BINOMIAL THEOREM

8

8.1 INTRODUCTION

We have already learnt how to multiply a binomial by another binomial or a binomial by itself. Finding squares and cubes of a binomial by actual multiplication is not difficult. For example,

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2.$$

$$(a + b)^3 = (a + b)(a + b)^2 = a^3 + 3a^2b + 3ab^2 + b^3$$

But the process of finding higher powers of binomials such as $(a + b)^5$, $(a + b)^{50}$, $(a + b)^{100}$ etc., becomes more difficult. Therefore, we look for a general formula which will help us in finding higher powers of a binomial.

In this chapter, we shall study *Binomial Theorem*, which gives us a general method for finding the expansion of $(a+b)^n$, where the index n is a positive integer.

8.2 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDICES

The expansion of $(a+b)^n$ for different positive integers 1, 2, 3, ...are obtained by actual multiplication as follows

We know that $(a+b)^0 = 1$, $(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ and so on

The coefficients of the expansions are arranged as follows:

Index	*Coefficients*
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Fig. 8.1

From the pattern of coefficients, we observe the following.

i. The number of coefficients is 1 more than the index.

ii. The first and last coefficient in each row is 1.

iii. Except the first and last entry, any coefficient in a row is the sum of two coefficients in the preceding row, one on the immediate left and one on the immediate right. Proceeding like this we can write the coefficients in the expansion of $(a+b)^{10}$, we have to write all 10 rows, since the coefficients in each row is obtained from its preceding row. The above pattern is known as Pascal's triangle.

The concept of combination makes us to write the coefficients in each row of a Pascal's triangle in terms of ${}^nC_r$. The Pascal's triangle is re-written as

Index	*Coefficients*	*Coefficients*
0	1	${}^0C_0$
1	1 1	${}^1C_0$ ${}^1C_1$
2	1 2 1	${}^2C_0$ ${}^2C_1$ ${}^2C_2$
3	1 3 3 1	${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$
4	1 4 6 4 1	${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$
5	1 5 10 10 5 1	${}^5C_0$ ${}^5C_1$ ${}^5C_2$ ${}^5C_3$ ${}^5C_4$ ${}^5C_5$

Fig. 8.2

From this pattern, we can directly write the coefficients for any index without writing the preceedings rows.

For example, the coefficients of $(a + b)^{16}$ are $^{16}C_0$, $^{16}C_1$, $^{16}C_2$, ..., $^{16}C_{16}$.

Pascal's triangle

The pattern given in Fig. 8.1 gives the coefficients in the expansion of $(a + b)^n$, $n = 0, 1, 2, \ldots$. The pattern looks like a triangle with 1 on the top vertex. This pattern of coefficients is known as **Pascal's triangle.** Fig. 8.2 gives the Pascal's triangle by using ^nC_r.

Now the expression given above can be rewritten as

$$(a + b)^0 = {}^0C_0$$

$$(a + b)^1 = {}^1C_0 a + {}^1C_1 b$$

$$(a + b)^2 = {}^2C_0 a^2 + {}^2C_1 ab + {}^2C_2 b^2$$

$$(a + b)^3 = {}^3C_0 a^3 + {}^3C_1 a^2 b + {}^3C_2 ab^2 + {}^3C_3 b^3$$

$$(a + b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$$

$$(a + b)^5 = {}^5C_0 a^5 + {}^5C_1 a^4 b + {}^5C_2 a^3 b^2 + {}^5C_3 a^2 b^3 + {}^5C_4 ab^4 + {}^5C_5 b^5 \text{ and so on}$$

Continuing this way, we can easily write the expansion of $(a + b)^n$, where n is a positive integer. This is given in the following theorem known as **Binomial Theorem.**

Theorem 1 For any positive integer n, **(August 2014)**

$$(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \ldots\ldots\ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n$$

for $0 \le r \le n$ and $^nC_r = \dfrac{n!}{r!(n-r)!}$

Proof

We can prove the result by the principle of Mathematical Induction

Let P(n) be the statement:

$$(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n$$

First, we verify the truth of P(1) .

