Example 12 (March 2011)

Consider the expansion of $(x + 3)^8$.

i. Write its general term.

ii. Hence find the third term.

Solution

i. General term $T_{r+1} = {}^nC_r a^{n-r} b^r$

$= {}^8C_3 x^{8-r} 3^r$

ii. Put $r = 2$ in T_{r+1}, we get the third term $T_3 = {}^8C_2 x^{8-2} 3^2$

$= 28 \times 9x^6$

$= 252\, x^6$

Example 13

i. Write the general term in the expansion of $\left(\frac{x}{2}+\frac{1}{x}\right)^6$.

ii. What is the fourth term in the expansion of $\left(\frac{x}{2}+\frac{1}{x}\right)^6$? (March 2010)

Solution

i. The general term $T_{r+1} = {}^6C_r\left(\frac{x}{2}\right)^{6-r}\left(\frac{1}{x}\right)^r = {}^6C_r\left(\frac{1}{2}\right)^{6-r} x^{6-2r}$

ii. Fourth term = T_4

Put $r = 3$ in T_{r+1}

$\therefore T_4 = {}^6C_3\left(\frac{1}{2}\right)^{6-3} x^{6-2(3)}$ or $T_4 = 20\left(\frac{1}{2}\right)^3 = \frac{20}{8} = \frac{5}{2}$

Example 14

Find the r^{th} term from the end in the expansion of $(x + a)^n$

Solution

There are $(n + 1)$ terms in the expansion.

The first term from the end = $(n + 1)^{th}$ term = $(n + 2 - 1)^{th}$ term

The second term from the end = n^{th} term = $(n + 2 - 2)^{th}$ term

The third term from the end = $(n - 1)^{th}$ term = $(n + 2 - 3)^{th}$ term and so on.

$\therefore$ The r^{th} term from the end = $(n + 2 - r)^{th}$ term

$= (n - r + 2)^{th}$ term

$= T_{n-r+2}$

$= T_{(n-r+1)+1} = {}^nC_{n-r+1} x^{n-(n-r+1)}.a^{(n-r+1)}$

$= {}^nC_{n-r+1} x^{r-1}.a^{n-r+1}$

Example 15

Find the coefficient of a^4 in the product $(1 + 2a)^4(2 - a)^5$ using binomial theorem. **(NCERT)**

Solution

$$(1 + 2a)^4 = {}^4C_0 + {}^4C_1(2a) + {}^4C_2(2a)^2 + {}^4C_3(2a)^3 + {}^4C_4(2a)^4$$
$$= 1 + 4(2a) + 6(4a^2) + 4(8a^3) + 16a^4$$
$$(1 + 2a)^4 = 1 + 8a + 24a^2 + 32a^3 + 16a^4$$
$$(2 - a)^5 = {}^5C_0 2^5 - {}^5C_1 2^4(a) + {}^5C_2 2^3(a)^2 - {}^5C_3 2^2(a)^3 + {}^5C_4 2(a)^4 - {}^5C_5(a)^5$$
$$= 32 - 5 \times 16a + 10 \times 8a^2 - 10 \times 4a^3 + 5 \times 2a^4 - a^5$$
$$(2 - a)^5 = 32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5$$
$$\therefore (1 + 2a)^4(2 - a)^5 = (1 + 8a + 24a^2 + 32a^3 + 16a^4)(32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5)$$

Term containing $a^4 = 1 \times 10a^4 + 8a(-40a^3) + 24a^2(80a^2) + 32a^3(-80a) + 16a^4(32)$

$$= 10a^4 - 320a^4 + 1920a^4 - 2560a^4 + 512a^4 = -438a^4$$

$\therefore$ Coefficient of $a^4 = -438$

Example 16

The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio $1 : 7 : 42$. Find n **(NCERT)**

Solution

Let T_{r-1}, T_r and T_{r+1} be the three consecutive terms.

$T_{r-1} = {}^nC_{r-2}\,a^{r-2}$ $\therefore$ Coefficient of $T_{r-1} = {}^nC_{r-2}$

$T_r = {}^nC_{r-1}\,a^{r-1}$ $\therefore$ Coefficient of $T_r = {}^nC_{r-1}$

$T_{r+1} = {}^nC_r\,a^r$ $\therefore$ Coefficient of $T_{r+1} = {}^nC_r$

The coefficients of T_{r-1}, T_r and T_{r+1} are in the ratio $1 : 7 : 42$.

