7. PERMUTAIONS AND COMBINATIONS

A combination focuses on the selection of objects without regard to the order in which they are selected. A permutation, in contrast, focuses on the arrangement of objects with regard to the order in which they are arranged.

FUNDAMENTALS PRINCIPLES OF COUNTING

There are two fundamental principles in counting:

1. Fundamental principles of addition
2. Fundamental principles of multiplication.

Fundamental principle of Addition

If there are two events such that, they can be performed independently in m and n ways, then either of the two events can be performed in (m+n) ways.

For example, suppose a teacher wants to select a boy from 6 boys or a girl from 7 girls to represent a class in a function:

The teacher is to be selected either a boy from 6 boys or a girl from 7 girls.

Therefore, total number of selection = 6 + 7=13, is known as fundamental principle of addition.

Fundamental principle of Multiplication

If there are two events such that, one event can occur in m ways, following which, another event can occur in n different ways, then the total number of occurrence of the events is mn ways.

For example, suppose a teacher wants to select a boy from 7 boys and a girl from 6 girls to represent a class in a function, she has to do:

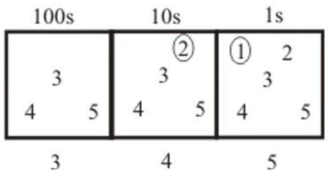
select a boy from 7 boys and select a girl from 6 girls. Thus the total number of sections
is (7)(6)=42 ways.

Q: How many three digit numbers can be formed from the digits 1,2,3,4 and 5 assuming that:

i) repetition is not allowed?

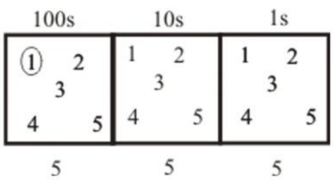
ii) repetition is allowed?

Ans: i) If repetition is not allowed,



In the first place, we can fill all numbers. If 1 is filled in the 1's place, then we can fill 4 numbers (except 1) in the 10's place and if 2 is filled in the 10's place, the we can fill 3 numbers (except 1 and 2) in the 100's place. Therefore, by FPC, the number of three digit numbers = $3\times4\times5=60$.

ii) If repetition is not allowed,



Here, since repetition is not allowed, the 1's place is filled by 5 numbers, 10's place is filled by 5 numbers and 100's place is filled by 5 numbers.

Therefore, by FPC, the number of 3 digit numbers = $5\times5\times5=125$

Factorial of a number

Let n be a positive natural number. The product of first n natural numbers is known as factorial of n. It is denoted by $n!\ or\ \lfloor\underline{n}$.

$$n!\ or\ \lfloor\underline{n}=1\times2\times3\times...\times n$$

$$n(n-1)(n-2)...\times3\times2\times1$$

$$1!=1$$

$$2!=1\times2=2$$

$$3!=1\times2\times3=6$$

$$4!=1\times2\times3\times4=24$$

$$5!=1\times2\times3\times4\times5=120$$

$$6!=1\times2\times3\times4\times5\times6=720$$

$$0!=1\ and\ (-n)!=\infty$$

$$5!=1\times2\times3\times4\times5=4!\times5$$

$$5!=(5-1)!\times5$$

$$\boxed{\begin{aligned}&Thus,\ n!=(n-1)!\times n\\&n!=(n-2)!\times(n-1)\times n\end{aligned}}$$

Q.: Find $x, if\ \frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$

$\frac{1}{6!}+\frac{1}{6!\times 7}=\frac{x}{6!\times 7\times 8}$

$\times ing\ by\ 6!$

$1+\frac{1}{7}=\frac{x}{7\times 8}\Rightarrow \frac{7+1}{7}=\frac{x}{7\times 8}$

$\Rightarrow \frac{8}{1}=\frac{x}{8}\Rightarrow x=8\times 8=64$

Q: Is $7!-4!=3!$?

No.

$7!-4!=5!\times 6\times 7-24=120\times 42=5040$

But $3!=6$

$\therefore\ 7!-4!\neq 3!$

Permutation

Permutation is an arrangement in a particular order of a number of object taken some or all at a time. The number of permutations or arrangements of 'n' different things, taken 'r' at a time ($0<r\leq n$) is ${}^{n}P_{r}$.

