

$\therefore S = \frac{a}{1-r}$, since $|r| < 1$

$$S=\frac{1}{1-\left(\frac{-1}{2}\right)}=\frac{1}{1+\frac{1}{2}}=\frac{1}{\left(\frac{3}{2}\right)}=\frac{2}{3}$$

Example 52

i. Find the sum to infinity 6, 1.2, 0.24, ... ∞

ii. The first term of a G.P. is 2 and sum to infinity is 6. Find the common ratio.

Solution

i. Here $a=6$, $r=\frac{1.2}{6}=0.2$

$$\therefore \text{Sum to infinity } S_\infty=\frac{a}{1-r}=\frac{6}{1-0.2}$$

$$=\frac{6}{0.8}=7.5$$

ii. Given that $a=2$ and $S_\infty=6$

$$\Rightarrow \frac{a}{1-r}=6$$

$$\Rightarrow \frac{2}{1-r}=6 \Rightarrow 2=6-6r$$

$$\Rightarrow 6r=4 \Rightarrow r=\frac{4}{6}=\frac{2}{3}$$

Solutions to NCERT Exercise 9.4

Find the sum to infinity in each of the following Geometric Progression (Questions 1 to 4)

1. $1, \frac{1}{3}, \frac{1}{9}, \dots\dots\dots$

Solution

Here $a=1$, $r=\frac{1}{3}$

$$\therefore S=\frac{a}{1-r}, \text{ since } |r|<1$$

$$=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}=1.5$$

2. 6, 1.2, 0.24, ...

Solution

Refer Solution to Example 52(i)

3. $5, \frac{20}{7}, \frac{80}{49}, \dots\dots$

Solution

Here $a=5$, $r=\frac{4}{7}$

$$S=\frac{a}{1-r}, \text{ since } |r|<1$$

$$=\frac{5}{1-\frac{4}{7}}=\frac{5}{\frac{3}{7}}=\frac{35}{3}$$

4. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \ldots\ldots$

Solution

Here $a = \frac{-3}{4}$, $r = \frac{-1}{4}$

$S = \frac{a}{1-r}$, since $|r|<1$

$$= \frac{\frac{-3}{4}}{1-\frac{-1}{4}} = \frac{\frac{-3}{4}}{\frac{5}{4}} = \frac{-3}{5}$$

5. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times \ldots\ldots = 3$

LHS $= 3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times \ldots\ldots = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\ldots}$ ......(1)

Consider $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$

This is a GP with $a = \frac{1}{2}$, $r = \frac{1}{2}$

$$\therefore \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots = \frac{\frac{1}{2}}{1-\frac{1}{2}}, \text{ since } |r|<1$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$\therefore$ (1) gives $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times \ldots\ldots = 3^1 = 3$

6. Let $x = 1 + a + a^2 + \ldots\ldots$; $|a|<1$

$\Rightarrow y = 1 + b + b^2 + \ldots\ldots$; $|b|<1$

Prove that $1 + ab + a^2b^2 + \ldots\ldots = \frac{xy}{x+y-1}$

Solution

$1, a, a^2, \ldots\ldots$ form an infinite GP with $|a|<1$

$1, b, b^2, \ldots\ldots$ form an infinite GP with $|b|<1$

$1, ab, a^2b^2 \ldots\ldots$ form an infinite GP with $|ab|<1$, since $|a|<1$ & $|b|<1 \Rightarrow |ab|<1$

$\therefore x = 1 + a + a^2 + \ldots\ldots$

$$x = \frac{1}{1-a} \Rightarrow 1-a = \frac{1}{x} \Rightarrow a = 1-\frac{1}{x}$$

Similarly $y = \frac{1}{1-b} \Rightarrow 1-b = \frac{1}{y} \Rightarrow b = 1-\frac{1}{y}$

Now $1 + ab + a^2b^2 + \ldots\ldots = \frac{1}{1-ab}$

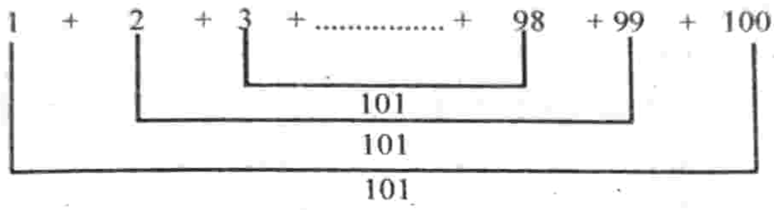
$$= \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)}$$

$$= \frac{1}{1-\frac{(x-1)(y-1)}{xy}}$$

$$= \frac{xy}{xy-(xy-x-y+1)} = \frac{xy}{x+y-1}$$

9.8 SUM TO *n* TERMS OF SPECIAL SERIES

As a school boy, the German mathematician Carl Friedrich Gauss calculated the sum of the first 100 natural numbers, i.e., $1 + 2 + 3 + ... + 99 + 100$ in a simple way as given below



In this way of addition by grouping terms, we get 50 sums of 101

i.e., $1 + 2 + ... + 99 + 100 = (50)\ 101 = \left(\frac{100}{2}\right) 101$ (100 → Number of terms; 101 → Sum of 1st and last term)

This formula can be extended to any number of natural numbers.