i.e., $\dfrac{\text{coefficient of } T_{r-1}}{\text{coefficient of } T_r} = \dfrac{1}{7}$ and $\dfrac{\text{coefficient of } T_r}{\text{coefficient of } T_{r+1}} = \dfrac{7}{42}$

$$\Rightarrow \frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}$$

$$\Rightarrow \frac{\dfrac{n!}{(r-2)!(n-r+2)!}}{\dfrac{n!}{(r-1)!(n-r+1)!}} = \frac{1}{7}$$

$$\Rightarrow \frac{(r-1)!(n-r+1)!}{(r-2)!(n-r+2)!} = \frac{1}{7}$$

$$\Rightarrow \frac{r-1}{n-r+2} = \frac{1}{7}$$

$$\Rightarrow \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42} = \frac{1}{6}$$

$$\frac{\dfrac{n!}{(r-1)!(n-r+1)!}}{\dfrac{n!}{r!(n-r)!}} = \frac{1}{6}$$

$$\frac{r!(n-r)!}{(r-1)!(n-r+1)!} = \frac{1}{6}$$

$$\frac{r}{n-r+1} = \frac{1}{6} \Rightarrow 6r = n - r + 1$$

$$n - 7r + 1 = 0 \quad \text{.........(ii)}$$

$\Rightarrow 7r - 7 = n - r + 2$

$\Rightarrow n - 8r + 9 = 0$ ....(i)

(i) – (ii) gives $-r + 8 = 0 \Rightarrow r = 8$

(i) $\rightarrow n - 8 \times 8 + 9 = 0 \Rightarrow n = 55$

Hence $n = 55$

Middle term

The expansion of $(a + b)^n$ contains $(n + 1)$ terms.

Case i **:** If n is even, then the number of terms in the expansion is odd.

Therefore $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

For example, the middle term in the expansion of $(a + b)^6$ is $\left(\frac{6}{2}+1\right)^{th}$

i.e., 4^{th} term.

Case ii **:** If n is odd, then the number of terms in the expansion is even. Therefore there will be two middle terms.

The $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms are the two middle terms.

For example, the middle terms in the expansion of $(a + b)^7$ are $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$ i.e., 4^{th} and 5^{th} terms are the middle terms.

Example 17

Consider the expansion of $(2x - 3)^6$.

a. How many terms are there in the expansion?

b. Write the general term of the expansion.

c. Find the middle term of the expansion. **(March 2010)**

Solution

a. The expansion of $(a + b)^n$, $n \in \mathbf{N}$ contains $n + 1$ terms.

Hence the expansion of $(2x - 3)^6$ contains 7 terms.

b. The general term of $(a + b)^n$ is $T_{r+1} = {}^nC_r a^{n-r} b^r$

$\therefore$ General term $T_{r+1} = {}^6C_r (2x)^{6-r}(-3)^r = (-1)^r\, {}^6C_r\, 2^{6-r}\, 3^r\, (x)^{6-r}$

c. Since the exponent 6 is even, there is only one middle term.

$\therefore \left(\frac{6}{2}+1\right)^{th} = 4^{th}$ term is the middle term.

$\therefore T_4 = (-1)^3\, {}^6C_3 2^{6-3}\, 3^3 x^{6-3} = (-1)\,(20)\,(8)\,(27)\, x^3 = -4320\, x^3$

Example 18

i. The number of terms in the expansion of $\left(x+\frac{2}{x^2}\right)^6$ is **(March 2015)**

a. 7 b. 6 c. 12 d. 3

ii. Find the middle term in the expansion of $\left(x+\frac{2}{x^2}\right)^6$

Solution

i. 7

ii. The general term $T_{r+1} = {}^6C_r x^{6-r}\left(\frac{2}{x^2}\right)^r$

There are 7 terms.

The middle term is $\left(\frac{7+1}{2}\right)^{th}$ term = 4^{th} term

$\therefore$ Put $r = 3$ in the T_{r+1}, we get

$$T_4 = {}^6C_3 x^{6-3}\left(\frac{2}{x^2}\right)^3 = 20x^3 \times \frac{8}{x^6} = \frac{160}{x^3}$$

Example 19

Consider the expansion of $\left(\frac{x}{9}+9y\right)^{2n}$.

i. The number of terms in the above expansion is ....

ii. What is its $(n + 1)^{th}$ term?

iii. If $n = 5$, find its middle term.

Solution

i. $2n + 1$

ii. The general term $T_{r+1} = {}^nC_r a^{n-r} b^r = {}^{2n}C_r\left(\frac{x}{9}\right)^{2n-r} (9y)^r$

Put $r = n$ in T_{r+1}, we get

$$T_{n+1} = {}^{2n}C_n\left(\frac{x}{9}\right)^{2n-n} (9y)^n = {}^{2n}C_n x^n y^n$$

iii. When $n = 5$, the exponent of $\left(\frac{x}{9}+9y\right)$ is 10, an even integer.