By theorem, ${}^{n}P_{r}=\frac{n!}{(n-r)!}$

OR

${}^{n}P_{r}=n(n-1)(n-2)(n-3)...(n-r+1)$

E.g.:

${}^{5}P_{3}=\frac{5!}{(5-3)!}=\frac{120}{2!}=\frac{120}{2}=60$

OR

${}^{5}P_{3}=5(5-1)(5-2)=5(4)(3)=60$ $||\ n-r+1=5-3+1=3$. So take the reverse multiple of 5 in 3 numbers. i.e., $5\times 4\times 3$

Note:

1. ${}^{n}P_{1}=\frac{n!}{(n-1)!}=\frac{(n-1)!\times n}{(n-1)!}=n$

2. $^{n}P_{0}=\dfrac{n!}{(n-0)!}=\dfrac{n!}{n!}=1$

3. $^{n}P_{n}=\dfrac{n!}{(n-n)!}=\dfrac{n!}{0!}=\dfrac{n!}{1}=n!$

4. The number of permutations of n objects, where p objects are of the same kind and rest are all different $=\dfrac{n!}{p!}$.

   E.g.: How many words with or without meaning can be formed using all the letters of the word RADIAL?

   The word 'RADIAL' has

R	-	1
A	-	2
I	-	1
D	-	1
L	-	1

   Total = 6

   $\therefore$ Total no. of arrangements $=\dfrac{6!}{2!}=\dfrac{720}{2}=360$

5. The number of permutations of n objects, where p_1 objects are of the first kind, p_2 are of the 2nd kind, …, p_k are of the 3rd kind, then number of permutations $=\dfrac{n!}{p_1!\,p_2!\ldots p_k!}$.

E.g.: 1) How many words with or without meaning can be formed using all the letters of the word MALAYALAM?

The word 'MALAYALAM' has

M	-	2
A	-	4
L	-	2
Y	-	1

Total = 9

$\therefore$ Total no. of arrangements $=\dfrac{9!}{2!\times 4!\times 2!}=\dfrac{4!\times 5\times 6\times 7\times 8\times 9}{2\times 4!\times 2}=60\times 63=3780$

2) How many 4 digit numbers are there with no digit repeated?

Let the total number of digits = 10 (0-9).

No. of 4 digits nos including '0' in the 1000's place =

$$^{10}P_{4}=\frac{10!}{(10-4)!}=\frac{10!}{6!}=7\times 8\times 9\times 10=5040$$

If '0' comes in the 1000's place, the number becomes 3 digits.

No. of 3 digits nos if '0' comes in the 1000's place =

$$^9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 7\times8\times9 = 504$$

$\therefore$ Required number of 4 digits numbers = $5040 - 504 = 4536$

3) In how many words, with or without meaning can be formed using all the letters of the word MONDAY, assuming that no letter is repeated if all letters are used but first letter is a vowel?

The word MONDAY has two vowels – A and O and these two vowels can be arranged in $^2P_1$ ways = 2! ways.

If one vowel is fixed at the first position, the remaining 5 letters can be arranged in $^5P_5$ ways = 5! ways.

$\therefore$ Total number of arrangements= $2!\times5! = 2\times120 = 240$

4) Find the value of 'n' such that

a) $^nP_5 = 42\,^nP_3,\ n>4$

$^nP_5 \to n-r+1 = n-5+1 = n-4$

$^nP_3 \to n-r+1 = n-3+1 = n-2$

$\therefore\ ^nP_5 = 42\,^nP_3$

$n(n-1)(n-2)(n-3)(n-4) = 42\times n(n-1)(n-2)$

$(n-3)(n-4) = 42$

$n^2 - 7n + 12 - 42 = 0$

$n^2 - 7n - 30 = 0 \Rightarrow (n-10)(n+3) = 0$

$n = 10\ or\ n = -3$

But n cannot be negative.

$\therefore\ n = 10$.

5) Find the value of 'r' if $5\,^4P_r = 6\,^5P_{r-1}$

$$5\times\frac{4!}{(4-r)!} = 6\times\frac{5!}{[5-(r-1)]!}$$

$$\frac{5!}{(4-r)!} = \frac{5!\times6}{(6-r)!} \Rightarrow \frac{1}{(4-r)!} = \frac{6}{(4-r)!\times(5-r)(6-r)}$$

$$\frac{1}{1} = \frac{6}{(5-r)(6-r)} \Rightarrow (5-r)(6-r) = 6$$

$30 - 5r - 6r - r^2 - 6 = 0 \Rightarrow r^2 - 11r + 24 = 0$

$(r-8)(r-3) = 0$

$r = 8\ or\ r = 3$

But $r \le n,\ \therefore r = 3$

6) How many ways can the letters of the word PERMUTATIONS be arranged if there are always 4 letters between P and S?

