Example 46

i. Show that $\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=\tan x$.

ii. Hence deduce $\tan\frac{\pi}{8}$.

Solution

i. $$\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=\frac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}=\frac{2\sin x(\sin x+\cos x)}{2\cos x(\sin x+\cos x)}=\tan x$$

ii. In (i) put $x=\frac{\pi}{8}$

i.e., $$\tan\frac{\pi}{8}=\frac{1-\cos\pi/4+\sin\pi/4}{1+\cos\pi/4+\sin\pi/4}=\frac{1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}$$

$$=\frac{\sqrt{2}}{2+\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2}\left(\sqrt{2}+1\right)}$$

$$=\frac{1}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}=\sqrt{2}-1$$

Example 47

Prove that $\tan 4x = \dfrac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$.

Solution

$$\tan 4x = \tan[2(2x)] = \frac{2\tan 2x}{1-\tan^2 2x}$$

$$= \frac{2\left[\dfrac{2\tan x}{1-\tan^2 x}\right]}{1-\left(\dfrac{2\tan x}{1-\tan^2 x}\right)^2} = \frac{\dfrac{4\tan x}{1-\tan^2 x}}{\dfrac{(1-\tan^2 x)^2-4\tan^2 x}{(1-\tan^2 x)^2}}$$

$$= \frac{4\tan x(1-\tan^2 x)}{1-2\tan^2 x+\tan^4 x-4\tan^2 x} = \frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$$

Example 48

Prove that $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

Solution

$$\text{L.H.S} = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right) = 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-\left(\frac{\pi}{4}-x\right)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S}$$

Another Method

$$\text{LHS} = \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{RHS}$$

Example 49

Prove that $\frac{\sin 5x-\sin 3x}{\cos 5x+\cos 3x}=\tan x$

Solution

$$\frac{\sin 5x-\sin 3x}{\cos 5x+\cos 3x}=\frac{2\cos 4x\sin x}{2\cos 4x\cos x}=\tan x$$

Example 50

Prove that $\frac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$

Solution

$$\text{LHS}=\frac{2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}=\frac{\sin 4x.\cos x}{\cos 4x.\cos x}=\tan 4x=\text{RHS}$$

Example 51

Show that $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$.

Solution

$$\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\cos\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}{-(\cos^2 x - \sin^2 x)}$$

$$= \frac{2\cos 2x \sin(-x)}{-\cos 2x} = \frac{-2\cos 2x \sin x}{-\cos 2x} = 2\sin x$$

Example 52

Show that $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\dfrac{x+y}{2}$.

Solution

$$\text{L.H.S.} = \frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \frac{\sin\left(\frac{x+y}{2}\right)}{\cos\left(\frac{x+y}{2}\right)} = \tan\left(\frac{x+y}{2}\right) = \text{R.H.S}$$

Example 53

Prove that $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

Solution

LHS $= (\sin 6x + \sin 4x)(\sin 6x - \sin 4x)$

$$= 2\sin\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right) 2\cos\left(\frac{6x+4x}{2}\right)\sin\left(\frac{6x-4x}{2}\right)$$

$= 2\sin 5x \cos x \,.\, 2\cos 5x \sin x = (2\sin x \cos x)(2\sin 5x \cos 5x) = \sin 2x \sin 10x =$ RHS

Example 54

Show that $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4\cos^2\left(\frac{x-y}{2}\right)$.

Solution

LHS $= (\cos x + \cos y)^2 + (\sin x + \sin y)^2$

$= \cos^2 x + 2\cos x \cos y + \cos^2 y + \sin^2 x + 2\sin x \sin y + \sin^2 y$

$= \cos^2 x + \sin^2 x + \cos^2 y + \sin^2 y + 2[\cos x \cos y + \sin x \sin y]$

$= 1 + 1 + 2\cos(x - y)$

$= 2 + 2\cos(x - y)$

$= 2[1 + \cos(x - y)]$

$= 2.2\cos^2\left(\frac{x-y}{2}\right) = 4\cos^2\left(\frac{x-y}{2}\right) =$ RHS

Another Method

$$LHS = (\cos x + \cos y)^2 + (\sin x + \sin y)^2$$

$$= \left(2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right)^2 + \left(2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right)^2$$

$$= 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)\left[\cos^2\left(\frac{x+y}{2}\right) + \sin^2\left(\frac{x+y}{2}\right)\right] = 4\cos^2\left(\frac{x-y}{2}\right)(1)$$

$$= 4\cos^2\left(\frac{x-y}{2}\right) = RHS$$

Example 55

Prove that $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\frac{(x-y)}{2}$

Solution

$$(\cos x - \cos y) = -2\sin\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$$

$$(\sin x - \sin y) = 2\cos\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$$

$$\therefore (\cos x - \cos y)^2 + (\sin x - \sin y)^2$$

$$= 4\sin^2\frac{(x+y)}{2}\sin^2\frac{(x-y)}{2} + 4\cos^2\frac{(x+y)}{2}\sin^2\frac{(x-y)}{2}$$

$$= 4\sin^2\frac{(x-y)}{2}\left[\sin^2\frac{(x+y)}{2} + \cos^2\frac{(x+y)}{2}\right] = 4\sin^2\frac{(x-y)}{2}$$

Example 56

Show that $\dfrac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = \sec x + \tan x$.

Solution

$$L.H.S = \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)} = \frac{\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{1 + \sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x = R.H.S$$

Example 57

Show that $\dfrac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x} = 4\cos 2x \cos 4x$:

Solution

$$\text{L.H.S} = \frac{\left(\dfrac{\sin 5x}{\cos 5x} + \dfrac{\sin 3x}{\cos 3x}\right)}{\left(\dfrac{\sin 5x}{\cos 5x} - \dfrac{\sin 3x}{\cos 3x}\right)} = \frac{\left(\dfrac{\sin 5x \cos 3x + \cos 5x \sin 3x}{\cos 5x \cos 3x}\right)}{\left(\dfrac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\cos 5x \cos 3x}\right)}$$

$$= \frac{\sin 5x \cos 3x + \cos 5x \sin 3x}{\sin 5x \cos 3x - \cos 5x \sin 3x} = \frac{\sin(5x+3x)}{\sin(5x-3x)}$$

$$= \frac{\sin 8x}{\sin 2x} = \frac{2\sin 4x \cos 4x}{\sin 2x} = \frac{2\times 2\sin 2x \cos 2x \cos 4x}{\sin 2x}$$

$$= 4\cos 2x \cos 4x = \text{R.H.S}$$

Example 58

Prove that $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

Solution

$$\text{L.H.S} = \frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \frac{\sin 5x + \sin x - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \frac{2\sin 3x \cos 2x - 2\sin 3x}{-2\sin 3x \sin 2x} = \frac{\sin 3x(\cos 2x - 1)}{\sin 3x \sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x} = \tan x = \text{R.H.S}$$

Example 59

Prove that $\dfrac{\cos 3x + \cos 7x - \cos 2x}{\sin 7x - \sin 3x - \sin 2x} = \cot 2x$

Solution

$$\text{LHS} = \frac{(\cos 7x + \cos 3x) - \cos 2x}{(\sin 7x - \sin 3x) - \sin 2x} = \frac{2\cos 5x \cos 2x - \cos 2x}{2\cos 5x \sin 2x - \sin 2x}$$

$$= \frac{\cos 2x(2\cos 5x - 1)}{\sin 2x(2\cos 5x - 1)} = \frac{\cos 2x}{\sin 2x}$$

$$= \cot 2x = \text{RHS}$$