Inverse Trigonometric Functions
Focus area class-4

1, Simplify $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, $x<\pi$

$$\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}$$

$$= \tan^{-1}\sqrt{\tan^2 x/2}$$

$$= \tan^{-1}(\tan x/2) = \underline{x/2}$$

2, Simplify $\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right]$, $0<x<\pi$

$$\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right] = \tan^{-1}\left[\frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan x}{1+\tan x}\right]$$

$$= \tan^{-1}\left[\tan(\pi/4 - x)\right]$$

$$= \underline{\underline{\pi/4 - x}}$$

4, S.T $\sin^{-1}\left[2x\sqrt{1-x^2}\right] = 2\sin^{-1}x, \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$LHS = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

Put
$x = \sin\theta$
$\theta = \sin^{-1}x$

$= \sin^{-1}\left[2\sin\theta\sqrt{1-\sin^2\theta}\right]$

$= \sin^{-1}\left[2\sin\theta \cdot \cos\theta\right]$

$= \sin^{-1}\left[\sin 2\theta\right]$

$= 2\theta = 2\sin^{-1}x = RHS$

5, S.T $3\sin^{-1}x = \sin^{-1}\left[3x - 4x^3\right], x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$

$RHS = \sin^{-1}\left[3x - 4x^3\right]$

$x = \sin\theta$
$\theta = \sin^{-1}x$

$= \sin^{-1}\left[3\sin\theta - 4\sin^3\theta\right]$

$= \sin^{-1}\left[\sin 3\theta\right]$

$= 3\theta = 3\sin^{-1}x = LHS$

6, Simplify $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0$

$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

③ write the simplest form of

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \quad -\pi/2 < x < 3\pi/2$$

$$\tan^{-1}\left[\frac{\cos x}{1-\sin x}\right] = \tan^{-1}\left[\frac{\sin(\pi/2 - x)}{1-\cos(\pi/2 - x)}\right]$$

$$= \tan^{-1}\left[\frac{\cancel{2}\sin\left(\frac{\pi/2 - x}{2}\right)\cos\left(\frac{\pi/2 - x}{2}\right)}{\cancel{2}\sin^2\left(\frac{\pi/2 - x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\cos\left(\frac{\pi/2 - x}{2}\right)}{\sin\left(\frac{\pi/2 - x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{\pi/2 - x}{2}\right)\right]$$

$$\boxed{\cot x = \tan(90 - x)}$$

$$= \tan^{-1}\left[\tan\left\{\frac{\pi}{2} - \left(\frac{\pi/2 - x}{2}\right)\right\}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} + \frac{x}{2}$$

$$\boxed{\tan^{-1}(\tan x) = x}$$

$$= \frac{\pi}{4} + \frac{x}{2}$$

$$\boxed{\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ 1-\cos x &= 2\sin^2 x/2 \end{aligned}}$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

Put
$x = \tan\theta$
$\theta = \tan^{-1}x$

$$= \tan^{-1}\left[\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left[\frac{1 - \cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\theta/2}{2\sin\theta/2\,\cos\theta/2}\right]$$

$$= \tan^{-1}\left[\tan\theta/2\right]$$

$$= \theta/2 = \underline{\underline{\frac{1}{2}\tan^{-1}x}}$$

7, S.T $\underline{2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)}, |x| \leq 1$

$$\text{RHS} = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Put
$x = \tan\theta$
$\theta = \tan^{-1}x$

$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta$$

$$= \underline{\underline{2\tan^{-1}x = \text{LHS}}}$$

Note

1, $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x| \leq 1$

2, $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \geq 0$

3, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $-1 < x < 1$