Hence the middle term is $\left(\frac{10}{2}+1\right)^{th}$ term. i.e, 6^{th} term.

Put $n = 5$ in T_{n+1}, we get

$$T_6 = {}^{10}C_5 x^5 y^5 = 252\, x^5 y^5$$

Term independent of variable

If the general term in the binomial expansion is a constant, then that term is called the term **independent of the variable.**

Example 20

i. Find the general term in the expansion of $\left(3x^2 - \frac{1}{3x}\right)^9$.

ii. Find the term independent of x in the above expansion. **(March 2010)**

Solution

i. The general term $T_{r+1} = {}^nC_r a^{n-r} b^r$

$$= {}^9C_r (3x^2)^{9-r}\left(\frac{-1}{3x}\right)^r = (-1)^r\, {}^9C_r 3^{9-r} x^{18-2r}\left(\frac{1}{3}\right)^r\left(\frac{1}{x}\right)^r$$

$$= (-1)^r\, {}^9C_r 3^{9-2r} x^{18-3r}$$

ii. The term will be independent of x, if index of x in T_{r+1} is zero.
i.e., $18 - 3r = 0 \Rightarrow r = 6$

$\therefore T_7 = (-1)^6\, {}^9C_6 3^{9-12} = \frac{84}{27} = \frac{28}{9}$ is independent of x.

Example 21

Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$.

(March 2009, August 2009, NCERT)

Solution

In the Binomial expansion of $(a + b)^n$, the general term $T_{r+1} = {}^nC_r a^{n-r} b^r$

$\therefore$ The general term in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$

$$T_{r+1} = {}^6C_r\left(\frac{3}{2}x^2\right)^{6-r}\left(\frac{-1}{3x}\right)^r = (-1)^r\, {}^6C_r\left(\frac{3}{2}\right)^{6-r}\left(\frac{1}{3}\right)^r (x^2)^{6-r}\left(\frac{1}{x}\right)^r.$$

$$= (-1)^r\, {}^6C_r\left(\frac{3}{2}\right)^{6-r}\left(\frac{1}{3}\right)^r x^{12-2r}.x^{-r} = (-1)^r\, {}^6C_r\left(\frac{3}{2}\right)^{6-r}\left(\frac{1}{3}\right)^r .x^{12-3r},$$

The term will be independent of x if the index of x in T_{r+1} is zero.
i.e., $12 - 3r = 0 \Rightarrow r = 4$
i.e., T_5 is independent of x.

$$T_5 = (-1)^4\, {}^6C_4\left(\frac{3}{2}\right)^{6-4}\left(\frac{1}{3}\right)^4 = 15 \times \frac{3^2}{2^2} \times \frac{1}{3^4} = \frac{5}{12}$$

Example 22

i. Find the number of terms in the expansion of $\left(x-\frac{1}{x}\right)^{14}$

ii. Find the general term in the expansion of $\left(x-\frac{1}{x}\right)^{14}$

iii. Find the term independent of x in the above expansion. **(March 2013)**

Solution

i. 15 terms

ii. The general term $T_{r+1} = (-1)^r\ {}^{14}C_r x^{14-r}\left(\frac{1}{x}\right)^r = (-1)^r.{}^{14}C_r x^{14-r}\left(\frac{1}{x^r}\right)$

$T_{r+1} = (-1)^r\ {}^{14}C_r x^{14-2r}$

iii. T_{r+1} is independent of x if the index of x is zero.

i.e., $14-2r = 0 \Rightarrow r = 7$

$\therefore$ Term independent of $x = T_{7+1} = (-1)^7\ {}^{14}C_7\ x^{14-2\times7} = -({}^{14}C_7)$

Example 23.

Find the term independent of x in the expansion of $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x > 0$. **(NCERT)**

Solution

We have $T_{r+1} = {}^{18}C_r(\sqrt[3]{x})^{18-r}\left(\frac{1}{2\sqrt[3]{x}}\right)^r = {}^{18}C_r x^{\frac{18-r}{3}}.\frac{1}{2^r x^{\frac{r}{3}}} = {}^{18}C_r\frac{1}{2^r}.x^{\frac{18-2r}{3}}$

Since we have to find a term independent of x, i.e., term not having x, so take $\frac{18-2r}{3} = 0$

We get $r = 9$. $\therefore$ The required term is ${}^{18}C_9\frac{1}{2^9}$.

Example 24

Find a if the 17^{th} and 18^{th} terms of the expansion $(2 + a)^{50}$ are equal.

