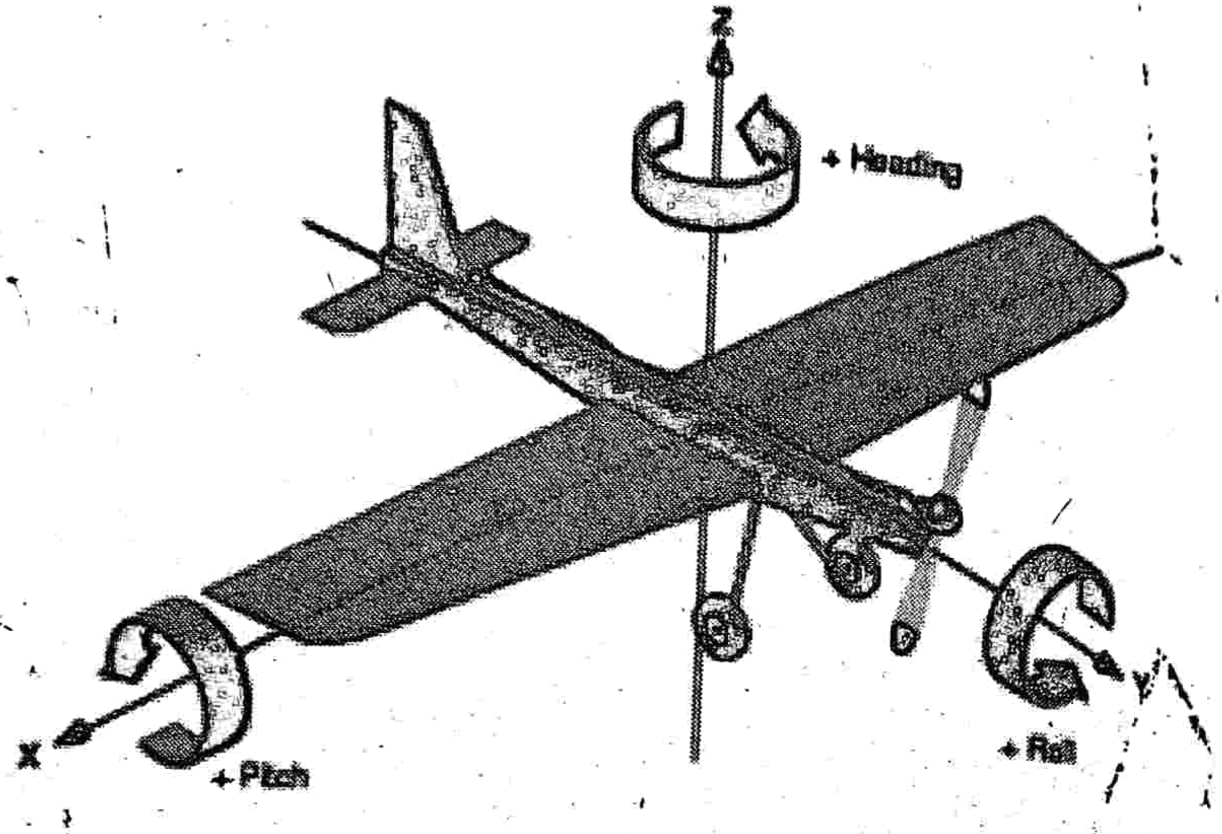


7.1. Introduction

Before going to the motion of system of particles, you must know about a single particle or a point object and its motion on the application of a force. Translatory motion of point objects were dealt in the earlier chapters. An extended body can be considered as the collection of point objects. In order to understand the motion of an extended body, we introduce the idea of centre of mass. The result of application of force on a system of particles is identical with the result of force applied to a point mass at the centre of mass of the system. This shows that the motion of a complicated system can be reduced to one of the motions of its centre of mass. The properties and motion of centre of mass, the idea of moment of inertia and its calculation are all involved in this chapter.

CURRICULUM OBJECTIVES

- To create the concept of centre of mass through discussion, experiments and IT.
- To develop the idea of moment of inertia and radius of gyration through demonstration and discussion.
- To create the concept of torque and angular momentum through experiments and discussion.
- To compare linear motion and angular motion by solving problems.
- To develop the concept of parallel axis and perpendicular axis theorem through derivation.
- To develop the concept and equation of the moment of inertia of circular ring, disc, cylinder rolling without slipping by the application of parallel axis and perpendicular axis theorem.
- To identify the binary systems in nature through discussions.

