8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ *n* terms.

Solution

$$a=\sqrt{7}, r=\frac{\sqrt{21}}{\sqrt{7}}=\frac{(\sqrt{7})(\sqrt{3})}{(\sqrt{7})}=\sqrt{3}$$

$$S_n=\frac{a(r^n-1)}{r-1}=\frac{\sqrt{7}((\sqrt{3})^n-1)}{(\sqrt{3}-1)}$$

$$=\frac{\sqrt{7}(\sqrt{3}+1)((\sqrt{3})^n-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{\sqrt{7}}{2}(\sqrt{3}+1)(3^{n/2}-1)$$

9. $1, -a, a^2, -a^3, \ldots$ *n* terms (if $a \neq -1$).

Solution

Let $a=1$, $r=-a$, where $a \neq -1$

$$\therefore S_n=\frac{a(1-r^n)}{1-r}=\frac{1(1-(-a)^n)}{1-(-a)}$$

$$=\frac{[1-(-a)^n]}{1+a}$$

10. $x^3, x^5, x^7, \ldots$ *n* terms (if $x \neq \pm 1$).

Solution

Let $a=x^3$, $r=x^2$, where $x \neq \pm 1$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{x^3(1-(x^2)^n)}{1-x^2}$$

$$= \frac{x^3(1-x^{2n})}{1-x^2}$$

11. Evaluate $\sum_{k=1}^{11} (2+3^k)$

Solution

We have $\sum_{k=1}^{11}(2+3^k)$

$= [(2 + 3^1) + (2 + 3^2) + ... + (2 + 3^{11})]$

$= [2 + 2 + ... 11 \text{ terms}]$

$+ [3 + 3^2 + ... + 3^{11}]$

$= [22 + S_{11}]$

[where $S_{11} = 3 + 9 + 27 + ... 11$ terms]

Now $S_{11} = \left[\frac{3(3^{11}-1)}{3-1}\right] = \frac{3}{2}(3^{11}-1)$

Here $a = 3$, $r = 3$ and $n = 11$

$\therefore \sum_{k=1}^{11}(2+3^k) = 22 + \frac{3}{2}(3^{11}-1)$

Another Method

$\sum_1^{11}(2+3^k) = \sum_1^{11} 2 + \sum_1^{11} 3^k$

$= 2\sum_1^{11}(1) + (3^1 + 3^2 + ... + 3^{11})$

$= 2 \times 11 + \frac{3(3^{11}-1)}{3-1}$ since $3, 3^2, ...3^{11}$

is a G.P with $a = 3$, $r = 3$, $n = 1$

$= 22 + \frac{3}{2}(3^{11}-1)$

12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

(March 2008, 2011, September 2013)

Solution

Let $\frac{a}{r}$, a, ar be the first three terms of a G.P.

$\therefore \frac{a}{r} + a + ar = \frac{39}{10}$ ——— (1) and

$\left(\frac{a}{r}\right) a(ar) = 1 \Rightarrow a^3 = 1 \Rightarrow a = 1$ ——— (2)

Substituting the value of a in (1) we get

$\frac{1}{r} + 1 + r = \frac{39}{10}$

$\Rightarrow 10 + 10r + 10r^2 - 39r = 0$

$\Rightarrow 10r^2 - 29r + 10 = 0$

$\Rightarrow 10r^2 - 25r - 4r + 10 = 0$

$\Rightarrow 5r(2r - 5) - 2(2r - 5) = 0$

$\Rightarrow (2r - 5)(5r - 2) = 0$

$\Rightarrow r = \frac{5}{2}$ or $\frac{2}{5}$

When $r = \frac{5}{2}$ the terms are $\frac{2}{5}$, 1, $\frac{5}{2}$ and

when $r = \frac{2}{5}$ the terms are $\frac{5}{2}$, 1, $\frac{2}{5}$

13. How many terms of G.P. 3, 3^2, 3^3, ... are needed to give the sum 120?

Solution

Let $a = 3$, $r = 3$ and $S_n = 120$

$\therefore \frac{a(r^n - 1)}{r - 1} = 120$

$\Rightarrow \frac{3(3^n - 1)}{3 - 1} = 120$

$\Rightarrow \frac{3}{2}(3^n - 1) = 120$

$\Rightarrow 3^n - 1 = \frac{120 \times 2}{3} = 80$

$\Rightarrow 3^n = 81 \Rightarrow 3^n = 3^4 \Rightarrow n = 4$

$\therefore$ Number of terms = 4

15. Given a G.P. with $a = 729$ and 7^{th} term 64, determine S_7.

Solution

Given that $a = 729$ and $a_7 = 64$

i.e., $ar^6 = 64 \Rightarrow (729)r^6 = 64$

$$\Rightarrow r^6 = \frac{64}{729} = \left(\frac{2}{3}\right)^6 \Rightarrow r = \frac{2}{3}$$