(NCERT, March 2014, March 2015)

Solution

The general term, $T_{r+1} = {}^{50}C_r \times 2^{50-r} a^r$

$\therefore T_{17} = T_{16+1} = {}^{50}C_{16}\times 2^{50-16}a^{16} = {}^{50}C_{16} \times 2^{34}\ a^{16}$

$T_{18} = T_{17+1} = {}^{50}C_{17}\times 2^{50-17}a^{17} = {}^{50}C_{17}2^{33}a^{17}$

$T_{17} = T_{18} \Rightarrow {}^{50}C_{16}\times 2^{34}a^{16} = {}^{50}C_{17}2^{33}a^{17}$

$\Rightarrow \frac{50!}{16!\times34!}\times2^{34}a^{16} = \frac{50!}{17!33!}\times2^{33}a^{17}$

$$\Rightarrow \frac{17! \times 33! \times 2^{34}}{16! \times 34! \times 2^{33}} = \frac{a^{17}}{a^{16}}$$

$$\Rightarrow a = \frac{17 \times 2}{34} = 1$$

$$\therefore a = 1$$

Example 25

i. Write the general term in the expansion of $(1 + x)^{44}$.

ii. Write the 21^{st} and 22^{nd} terms in the expansion of $(1 + x)^{44}$.

iii. If the 21^{st} and 22^{nd} terms in the expansion of $(1 + x)^{44}$ are equal, then find the value of x.

(October 2011)

Solution

i. General term $T_{r+1} = {}^nC_r a^{n-r} b^r$

$$= {}^{44}C_r (1)^{44-r} x^r$$

$$= {}^{44}C_r x^r$$

ii. $T_{21} = {}^{44}C_{20} x^{20}$

$T_{22} = {}^{44}C_{21} x^{21}$

iii. Given $T_{21} = T_{22}$

$${}^{44}C_{20} x^{20} = {}^{44}C_{21} x^{21}$$

$$\frac{44!\,.\,x^{20}}{(44-20)!\,20!} = \frac{44!\,.\,x^{21}}{(44-21)!\,21!}$$

$$\frac{44!\,x^{20}}{24!\,20!} = \frac{44!\,x^{21}}{23!\,21!}$$

$$\frac{1}{24.23!\,20!} = \frac{x}{23!.21.20!}$$

$$\frac{1}{24} = \frac{x}{21}$$

$$x = \frac{21}{24} \Rightarrow x = \frac{7}{8}$$

Example 26

i. What is the second term in the expansion of $(1 + x)^n$?

ii. Write the 3^{rd} and 4^{th} terms in the expansion of $(1 + x)^n$.

iii. If the coefficients of 2^{nd}, 3^{rd} and 4^{th} terms in the expansion of $(1 + x)^n$ are in A.P, then show that $n^2 - 9n + 14 = 0$.

(March 2011)

Solution

i. General term $T_{r+1} = {}^nC_r(1)^{n-r}x^r = {}^nC_r x^r$

Put $r = 1$ in T_{r+1}, we get $T_2 = {}^nC_1 x^1 = nx$

$\therefore$ Second term $= nx$

ii. Put $r = 2$ and $r = 3$ in T_{r+1}, we get

$T_3 = {}^nC_2 x^2 = \dfrac{n(n-1)}{2}x^2$

$T_4 = {}^nC_3 x^3 = \dfrac{n(n-1)(n-2)}{6}x^3$

iii. The coefficients of the 2nd, 3rd and 4th terms are n, $\dfrac{n(n-1)}{2}$ and $\dfrac{n(n-1)(n-2)}{2}$

Since these coefficients are in A.P, we get

$\dfrac{2n(n-1)}{2} = n + \dfrac{n(n-1)(n-2)}{6}$, since $2b = a + c$

$n - 1 = 1 + \dfrac{(n-1)(n-2)}{6}$

$6n - 6 = 6 + n^2 - 3n + 2$

$n^2 - 9n + 14 = 0$

Example 27

Show that the coefficient of the middle term of $(1 + x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1 + x)^{2n-1}$. **(NCERT)**

Solution

The middle term in the expansion of $(1 + x)^{2n}$ is the $\left(\dfrac{2n}{2}+1\right)^{th}$ term i.e., the $(n + 1)^{th}$ term

$T_{n+1} = {}^{2n}C_n x^n$

$\therefore$ The coefficient of $T_{n+1} = {}^{2n}C_n$ .... (1)

The middle terms in the expansion of $(1 + x)^{2n-1}$ are the $\left(\dfrac{2n-1+1}{2}\right)^{th}$ and $\left(\dfrac{2n-1+1}{2}+1\right)^{th}$ terms. i.e., the n^{th} and $(n + 1)^{th}$ terms