1. Sum of first *n* natural numbers.

$\sum n = 1 + 2 + 3 + ... + n$, an A.P. with $a = 1, d = 1$

$\therefore\ 1 + 2 + ...\ n = \frac{n}{2}(n+1) = \frac{n(n+1)}{2}$ **(September 2012, March 2014)**

STUDY TIP

$1 + 2 + ... + n$

$= \sum n = \frac{n(n+1)}{2}$

2. Sum of squares of first *n* natural numbers.

Consider the identity

$x^3 - (x - 1)^3 = 3x^2 - 3x + 1$

Put $x = 1, 2, 3, ...\ n$ successively we get

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$
$$2^3 - 1^3 = 3(2^2) - 3(2) + 1$$
$$3^3 - 2^3 = 3(3^2) - 3(3) + 1$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$
$$n^3 - (n-1)^3 = 3(n^2) - 3n + 1$$

Adding we get, $n^3 = 3[1^2 + 2^2 + ... + n^2] - 3[1 + 2 + ... + n] + n$

i.e., $n^3 = 3\sum n^2 - 3\sum n + n$

i.e., $n^3 = 3\sum n^2 - 3\sum n + n$ $\qquad 3\sum n^2 = n^3 + 3\frac{n(n+1)}{2} - n$

$3\sum n^2 = \frac{2n^3 + 3n(n+1) - 2n}{2} = \frac{n(n+1)(2n+1)}{2}$ $\qquad \therefore \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

ie., $1^2+2^2+3^2+\ldots\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

3. Sum of cubes of first n natural numbers

Consider the identity

$x^4-(x-1)^4=4x^3-6x^2+4x-1$

Put $x = 1, 2, 3, \ldots n$ successively we get,

$$\begin{array}{llllll}
1^4-0^4 & =4(1)^3 & -6(1)^2 & +4(1) & -1 \\
2^4-1^4 & =4(2)^3 & -6(2)^2 & +4(2) & -1 \\
3^4-2^4 & =4(3)^3 & -6(3)^2 & +4(3) & -1 \\
\ldots & & & & \\
\ldots & & & & \\
(n-1)^4-(n-2)^4 & =4(n-1)^3 & -6(n-1)^2 & +4(n-1) & -1 \\
n^4-(n-1)^4 & =4(n^3) & -6n^2 & +4n & -1 \\
\hline
\text{Adding we get, } n^4 & =4\Sigma n^3 & -6\Sigma n^2 & +4\Sigma n & -n
\end{array}$$

$\therefore 4\Sigma n^3=n^4+6\Sigma n^2-4\Sigma n+n$

$=n^4+n(n+1)(2n+1)-2n(n+1)+n$

$=n[n^3+2n^2+2n+n+1-2n-2n+1]$

$=n[n^3+2n^2+n)=n^2(n^2+2n+1)$

$=n^2(n+1)^2$

$\therefore \Sigma n^3=\dfrac{n^2(n+1)^2}{4}=\left[\dfrac{n(n+1)}{2}\right]^2$

ie., $1^3+2^3+3^3+\ldots\ldots+n^3=\left[\dfrac{n(n+1)}{2}\right]^2$

Example 53

Find the sum to n terms of the series whose n^{th} term is $n(n+3)$. **(March 2015)**

Solution

$a_n=n(n+3)=n^2+3n$

Sum to n terms $=\Sigma a_n$

$=\Sigma(n^2+3n)$

$=\Sigma n^2+3\Sigma n$

$=\dfrac{n(n+1)(2n+1)}{6}+\dfrac{3n(n+1)}{2}$

$$= n(n+1)\left[\frac{2n+1}{6}+\frac{3}{2}\right]$$

$$= n(n+1)\left[\frac{(2n+1+9)}{6}\right]$$

$$= \frac{n(n+1)2(n+5)}{6}$$

$$= \frac{n(n+1)(n+5)}{3}$$

Example 54

Find the sum to n terms of the series

$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$ **(NCERT)**

Solution

Let a_n be the n^{th} term.

