9. Sequences & Series

Sequence:

Let X be a set of numbers and $f : N \to X$ be a function, then the ordered set $\{f(1), f(2), \ldots, f(n)\}$ is called a finite sequence where as $\{f(1), f(2), \ldots\}$ is called an infinite sequence.

If X =R, then the sequence is called a real sequence and if X = C, then the sequence is called a complex sequence.

Thus, the sequence is an arrangement of numbers in a definite order.

Note1: The numbers $f(1), f(2), \ldots, f(n)$ are called first term, second term, 3rd term, ……, nth term of the sequence. The nth term of a sequence is also denoted by $T_n, t_n, a_n, a(n), u_n$, etc.

Note2: The sequence $\{f(1), f(2), \ldots, f(n)\}$ are generally denoted as $f(1), f(2), \ldots, f(n)$.

Progression: A sequence is said to be a progression if its terms increases (decreases) numerically

E.g.: 4, 9, 14,19, …

8, 4, 0, −4, …

Series: If $\{a_n\}$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n$ is called the series corresponding to the sequence $\{a_n\}$. Or in other words, sum of all terms of a sequence is called a series.

Depending on the sequence, a series is called finite or infinte.

E.g.: i. 1 + 2 + 3 + ……. +100

ii. 2 + 6 + 18 + ……

iii. 5 + 10 + 15 + …….

iv. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \ldots$

Arithmetic Progression (A.P)

A sequence in which, its terms after the 1st term is obtained by adding a fixed number to its just preceding terms. Such sequence is called Arithmetic Progression. The fixed number is called common difference and is denoted by 'd'.

E.g: −4, −2, 0, 2, 4, …

6, 10, 14, …

Thus $d = \textit{any term} - \textit{the term just preceding it}$

i.e., $\boxed{\mathbf{d = a_{n+1} - a_n}}$

Note:

1. If in a sequence, the terms are alternatively +ve and −ve, then it cannot be an A.P.
2. The standard form of an A.P **is a, a+d, a+2d, a + 3d, …….**
3. If a, b, c be three consecutive terms of an A.P then 2nd term − 1st term = 3rd term − 2nd term. i.e., b − a = c − b.
4. If three numbers a, b, c are in A.P. then $\mathbf{2b = a + c}$

nth term of an A.P.[a_n]

The first term of the sequence is denoted by 'a', common difference by 'd', then nth term is defined as $\mathbf{a_n = a + (n-1)d}$

Number of terms of an A.P.[n]

Let 'n' be the number of terms of an A.P, then

$$\mathbf{n = \frac{a_n - a}{d} + 1}$$

Where	a	1st term
	a_n	last term
	d	Common difference

Sum of 'n' terms of an A.P.[S_n]

Let 'a' be the 1st term, 'd' the common difference, 'n' the number of terms and 'a_n' be the last term then sum of n terms is

$$S_n = \frac{n}{2}[2a+(n-1)d] \quad \text{(or)} \quad S_n = \frac{n}{2}[a+a_n]$$

Arithmetic mean between two quantities.

Let 'x' be the arithmetic mean between two quantities 'a' and 'b' then $x = \frac{a+b}{2}$

Notes:

1. If each term of a given A.P be increased, decreased, multiplied or divided by the same non-zero quantity then the resulting series thus obtained will also be in A.P.
2. **No. of Terms** **A.P.**

No. of Terms	A.P.
3	$a-d, a, a+d$
4	$a-3d, a-d, a+d, a+3d$
5	$a-d,\ a-d,\ a,\ a+d,\ a+2d$
6	$a-5d,\ a-3d, a-d,\ a+d,\ a+3d, a+5d$

3. In an A.P., the sum of terms equidistant from the beginning and end is constant and equal to the sum of 1st and last terms.
4. Any term except the 1st term of an A.P. is equal to half the sum of terms which are equidistant from it.

   i.e., $a_n = \frac{1}{2}(a_{n-1} + a_{n+1})$
5. nth term, T_n **or** $a_n = S_n - S_{n-1} \quad (n \geq 2)$
6. Sum and difference of corresponding terms of two A.Ps will form a series in A.P.
7. If a, b, c are in A.P then
   i. **a+k, b+k, c+k** are in A.P
   ii. **a−k, b−k, c−k** are in A.P
   iii. **ak, bk, ck** are in A.P
   iv. $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are in A.P $(k \neq 0)$

Geometrical Progression

A finite or infinite sequence is said to be a G.P. if its terms after the first term is obtained by multiplying a fixed number to its just preceding terms. The fixed number is called common ratio (r).

