8.2.1 Some particular cases of Binomial theorem

$(a-b)^n = (a+(-b))^n = {}^nC_0a^n + {}^nC_1a^{n-1}(-b) + {}^nC_2a^{n-2}(-b)^2 + {}^nC_3a^{n-3}(-b)^3 + \dots + {}^nC_n(-b)^n$

$(a-b)^n = {}^nC_0a^n - {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 - {}^nC_3a^{n-3}b^3 + \dots\dots\dots + (-1)^n\,{}^nC_nb^n$

Taking $a = 1$ and $b = x$ in the expansion of $(a + b)^n$, we get

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_nx^n$ **(March 2008)**

Taking $a = 1$ and $b = x$ in the expansion of $(a - b)^n$, we get

$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \dots + (-1)^n\,{}^nC_nx^n$

Take $a = 1$ and $b = 1$ in the expansion of $(a + b)^n$, we get

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$

The sum of coefficients of $(1 + x)^n$ is 2^n.

Take $a = 1$ and $b = 1$ in the expansion of $(a - b)^n$, we get

$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots\dots\dots + (-1)^n\,{}^nC_n$

The sum of coefficients of $(1 - x)^n$ is zero.

Hence ${}^nC_1 + {}^nC_3 + {}^nC_5 + \dots\dots\dots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \dots\dots\dots$

Example 4

Expand the expression $(1-2x)^5$. **(NCERT)**

Solution

By using the binomial theorem, we have

$(1-2x)^5 = {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)(2x)^4 - {}^5C_5(2x)^5$

$= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - 1(32x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Example 5

Find $(x + y)^4 - (x - y)^4$. Hence evaluate $\left(\sqrt{5}+\sqrt{6}\right)^4 - \left(\sqrt{5}-\sqrt{6}\right)^4$

(September 2012)

Solution

$(x + y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3y + {}^4C_2 x^2y^2 + {}^4C_3 xy^3 + {}^4C_4 y^4$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$\therefore (x + y)^4 - (x - y)^4 = 2[4x^3y + 4xy^3]$

$(x + y)^4 - (x - y)^4 = 8xy(x^2 + y^2)$ ....(i)

Put $x = \sqrt{5}$ and $y = \sqrt{6}$ in (i) we get

$$\left(\sqrt{5}+\sqrt{6}\right)^4 - \left(\sqrt{5}-\sqrt{6}\right)^4 = 8\sqrt{5}\sqrt{6}\left[\left(\sqrt{5}\right)^2 + \left(\sqrt{6}\right)^2\right]$$

$$= 8\sqrt{30}(5+6) = 88\sqrt{30}$$

Example 6

Prove that $\sum_{r=0}^{n} 3^r\,{}^nC_r = 4^n$ **(NCERT)**

Solution

$$\sum_{r=0}^{n} 3^r\,{}^nC_r = \sum_{r=0}^{n} {}^nC_r 3^r$$

$$= {}^nC_0(3)^0 + {}^nC_1(3)^1 + {}^nC_2(3)^2 + \ldots\ldots\ldots\ldots + {}^nC_n(3)^n$$

$$= {}^nC_0 1^n(3)^0 + {}^nC_1 1^{n-1}(3)^1 + {}^nC_2 1^{n-2}(3)^2 + \ldots\ldots + {}^nC_n(3)^n = (1+3)^n = 4^n$$

Example 7

Using binomial theorem, prove that $6^n - 5n$ always leaves remainder 1 when divided by 25, whenever n is a positive integer. **(NCERT)**

Solution

To prove $6^n - 5n$ leaves remainder 1 when divided by 25, it is enough to show

$6^n - 5n = 1 + a$ multiple of 25

$(1 + 5)^n = {}^nC_0 + {}^nC_1 \times 5 + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \ldots\ldots + {}^nC_n \times 5^n$

i.e., $6^n = 1 + 5n + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \ldots\ldots + {}^nC_n \times 5^n$

$= 1 + 5n + 25[{}^nC_2 + 5 \times {}^nC_3 + 5^2 \times {}^nC_4 + \ldots\ldots + 5^{n-2} \times {}^nC_n]$

$\therefore 6^n - 5n = 1 + 25k$, where $k = [{}^nC_2 + 5 \times {}^nC_3 + \ldots\ldots + 5^{n-2} \times {}^nC_n]$ and is a positive integer.

Hence $6^n - 5n$ leaves remainder 1 when divided by 25.

Example 8

Which is larger $(1.01)^{1000000}$ or 10,000?

Solution

$$(1.01)^{1000000} = (1 + 0.01)^{1000000}$$

$$= 1 + {}^{1000000}C_1(0.01) + {}^{1000000}C_2(0.01)^2 + \ldots$$

$$= 1 + 1000000 \times 0.01 + \text{positive number}$$

$$= 1 + 10000 + \text{a positive number}$$

$$(1.01)^{1000000} = 1 + 10000 + \text{a positive number}$$

$$\therefore (1.01)^{1000000} > 10000$$

Example 9

Consider the expansion of $\left(x^3 + \frac{1}{x}\right)^8$

i. Write the general term in the expansion.

ii. Find the coefficient of the term containing x^8.

Solution

i. General term $T_{r+1} = {}^nC_r a^{n-r} b^r = {}^8C_r (x^3)^{8-r} \left(\frac{1}{x}\right)^r$

$$= {}^8C_r\, x^{24-3r} \cdot \frac{1}{x^r} = {}^8C_r\, x^{24-4r}$$

ii. To obtain the term containing x^8, equate the power of x in T_{r+1} as 8.

i.e., $24 - 4r = 8$

$24 - 8 = 4r \quad \therefore r = 4$

$\therefore T_5 = {}^8C_4 x^8$

Hence the coefficient of $x^8 = {}^8C_4 = 70$

Example 10

Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$ **(NCERT, March 2014)**

Solution

General term, $T_{r+1} = {}^9C_r (x)^{9-r} \cdot (2y)^r$

$T_{r+1} = 2^r \times {}^9C_r \times x^{9-r} y^r$

Equating the powers of T_{r+1} with x^6y^3,

we get $9 - r = 6$ and $r = 3$

$\Rightarrow r = 3$ and $r = 3$

$\therefore$ Coefficient of $x^6y^3 = 2^3\, {}^9C_3 = \frac{8 \times 9 \times 8 \times 7}{1 \times 2 \times 3} = 672$

Example 11

Write the general term in the expansion of $(x^2 - y)^6$

(NCERT, March 2013)

Solution

$(x^2 - y)^6 = [x^2 + (-y)]^6$

The general term $T_{r+1} = {}^6C_r (x^2)^{6-r} (-y)^r$

$$= {}^6C_r x^{12-2r} (-1)^r y^r$$

$$= {}^6C_r (-1)^r (x)^{12-2r} y^r$$