$$\therefore S_7 = \frac{a(1-r^7)}{1-r} = \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-\frac{2}{3}}$$

$$= \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{\left(\frac{1}{3}\right)}$$

$$= 2187\left[1-\left(\frac{2}{3}\right)^7\right] = 2187\left[1-\frac{128}{2187}\right]$$

$$= 2187\left[\frac{2059}{2187}\right] = 2059$$

16. Find a G.P. for which sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution

Given that $a + ar = -4$,

$a(1 + r) = -4$ ——— (1)

Also $a_5 = 4\,a_3$ i.e., $ar^4 = 4\,ar^2$

$\therefore r^2 = 4 \Rightarrow r = 2$ or $r = -2$

When $r = 2$, (1) $\Rightarrow a(1 + 2) = -4$

$\therefore\ a = \frac{-4}{3}$

Hence the required G.P. is

$$\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$$

When $r = -2$, (1) $\Rightarrow a(1-2) = -4$,

$$a = 4$$

Hence the required G.P. is

$$4, -8, 16, -32, 64, \ldots$$

17. If the 4th, 10th and 16th terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

Solution

Given that $a_4 = x$, $a_{10} = y$ and $a_{16} = z$

$$\frac{y}{x} = \frac{a_{10}}{a_4} = \frac{ar^9}{ar^3} = r^6 \text{ ——— (1) and}$$

$$\frac{z}{y} = \frac{a_{16}}{a_{10}} = \frac{ar^{15}}{ar^9} = r^6 \text{ ——— (2)}$$

From (1) and (2), $\frac{y}{x} = \frac{z}{y}$

$\therefore$ x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888 ...

Solution

We have $S_n = 8 + 88 + 888 + \ldots$ upto n terms

$= 8\,[1 + 11 + 111 + \ldots$ to n terms$]$

$= \frac{8}{9}\,[9 + 99 + 999 + \ldots$ to n terms$]$

$= \frac{8}{9}\,[(10-1) + (100-1) + (1000-1) + \ldots]$

$= \frac{8}{9}\,[(10 + 100 + 1000 + \ldots) - (1 + 1 + \ldots n \text{ terms})$

$$= \frac{8}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right]$$

$$= \frac{8}{9}\left[\frac{10}{9}(10^n - 1) - n\right] = \frac{80}{81}(10^n - 1) - \frac{8}{9}n$$

19. Find the sum of the products of the corresponding terms of the sequences

$$2, 4, 8, 16, 32 \text{ and } 128, 32, 8, 2, \frac{1}{2}$$

Solution

The sum of the product of the corresponding terms

$$S_n = 2(128) + 4(32) + 8(8) + 16(2) + 32\left(\frac{1}{2}\right)$$

$= 2\,[128 + 64 + 32 + 16 + 8]$

$= 2\,[8 + 16 + 32 + 64 + 128]$

$$= 2\left[\frac{8(2^5 - 1)}{(2-1)}\right] = 2\left[\frac{8(31)}{1}\right] = 496$$

20. Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and $A, AR, AR^2, \ldots AR^{n-1}$ form a G.P. and find the common ratio.

Solution

Consider the sequence, aA, $arAR$, ar^2AR^2, ...

$$\frac{arAR}{aA} = rR \text{ ——— (1)}$$

$$\frac{ar^2AR^2}{arAR} = rR \text{ ——— (2)}$$

Hence the above sequence forms a G.P. with common ratio rR.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

Solution

Let a be the first term and r be the common ratio.