7.1.1 What Kind of Motion Can a Rigid Body Have?

A rigid body is a body having definite shape and the particles of the body are situated in such a manner that there is no change for their separations even under the action of a force. Complicated extended bodies can be solved by treating them to be rigid bodies.

Let us consider the motion of a rigid body, say, a stone or a train bogey. When a force is applied, it moves from one point to another in such a way that all the constituent particles suffer the same displacement (fig. 1). This type of motion of a rigid body, in which all the particles of the body move with the same velocity either in a straight line or in a curved path is called pure ***translational motion.***

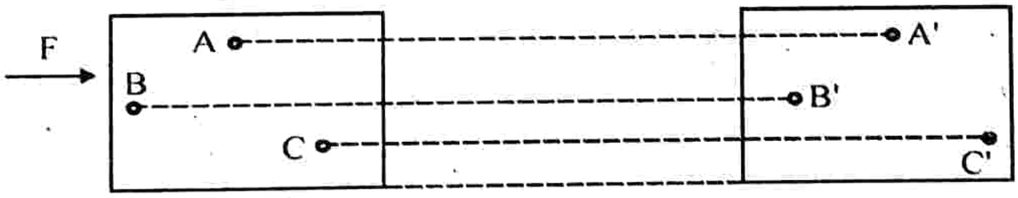


Fig. 1
Translational motion of a block

During translational motion all the particles suffer the same displacement.

Now we consider the motion of a ceiling fan or a giant wheel. The body is turning about a fixed axis. Here the rigid body is said to be ***rotating*** about the Y-axis. During rotation the particles of the body, say, A and B at different distances from the axis of rotation trace a circular path with different radii about the same axis YY' (fig. 2). Eventhough their angular displacement remains the same, their linear velocities are not the same. Their linear velocity always depends on the distance from the axis of rotation.

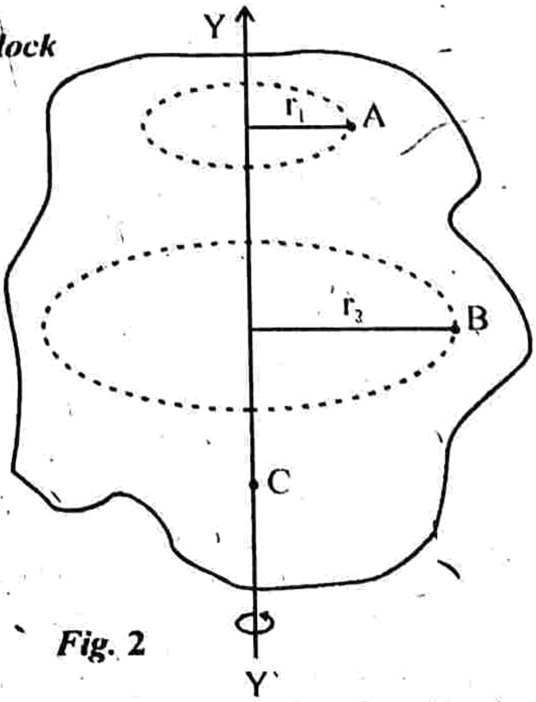


Fig. 2
A rigid body rotating about Y-axis

Consider a coin rolling down an inclined plane as in fig.3. Since the coin shifts from the top of the plane to bottom, it has translational motion. But during rolling each and every particle of the coin moves with different velocities as indicated by the arrows in the above figure. Here the motion is a combination of translation and rotation.