$\therefore a_n = (n^{th}$ term of A.P 1, 2, 3, ...) (n^{th} term of AP 2, 3, 4, ...) $= n(n+1) = n^2 + n$

Sum to n terms $= \Sigma(n^2 + n)$

$$= \Sigma n^2 + \Sigma n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]$$

$$= \frac{n(n+1)(2n+4)}{6} = \frac{n(n+1)(n+2)}{3}$$

Example 55

i. Find the n^{th} term of the series $2 \times 3 + 3 \times 4 + 4 \times 5 +$ ....

ii. Find the sum of first n terms of this series. **(March 2010)**

Solution

i. Let a_n be the n^{th} term.

$\therefore a_n = (n^{th}$ term of the A.P 2, 3, 4 ....) (n^{th} term of the A.P 3, 4, 5 ....)

$= (n+1)(n+2) = n^2 + 3n + 2$

ii. Sum to n terms $= \Sigma(n^2 + 3n + 2) = \Sigma n^2 + 3\Sigma n + 2\Sigma(1)$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n$$

$$= \frac{n}{6}[(n+1)(2n+1) + 9(n+1) + 12] = \frac{n}{6}(2n^2 + 12n + 22)$$

$$= \frac{n}{3}(n^2 + 6n + 11)$$

Example 56

Find the sum to n terms of the series $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$ **(NCERT)**

Solution

Let a_n be the n^{th} term.

$\therefore a_n = (n^{th}$ term of A.P 3, 5, 7, ...) (n^{th} term of $1^2, 2^2, 3^2, \ldots) = (2n+1)(n^2) = 2n^3 + n^2$

$$\therefore \text{Sum to } n \text{ terms} = 2\sum n^3 + \sum n^2 = 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[2\left(\frac{n(n+1)}{2}\right) + \frac{2n+1}{3}\right] = \frac{n(n+1)}{2}\left[\frac{3n(n+1)+2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+2n+1}{3}\right] = \frac{n(n+1)}{2}\left[\frac{3n^2+5n+1}{3}\right]$$

$$= \frac{n(n+1)(3n^2+5n+1)}{6}$$

Example 57

Find the sum to n terms of the series whose n^{th} term is $n(n+1)(n+4)$ **(NCERT)**

Solution

$a_n = n(n+1)(n+4) = n(n^2+5n+4) = n^3 + 5n^2 + 4n$

$$\therefore \text{Sum to } n \text{ terms} = \sum n^3 + 5\sum n^2 + 4\sum n$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} = \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)+10(2n+1)+24}{6}\right] = \frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right]$$

$$= \frac{n(n+1)}{2}\left(\frac{3n^2+23n+34}{6}\right) = \frac{n(n+1)}{12}\left(3n^2+23n+34\right)$$

Example 58

Find the sum to n terms of the series $5 + 11 + 19 + 29 + 41 + \ldots\ldots$ **(NCERT)**

Solution

Let $S_n = 5 + 11 + 19 + 29 + \ldots. + a_{n-1} + a_n$

Again $S_n = 5 + 11 + 19 + \ldots. + a_{n-2} + a_{n-1} + a_n$

Substracting, we get

$0 = 5 + 6 + 8 + 10 + \ldots.$ to $(n - 1)$ terms $- a_n$

$\therefore a_n = 5 + 6 + 8 + 10 + \ldots.$ to $n - 1$ terms

$= 5 + [6 + 8 + 10 + \ldots.$ to $n - 1$ terms$]$

$= 5 + \left(\frac{n-1}{2}\right)[2 \times 6 + (n-2)2]$, Since 6, 8, 10, ... is an AP with $a = 6, d = 2$

$$= 5 + \frac{(n-1)(12+2n-4)}{2}$$

$$= 5 + \frac{(n-1)(2n+8)}{2}$$

$= 5 + (n - 1)(n + 4)$

$= 5 + n^2 + 3n - 4$

$= n^2 + 3n + 1$

$\therefore$ Sum to n terms $= \sum a_n$

$$= \sum(n^2 + 3n + 1)$$

$$= \sum n^2 + 3\sum n + \sum 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n$$

$$= n\left[\frac{2n^2+3n+1}{6} + \frac{3n+3}{2} + 1\right]$$

$$= n\left[\frac{2n^2+3n+1+9n+9+6}{6}\right]$$

$$= \frac{n[2n^2+12n+16]}{6}$$

$$= \frac{n[n^2+6n+8]}{3} = \frac{n(n+2)(n+4)}{3}$$

SOLUTIONS TO NCERT TEXT BOOK EXERCISE 9.5

Find the sum to n terms of each of the series in Questions 1 to 7.

1. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$ **(March 2013, March 2014)**

Solution

Refer example 54

2. $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots$ **(August 2014)**

Solution

Let a_n be the n^{th} term

$a_n = (n^{th}$ term of A.P 1, 2, 3, ...)
$\times$ (n^{th} term of A.P 2, 3, 4, ...)
$\times$ (n^{th} term of A.P 3, 4, 5, ...)

$= n(n+1)(n+2)$

$= (n^2 + n)(n+2)$

$= n^3 + 2n^2 + n^2 + 2n$

$= n^3 + 3n^2 + 2n$

$\therefore$ Sum to n terms $= \sum(n^3 + 3n^2 + 2n)$

$$= \sum n^3 + 3\sum n^2 + 2\sum n$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 3\left[\frac{n(n+1)(2n+1)}{6}\right] + 2\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + (2n+1) + 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1) + 2(2n+1) + 4}{2}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n^2 + n + 4n + 6}{2}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n^2 + 5n + 6}{2}\right]$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$

Solution

Refer example 56

4. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots$

Solution

Let a_n be the n^{th} term

i.e., $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$a_1 = \frac{1}{1} - \frac{1}{2}$ $\quad a_2 = \frac{1}{2} - \frac{1}{3}$

$a_3 = \frac{1}{3} - \frac{1}{4}$ $\quad \ldots \quad \ldots$

$a_n = \frac{1}{n} - \frac{1}{n+1}$

Sum $= a_1 + a_2 + \ldots + a_n$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

5. $5^2 + 6^2 + 7^2 + \ldots + 20^2$

Solution

Sum $= (1^2 + 2^2 + \ldots + 20^2) - (1^2 + 2^2 + 3^2 + 4^2)$

$$= \frac{20(20+1)(40+1)}{6} - 30$$

$$= \frac{17220}{6} - 30 = 2870 - 30 = 2840$$

6. $3 \times 8 + 6 \times 11 + 9 \times 14 + \ldots$ **(September 2012)**

Solution

Let a_n be the n^{th} term

$a_n =$ (n^{th} term of 3, 6, 9 ...)
$\times$ (n^{th} term of 8, 11, ...)

$= [3 + (n-1)3]\,[8 + (n-1)3]$

$= 3n(3n+5) = 9n^2 + 15n$

Sum $= 9\sum n^2 + 15\sum n$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 15\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\,[3(2n+1)+15)]$$

$$= \frac{n(n+1)}{2}(6n+18)$$

$$= n(n+1)(3n+9)$$

$$= 3n(n+1)(n+3)$$

7. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$

Solution

Let a_n be the n^{th} term

$a_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

$\therefore$ Sum to n terms

$$= \frac{1}{6}\left[2\sum n^3 + 3\sum n^2 + \sum n\right]$$

$$= \frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2 + 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{6}\left[\frac{[n(n+1)]^2}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[n(n+1)+(2n+1)+1]$$

$$= \frac{n(n+1)}{12}(n^2+n+2n+2)$$

$$= \frac{n(n+1)}{12}(n^2+3n+2)$$

$$= \frac{n(n+1)(n+1)(n+2)}{12}$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

Find the sum to n terms of the series in Questions 8 to 10 whose n^{th} terms is given by

8. $n(n+1)(n+4)$

Solution ***Refer example 57***

9. $n^2 + 2^n$

Solution

Let $a_n = n^2 + 2^n$

$\therefore$ Sum to n terms $= \sum n^2 + \sum 2^n$

$$= \frac{n(n+1)(2n+1)}{6} + [2 + 2^2 + 2^3 + \ldots\text{ to } n \text{ terms}]$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(2^n-1)}{2-1}$$

$$= \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

10. $(2n-1)^2$

Solution

Let $a_n = (2n-1)^2 = 4n^2 - 4n + 1$

$\therefore$ Sum to n terms

$$= 4\sum n^2 - 4\sum n + \sum(1)$$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{n}{2}\left[\frac{4(n+1)(2n+1)}{3} - 4(n+1) + 2\right]$$

$$= \frac{n}{2}\left[\frac{4(2n^2+3n+1) - 12n - 12 + 6}{3}\right]$$

$$= \frac{n}{6}\left[8n^2 + 12n + 4 - 12n - 6\right]$$

$$= \frac{n}{6}(8n^2 - 2)$$

$$= \frac{n}{3}(4n^2 - 1) = \frac{n}{3}(2n+1)(2n-1)$$