Inverse Trigonometric Functions
Focus area class - 3

1, Show that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

if $xy < 1$

Let $\tan^{-1}x = \theta \Rightarrow x = \tan\theta$

$\tan^{-1}y = \phi \Rightarrow y = \tan\phi$

we have $\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}$

$\tan(\theta+\phi) = \frac{x+y}{1-xy}$

$\therefore \theta + \phi = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

Note

1, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

2, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$

1. Show that $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

$LHS = \tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right)$

$= \tan^{-1}\left[\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}}\right]$

$$= \tan^{-1}\left[\frac{\frac{48+77}{11\times 24}}{\frac{264-14}{11\times 24}}\right]$$

$$= \tan^{-1}\left[\frac{125}{250}\right] = \tan^{-1}\left(\frac{1}{2}\right) = RHS$$

(2) Show that

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$$

$$LHS = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\cdot\frac{1}{2}}\right] + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{1}{3/4}\right] + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right) = RHS$$

(3) Solve $\tan^{-1}(2x) + \tan^{-1}(3x) = \pi/4$

$$\tan^{-1}(2x) + \tan^{-1}(3x) = \pi/4$$

$$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \pi/4$$

$$\frac{5x}{1-6x^2} = \tan \pi/4 = 1$$

$$5x = 1 - 6x^2$$

$$6x^2 + 5x - 1 = 0$$

$$x = \frac{-5 \pm \sqrt{25 - 4 \times 6 \times -1}}{12}$$

$$= \frac{-5 \pm 7}{12} = \frac{-12}{12} \text{ or } \frac{2}{12}$$

$$\therefore x = \frac{1}{6} \text{ or } -1$$

$$\therefore x = \underline{\underline{1/6}}$$

(4) Show that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right) = x$ and $\cos^{-1}\left(\frac{12}{13}\right) = y$

$\therefore \cos x = \frac{4}{5}$ $\quad\quad$ $\therefore \cos y = \frac{12}{13}$

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

Now

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}$$

$$= \frac{48-15}{65}$$

$$\cos(x+y) = \frac{33}{65}$$

$$x+y = \cos^{-1}\left(\frac{33}{65}\right)$$

$$\therefore \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

5 Show that

$$\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{77}{36}\right)$$

Let $\sin^{-1}\left(\frac{8}{17}\right) = x$ and $\sin^{-1}\left(\frac{3}{5}\right) = y$

$$\frac{8}{17} = \sin x \quad \therefore \frac{3}{5} = \sin y$$

$$\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{8/17}{15/17} = 8/15$$

$$\tan y = \frac{3/5}{4/5} = 3/4$$

we have $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$

$$\tan(x+y) = \frac{8/15 + 3/4}{1 - 8/15 + 3/4}$$

$$= \frac{\frac{32+45}{15+4}}{\frac{60-24}{15+4}}$$

$$\tan(x+y) = \frac{77}{36}$$

$$\therefore x+y = \tan^{-1}\left(\frac{77}{36}\right)$$

$$\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{77}{36}\right)$$