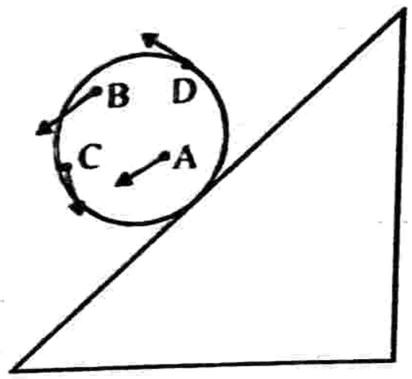


Fig. 3

Rolling motion of a coin

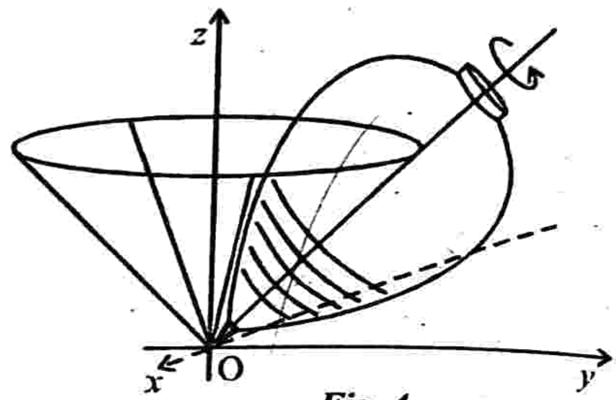


Fig. 4

A spinning top

Another type of motion executed by a rigid body is ***precession.*** The best example for this type is the motion of a spinning top (fig. 4). Here the axis of the top moves in a cone around the vertical through its point of contact with the ground.

7.2 Centre of Mass

***"Centre of mass** of a system of particles is a fixed point at which the entire mass of the system is assumed to be concentrated"*

When an external force is acted on the system, this point like mass would have the same type of translational motion as that of a single particle of same mass moves under the application of the same external force.

7.2.1 Centre of Mass of a Two Particle System

Consider two particles P_1 and P_2 of masses m_1 and m_2 in a reference frame with origin O as shown in figure (5). Let $\vec{r_1}$ and $\vec{r_2}$ be their position vectors. Let C be the centre of mass of this system. Its position vector $\vec{R}$ is given by

$$\boxed{\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}} \quad \text{................ (1)}$$

If $m_1 + m_2 = M$, the total mass of the system,

then $\vec{R} = \dfrac{m_1\vec{r_1} + m_2\vec{r_2}}{M}$ .... (2)

If $m_1 = m_2$, then $\vec{R} = \dfrac{\vec{r_1} + \vec{r_2}}{2}$ .... (3)

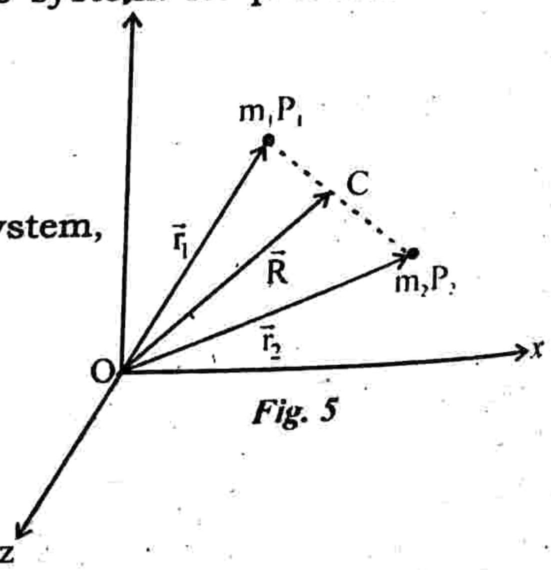


Fig. 5

Let the position coordinates of P_1 be (x_1, y_1, z_1),

P_2 be (x_2, y_2, z_2) and that of C be (x, y, z). Then

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \text{........ (4)}$$

$$y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \quad \text{............ (5)}$$

$$z = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} \quad \text{.......... (6)}$$

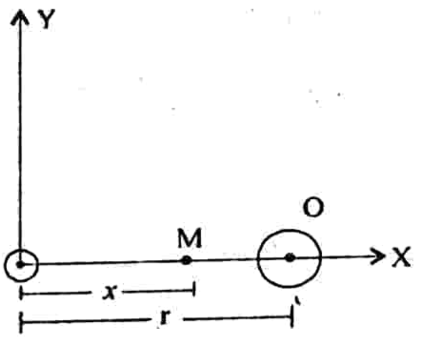
If two equal masses are considered, then $m_1 = m_2 = m$, and

$$\left.\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ y &= \frac{y_1 + y_2}{2} \\ z &= \frac{z_1 + z_2}{2} \end{aligned}\right\} \quad \text{.................. (7)}$$

Solved Examples

1. In a carbon monoxide molecule, the distance between carbon and oxygen atoms is 1.2 A°. If oxygen atom is 1.3 times as massive as carbon, find the location of the centre of mass of the molecule relative to the carbon atom.

Sol.