Hence the numbers are a, ar, ar^2, ar^3

$a_3 = a + 9 \Rightarrow ar^2 = a + 9$

i.e., $ar^2 - a = 9$

i.e., $a(r^2 - 1) = 9$ ——— (1)

$a_2 = a_4 + 18 \quad \Rightarrow ar = ar^3 + 18,$

$ar - ar^3 = 18,$

$ar(1 - r^2) = 18$

$-ar(r^2 - 1) = 18$ ——— (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{-ar(r^2-1)}{a(r^2-1)} = \frac{18}{9}$$

i.e., $-r = 2 \Rightarrow r = -2$

From (1) we get $a = \dfrac{9}{4-1} = \dfrac{9}{3} = 3$

$\therefore$ The required numbers are, a, ar, ar^2 and ar^3

i.e., 3, – 6, 12 and –24

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$

Solution

Let A be the first term and R be the common ratio of the G.P.

$a_p = a \Rightarrow AR^{p-1} = a$

$\therefore\ a^{q-r} = A^{q-r}.\ R^{(p-1)(q-r)}$

$a^{q-r} = A^{q-r}.\ R^{(pq-pr-q+r)}$ .... (1)

$a_q = b \Rightarrow AR^{q-1} = b$

$\therefore\ b^{r-p} = A^{r-p}.\ R^{(q-1)(r-p)}$

$b^{r-p} = A^{r-p}.\ R^{(qr-qp-r+p)}$............. (2)

$a_r = c \Rightarrow AR^{r-1} = c$

$\therefore\ c^{p-q} = A^{p-q}.\ R^{(r-1)(p-q)}$

$c^{p-q} = A^{p-q}.\ R^{(rp-rq-p+q)}$ .... (3)

(1) × (2) × (3) $\rightarrow$

$a^{q-r}.\ b^{r-p}.\ c^{p-q} = A^{q-r}.\ R^{(pq-pr-q+r)}$

$A^{r-p}.\ R^{(qr-qp-r+p)}\ A^{p-q}.\ R^{(rp-rq-p+q)}$

$= A^{(q-r+r-p+p-q)}.$

$R^{(pq-pr-q+r+qr-qp-r+p+rp-rq-p+q)}$

$= A^0 . R^0 = (1)(1) = 1$

23. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$

Solution

First term = a, $\quad n^{th}$ term $= b$

$a_n = b \Rightarrow ar^{n-1} = b$

$$\Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{1/n-1} \quad \text{——— (1)}$$

P = product of first n terms

$= a(ar)\,(ar^2)\,(ar^3)\ \dots\ (ar^{n-1})$

$$P = a^n r^{1+2+3+\dots+(n-1)} = a^n\, r^{\left(\frac{n(n-1)}{2}\right)}$$

$\therefore P^2 = a^{2n}\, r^{n(n-1)}$

$$= a^{2n}\left[\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right]^{n(n-1)}$$

$$= a^{2n}\left(\frac{b}{a}\right)^n = a^{2n}\,\frac{b^n}{a^n} = a^n b^n$$

$\Rightarrow P^2 = (ab)^n$

24. Show that the ratio of the sum of first n terms of a G.P to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\dfrac{1}{r^n}$

Solution

Consider a G.P. with first term a and common ratio r

$\therefore$ Sum of first n terms $= \dfrac{a(r^n - 1)}{r - 1}$

To find the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$, consider first term as ar^n with common ratio r

Number of terms from $(n+1)^{th}$ to $(2n)^{th}$

$$= [2n - (n+1)] + 1 = n$$

$\therefore$ Sum of terms from $(n+1)^{th}$ to $(2n)^{th}$

$$= \frac{ar^n(r^n - 1)}{r - 1}$$

$$\therefore \text{Ratio} = \frac{\frac{a(r^n - 1)}{r - 1}}{\frac{ar^n(r^n - 1)}{r - 1}} = \frac{1}{r^n}$$

25. If a, b, c and d are in G.P. show that

$$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2.$$

Solution

Let r be the common ratio

Given that a, b, c, d are in G.P.

$b = ar$, $c = ar^2$ and $d = ar^3$

$\therefore (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= a^2(1 + r^2 + r^4) \cdot a^2r^2(1 + r^2 + r^4)$

$= a^4r^2(1 + r^2 + r^4)^2$ ——— (1)

$(ab + bc + cd)^2 = [a(ar) + (ar)(ar^2)$

$+ (ar^2)(ar^3)]^2$

$= [a^2r + a^2r^3 + a^2r^5]^2$

$= [a^2r(1 + r^2 + r^4)]^2$

$= a^4r^2(1 + r^2 + r^4)^2$ ——— (2)

From (1) and (2) we get

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (ab + bc + cd)^2$