Taking $n = 1$, the statement P(1) is $(a + b)^1 = {}^1C_0 a + {}^1C_1 b = a + b$.

Thus P(1) is true.

Now, suppose that P(k) is true for some positive integer k. We shall prove that P($k + 1$) is true.

Now $(a + b)^{k+1} = (a + b)(a + b)^k$

$$= (a + b)[{}^kC_0 a^k + {}^kC_1 a^{k-1} b + {}^kC_2 a^{k-2} b^2 + {}^kC_3 a^{k-3} b^3 + \ldots\ldots\ldots + {}^kC_{k-1} ab^{k-1} + {}^kC_k b^k]$$

($\because$ P(k) is assumed to be true)

$= {}^kC_0a^{k+1} + {}^kC_1a^kb + {}^kC_2a^{k-1}b^2 + {}^kC_3a^{k-2}b^3 + ... + {}^kC_{k-1}a^2b^k + {}^kC_kab^k + {}^kC_0a^kb$

$+ {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-2}b^3 + {}^kC_3a^{k-3}b^4 + ... + {}^kC_{k-1}ab^k + {}^kC_kb^{k+1}$ (by actual multiplication)

$= {}^kC_0a^{k+1} + [{}^kC_1 + {}^kC_0]a^kb + [{}^kC_2 + {}^kC_1]a^{k-1}b^2 + [{}^kC_3 + {}^kC_2]a^{k-2}b^3 + ...$

$+ [{}^kC_k + {}^kC_{k-1}]ab^k + {}^kC_kb^{k+1}$ (by grouping like terms)

$= {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2a^{k-1}b^2 + {}^{k+1}C_3a^{k-2}b^3 + ... + {}^{k+1}C_kab^k + {}^{k+1}C_{k+1}b^{k+1}$

[Since ${}^{k+1}C_0 = {}^kC_0 = 1$, ${}^{k+1}C_{k+1} = {}^kC_k = 1$ and ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$]

This proves that P(k + 1) is true if P(k) is true. Therefore, by the Principle of Mathematical Induction, P(n) is true for each positive integer n. This proves the theorem.

NOTE

1. The Binomial Theorem is also true when $n = 0$.
2. Abbreviated form of the Binomial Theorem is $(a + b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$

Observations in the expansion of $(a + b)^n$

- The expansion contains $n + 1$ terms. **(March 2013)**
- In the expansion, the index of 'a' goes on decreasing by 1, starting from n, and ending with zero
- The index of 'b' goes on increasing by 1, starting from 0, and ending with n.
- In every term the sum of indices of 'a' and 'b' is equal to n.
- The binomial coefficients can be remembered with the help of the Pascal's Triangle.

Example 1

The number of terms in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is ....

a. $n + 1$ b. n c. $2n + 1$ d. $2n + 2$ **(March 2015)**

Solution

If n is a positive integer then the number of terms is one more than the index.

$\therefore$ Number of terms = $2n + 1$

$\therefore$ (c) is the correct option.

Example 2

Expand the expression $(2x - 3)^6$. **(NCERT)**

Solution

$(2x - 3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5$

$+ {}^6C_6(3)^6$

$= 1(64x^6) - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 1(729)$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Example 3

Expand the following : $\left(x^2 + \frac{2}{x}\right)^4$ **(March 2008)**

Solution

By Binomial Theorem,

$$\left(x^2 + \frac{2}{x}\right)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3\left(\frac{2}{x}\right) + {}^4C_2(x^2)^2\left(\frac{2}{x}\right)^2 + {}^4C_3(x^2)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$= x^8 + 4x^6\left(\frac{2}{x}\right) + 6x^4\left(\frac{2}{x}\right)^2 + 4x^2\left(\frac{2}{x}\right)^3 + \left(\frac{2}{x}\right)^4 = x^8 + 8x^5 + 24x^2 + \frac{32}{x} + \frac{16}{x^4}$$