Given, $r = 1.2\ A° = 1.2 \times 10^{-10}$ m

Mass of carbon atom, $m_1 = m$

$\therefore$ Mass of oxygen atom, $m_2 = 1.3$ m

Carbon atom is assumed to be at the origin.

Let x be the distance of carbon from the centre of mass 'M'

Now, $x_1 = 0$ and $x_2 = r = 1.2A°$

$\therefore$ centre of mass is given by,

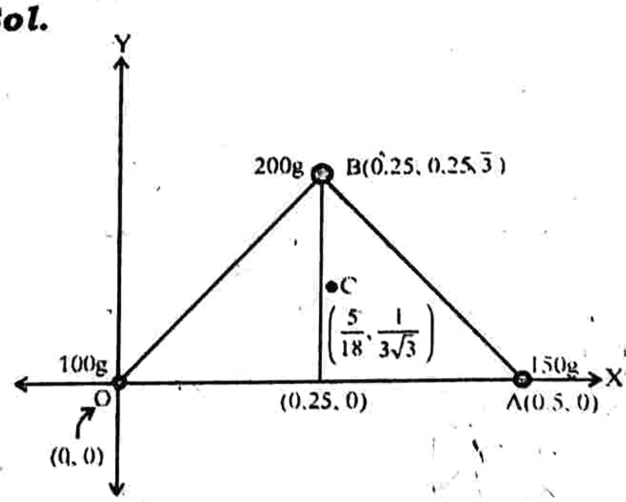
$$\vec{R} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

i.e., $$x = \frac{m \times 0 + 1.3m \times 1.2 \times 10^{-10}}{2.3m}$$

$$x = 0.678 \times 10^{-10}\ \text{m} = 0.678\ A°$$

2. Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100 g, 150 g, and 200 g respectively. Each side of the equilateral triangle is 0.5 m long.

Sol.



With the x - and y - axes chosen as shown in figure, the coordinates of points O, A and B forming the equilateral triangle are respectively (0,0), (0.5,0), (0.25, $0.25\sqrt{3}$). Let the masses 100g, 150 g and 200 g be located at O, A and B respectively. Then,

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100\,(0) + 150\,(0.5) + 200\,(0.25)]\text{ gm}}{(100 + 150 + 200)\text{ g}}$$

$$= \frac{75 + 50}{450}\text{m} = \frac{125}{450}\text{m} = \frac{5}{18}\text{m}$$

$$y = \frac{[100(0) + 150(0) + 200(0.25\sqrt{3})]\text{ gm}}{450\text{g}}$$

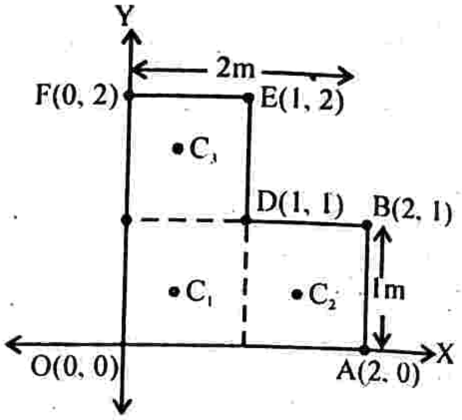
$$= \frac{50\sqrt{3}}{450}\text{m} = \frac{\sqrt{3}}{9}\text{m} = \frac{1}{3\sqrt{3}}\text{m}$$

The centre of mass C is shown in the figure. Note that it is not the geometric centre of the triangle OAB.

3. Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown in figure. The mass of the lamina is 3 kg.

Sol.

L-shaped lamina consists of three squares of 1kg each, since the lamina is uniform. By symmetry, centre of mass of each square lies at its geometric centres with coordinates $C_1\left(\frac{1}{2}, \frac{1}{2}\right)$, $C_2\left(\frac{3}{2}, \frac{1}{2}\right)$ and $C_3\left(\frac{1}{2}, \frac{3}{2}\right)$.



Let us imagine the mass of each square is concentrated at these points. Thus the coordinates of the centre of mass (X, Y) are given by

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{1 \times \frac{1}{2} + 1 \times \frac{3}{2} + 1 \times \frac{1}{2}}{1+1+1} = \frac{5}{6}\text{m}$$

$$Y = \frac{1 \times \frac{1}{2} + 1 \times \frac{1}{2} + 1 \times \frac{3}{2}}{1+1+1} = \frac{5}{6}\text{m}$$

7.2.2 Centre of Mass of a Rigid Body

For a rigid body, its centre of mass is a fixed point with respect to the body as a whole. Depending on the mass distribution and shape, the centre of mass may be inside the body or outside.