Thus $\text{common ratio} = \frac{\text{any term}}{\text{term just preceding it}}$

$$\boxed{\text{i.e., } r = \frac{a_n}{a_{n-1}}}$$

Standard form: $a, ar, ar^2, ar^3, \ldots\ldots, ar^{n-1}$.

nth term of a G.P. [a_n]

If a,r be the first term and common ratio of a G.P., then $a_n = ar^{n-1}$.

Note: If a, b, c are in G.P, then

i. ka, kb, kc are in G.P
ii. $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are in G.P $(k \neq 0)$
iii. $b^2 = ac$

If the product of n numbers are in G.P. is given,

No. of terms	G.P.
3	$\frac{a}{r}, a, ar$ (or) a, ar, ar^2
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$
6	$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$

Sum of n terms of a G.P [S_n]

If a, r be the first and common ratio of a G.P then

$$S_n = \frac{a(r^n-1)}{r-1},\ \text{if } r>1 \text{ and } S_n = \frac{a(1-r^n)}{1-r},\ \text{if } r<1$$

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Geometric mean between two numbers.

Let x is the GM between two numbers a and b. Then

$\mathbf{x = \sqrt{ab}}$

Sum to infinity of a G.P.

If a be the first term and r the common ratio, then sum of n terms of an infinite series is $\mathbf{S_\infty = \frac{a}{1-r}, |r| < 1}$.

Harmonic Progression (For JEE)

A sequence of non-zero numbers whose reciprocals form an arithmetic progression is called a Harmonic Progression.

If $\mathbf{a_1, a_2, a_3, \ldots\ldots\ldots, a_n, \ldots\ldots\ldots}$ is called a harmonic progression, then $\mathbf{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots\ldots}$ are in A.P.

Note: All problems given in H.P can easily be converted into an A.P and then solve.

Harmonic Mean: Let x be the H.M. between two quantities a and b. Then a, x, b are in H.P.

i.e., $\frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ are in A.P.

$$\therefore \frac{1}{x} - \frac{1}{a} = \frac{1}{b} - \frac{1}{x} \quad \Rightarrow \frac{1}{x} + \frac{1}{x} = \frac{1}{a} + \frac{1}{b}$$

$$\frac{2}{x} = \frac{1}{a} + \frac{1}{b} \Rightarrow \frac{2}{x} = \frac{a+b}{ab} \Rightarrow \frac{x}{2} = \frac{ab}{a+b} \Rightarrow \mathbf{x = \frac{2ab}{a+b}}$$

Note1: If A, G and H are the arithmetic, geometric and harmonic means between two positive real numbers then A, G, H form a geometric progression.

By definition, A = $\mathbf{\frac{a+b}{2}}$, G = $\mathbf{\sqrt{ab}}$, H= $\mathbf{\frac{2ab}{a+b}}$

A . H = $\mathbf{\frac{a+b}{2} \cdot \frac{2ab}{a+b}} = ab = G^2$

$\therefore$ A, G, H are in H.P.

Note2: Three nos. a,b,c are in A.P., G.P. or in H.P. then $\mathbf{\frac{a-b}{b-c} = \frac{a}{a}, \frac{a}{b} \text{ or } \frac{a}{c}}$ respectively.

Arithmetic-Geometric Progression (AGP)

(For JEE only)

A sequence in which each term is the product of an AP and a GP is called an AGP.

E.g.: $1 + 3x + 5x^2 + 7x^3 + \ldots$

$$1 - \frac{5}{2} + \frac{9}{2^2} - \frac{13}{2^3} + \ldots$$

Standard Form

$\mathbf{a, (a+d)r, (a+2d)r^2, \ldots [a+(n-1)d]r^{n-1}}$

Where a first term

d common difference

r common ratio

$\mathbf{n^{th}}$ **term of an AGP :** $\mathbf{a_n = [a+(n-1)d]r^{n-1}}$

Sum of n terms of an AGP

$$\mathbf{S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{(1-r)}}$$

Sum to infinity of an AGP, $\mathbf{S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}}$

Sum to n terms of Special series:

Sum of first n natural numbers 1 + 2 + 3 + …+ n

$$\mathbf{S_n = \frac{n(n+1)}{2}}$$

Sum of the squares of the first n natural numbers

$\mathbf{1^2 + 2^2 + 3^2 + \ldots + n^2}$ is $\mathbf{S_n = \frac{n(n+1)(2n+1)}{6}}$

Sum of the cubes of the first n natural numbers

$\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3}$ is $\mathbf{S_n = \left[\frac{n(n+1)}{2}\right]^2}$

Note: *From the above results we have Sum of the cubes of 1st n natural numbers is equal to square of the sum of the first n natural numbers*.

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