$\therefore T_n = {}^{(2n-1)}C_{(n-1)}x^{n-1}$ and $T_{n+1} = {}^{(2n-1)}C_n x^n$

$\therefore$ Sum of the coefficients of T_n and $T_{n+1} = {}^{(2n-1)}C_{n-1} + {}^{(2n-1)}C_n$

$= {}^{(2n-1+1)}C_n = {}^{2n}C_n$ (Since ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$) .... (2)

From (1) & (2), the coefficient of middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1 + x)^{2n-1}$

Example 28

If the coefficients of a^{r-1}, a^r and a^{r+1} in the expansion of $(1+a)^n$ are in Arithmetic Progression, prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$. **(NCERT)**

Solution

The $(r+1)^{th}$ term in the expansion is ${}^nC_r a^r$. a^r occurs in the $(r+1)^{th}$ term, and its coefficient is ${}^nC_r$. Hence the coefficients of a^{r-1}, a^r and a^{r+1} are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ respectively. Since these coefficients are in arithmetic progression, we get, ${}^nC_{r-1} + {}^nC_{r+1} = 2.{}^nC_r$

i.e., $$\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = 2\times\frac{n!}{r!(n-r)!}$$

i.e., $$\frac{1}{(r-1)!(n-r+1)(n-r)(n-r-1)!} + \frac{1}{(r+1)(r)(r-1)!(n-r-1)!}$$

$$= 2\times\frac{1}{r(r-1)!(n-r)(n-r-1)!}$$

or $$\frac{1}{(r-1)!(n-r-1)!}\left[\frac{1}{(n-r)(n-r+1)} + \frac{1}{(r+1)(r)}\right] = 2\times\frac{1}{(r-1)!(n-r-1)![r(n-r)]}$$

i.e., $$\frac{1}{(n-r+1)(n-r)} + \frac{1}{r(r+1)} = \frac{2}{r(n-r)} \text{ or } \frac{r(r+1)+(n-r)(n-r+1)}{(n-r)(n-r+1)r(r+1)} = \frac{2}{r(n-r)}$$

or $r(r+1) + (n-r)(n-r+1) = 2(r+1)(n-r+1)$

or $r^2 + r + n^2 - nr + n - nr + r^2 - r = 2(nr - r^2 + r + n - r + 1)$ or $n^2 - 4nr - n + 4r^2 - 2 = 0$

i.e., $n^2 - n(4r+1) + 4r^2 - 2 = 0$

Example 29

If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find r. **(NCERT)**

Solution

The coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms of the expansion $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$. Since they are equal we get ${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

Therefore, either $r-6 = 2r-2$ or $r-6 = 34-(2r-2)$

[if ${}^nC_r = {}^nC_p$, then either $r = p$ or $r = n - p$]

$\therefore r = -4$ or $r = 14$.

r being a natural number, $r = -4$ is not possible. $\therefore$ $r = 14$.

Example 30

The sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^m$ $x \neq 0$, m being a natural number, is 559. Find the term of the expansion containing x^3. **(NCERT)**

Solution

The coefficients of the first three terms of $\left(x-\frac{3}{x^2}\right)^m$ are ${}^mC_0$, $(-3)\,{}^mC_1$ and $9\,{}^mC_2$.

$\therefore$ By the given condition, we have ${}^mC_0 + (-3)\,{}^mC_1 + 9\,{}^mC_2 = 559$, i.e., $1-3m+\frac{9m(m-1)}{2}=559$

$\Rightarrow m = 12$ (m being a natural number).

Now $T_{r+1} = {}^{12}C_r\, x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r(-3)^r \,.\, x^{12-3r}$

Since we need the term containing x^3, put $12 - 3r = 3$ i.e., $r = 3$.

Thus, the required term T_3 is ${}^{12}C_3\,(-3)^3\,x^3 = -5940\,x^3$.

Example 31

Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5........(2n-1)}{n!}2^n x^n$, where n is a positive integer. **(NCERT)**

Solution

Since $2n$ is even, the middle term is $(n+1)^{th}$ term

$$T_{(n+1)}{}^{th}\text{ term} = {}^{2n}C_n(1)^{2n-n}x^n = \frac{1.2.3.4.......(2n-1)(2n)}{n!.n!}x^n$$

$$= \frac{[1.3.5.....(2n-1)][2.4.6....2n]}{n!.n!}x^n$$

$$= \frac{[1.3.5....(2n-1)]2^n[1.2.3.......n]}{n!.n!}x^n = \frac{1.3.5........(2n-1)}{n!}2^n x^n$$