TABLE 1

Centre of mass of some regular bodies

Sl. No.	Shape of the body	Position of centre of mass
1.	Uniform rod	Midpoint of the rod
2.	Circular ring	Centre of the ring
3.	Circular disc	Centre of the disc
4.	Sphere	Centre of the sphere
5.	Solid cone	3/4th height of the cone from the apex

6.	Cylinder	Centre of axis of the cylinder
7.	Triangular lamina	Point of intersection of medians
8.	Rectangular lamina (square)	Point of intersection of diagonals
9.	Cube	Point of intersection of diagonals.

7.3 Motion of Centre of Mass

Consider a system of n particles. Let $m_1, m_2, m_3, \ldots m_n$ be the masses of the particles with position vectors $r_1, r_2, r_3, \ldots\ldots\ldots\ldots r_n$ respectively.

The total mass of the system is given by $M = m_1 + m_2 + \ldots\ldots + m_n$ .... (1)

Let $\vec{R}$ be the position vector of the centre of mass. Then

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots + m_n\vec{r}_n}{M} \quad \text{or} \quad M\vec{R} = \sum_{i=1}^{n} m_i r_i \ \ldots\ldots\ldots (2)$$

The velocity of centre of mass is given by

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots\ldots + m_n\frac{d\vec{r}_n}{dt} \quad \text{or} \quad M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots + M_n\vec{v}_n$$

i.e., $M\vec{V} = \sum_{i=1}^{n} m_i\vec{v}_i$ .... (3)

The acceleration of the centre of mass is given by

$$M\frac{d\vec{V}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \ldots\ldots + m_n\frac{d\vec{v}_n}{dt} \quad \text{or}$$

$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots\ldots + m_n\vec{a}_n = \vec{F}_1 + \vec{F}_2 + \ldots\ldots + \vec{F}_n$$

$$M\vec{A} = \vec{F}_{total} \ \ldots\ldots\ldots\ldots (4)$$

But $\vec{F}_{total} = \vec{F}_{ext} + \vec{F}_{int} = \vec{F}_{ext}$ $\quad \because \vec{F}_{int} = 0$ by Newton's third law.

i.e., $M\vec{A} = \vec{F}_{ext}$ .... (5)

Eqn (5) tells that the centre of mass of a system of particles moves as if the whole mass of the system were concentrated at the centre of mass and all the external forces were applied at this point.

7.4 Linear Momentum of a System of Particles

Consider a system of n particles. Let $m_1, m_2, \ldots\ldots m_n$ be the masses of the particles of the system moving with velocities $v_1, v_2, \ldots\ldots\ldots\ldots v_n$ respectively.

The velocity of the centre of mass is given by,

$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots\ldots + m_n\vec{v}_n$ ....(1)

But the RHS of eqn (1) is the total momentum of the system

i.e., $\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots\ldots + m_n\vec{v}_n$ .... (2)

From (1) and (2), $M\vec{V} = \vec{P}$ .... (3)

Differentiating both sides with time, $\frac{d\vec{P}}{dt} = M\frac{d\vec{V}}{dt} = M\vec{A}$ .... (4)

But $M\vec{A} = \vec{F}_{ext}$ $\therefore$ (4) becomes, $\frac{d\vec{P}}{dt} = \vec{F}_{ext}$

if $\vec{F}_{ext} = 0$, then $\frac{d\vec{P}}{dt} = 0$ or $\vec{P}$ = a constant

i.e., $m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots + m_n\vec{v}_n$ = a constant

Thus *'if no external forces are acting on a system of particles, then their total linear momentum remains constant'. This is the law of* ***conservation of linear momentum.***

Also since $\frac{d\vec{P}}{dt} = 0$, $\frac{d}{dt}(M\vec{V}) = 0$ i.e., $M\frac{d}{dt}\vec{V} = 0$ $\because M \neq 0$,

$\frac{d}{dt}\vec{V} = 0$, $\vec{V}$ = a constant

Thus if no external force is acting on a system of particles, then the velocity of the centre of mass remains constant. That is, the system is not accelerating.