The word PERMUTATIONS has 12 letters, so there are 12 vacant places.

P	P	P	P	P	P	P					
1	2	3	4	5	6	7	8	9	10	11	12
					S	S	S	S	S	S	S

If P comes in the vacant places 1,2,3,4,5,6 and 7 and if S comes in the vacant places 6,7,8,9,10,11 and 12, we can insert 4 letters in between P and S.

Hence, we can insert 4 letters in between P and S or S and P in 7+7 = 14 ways.

Then the remaining 10 letters can be arranged in $\frac{10!}{2!}$ (T is repeated twice).

$\therefore$ the total permutations = $\frac{10!}{2!}\times 14 = \frac{6!\times 7\times 8\times 9\times 10}{2}\times 14 = 720\times 7\times 8\times 9\times 10\times 7$

$$= 720\times 49\times 720 = 25401600$$

6) Find the number of permutations of the word MATHEMATICS. In how many of these arrangements,

i. do the words start with E
ii. do all the vowels always occur together
iii. do all the vowels never occur together
iv. do the words begin with H and end in I.

The word MATHEMATICS has

M	- 2
A	- 2
T	- 2
H	- 1
E	- 1
I	- 1
C	- 1
S	- 1
Total	**- 11**

The total number of permutations $= \frac{11!}{2!\times 2!\times 2!} = 4989600$

v. If the words begin with H and end in I, let we can fix these letters and find the permutations of the remaining letters.

M	- 2
A	- 2
T	- 2
E	- 1
C	- 1
S	- 1
Total	**- 11**

Required permutations

i. If the words start with E, then fix the letter E and find the permutations of the remaining letters.

M	- 2
A	- 2
T	- 2
H	- 1
I	- 1
C	- 1
S	- 1
Total	**- 10**

Required permutations $= \dfrac{10!}{2!\times 2!\times 2!} = 453600$

ii) If all the vowels always occur together,

AAEI	- 1
M	- 2
T	- 2
H	- 1
C	- 1
S	- 1
Total	**- 8**

A	- 2
E	- 1
I	- 1
Total	**- 4**

Required permutations $= \dfrac{8!}{2!\times 2!}\times\dfrac{4!}{2!} = \dfrac{8!\times 24}{8} = 8!\times 3 = 120960$

iii) If all the vowels never occur together, Required permutations = Total permutations without any restriction -the number of permutations where all the vowels occur together.

$$4989600 - 120960 = 4868640$$

iv) If the words begin with H and end in I, let we can fix these letters and find the permutations of the remaining 9 letters.

$$\text{Required permutations } = \frac{9!}{2!\times 2!\times 2!} = 45360$$

COMBINATIONS

Combinations

A combination is a group or selection of a number of objects irrespective of the order in which they occur.

The number of combination of 'n' objects taken 'r' at a time is denoted by ${}^{n}C_{r}$ or $C_{(n,r)}$ or $\binom{n}{r}$, where n and r are whole numbers and $0 < r \leq n$.

Theorem(Factorial formula):

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}, \text{ for } 1 \leq r \leq n$$

$$\text{E.g.: } {}^{5}C_{3} = \frac{5!}{3!(5-3)!} = \frac{3!\times 4\times 5}{3!\times 2!} = \frac{20}{2} = 10$$

Note: $${}^{n}C_{r} = \frac{\text{Re verse multiple of n, r in numbers}}{\text{Factorial of r}}$$

$$\text{E.g.: } {}^{5}C_{3} = \frac{5\times 4\times 3}{1\times 2\times 3} = 10$$

Theorem(Reduction formula):

$${}^{n}C_{r} = {}^{n}C_{n-r}$$

E.g.:

i) $${}^{5}C_{3} = {}^{5}C_{5-3} = {}^{5}C_{2} = \frac{5\times 4}{1\times 2} = 10$$

ii) $${}^{12}C_{9} = {}^{12}C_{12-9} = {}^{12}C_{3} = \frac{12\times 11\times 10}{1\times 2\times 3} = 220$$

Theorem(Pascal's Rule):

$${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r} \quad ;1 \leq r \leq n$$

$$\text{LHS} = {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\left[n-(r-1)\right]!}$$

$$= \frac{n!}{(r-1)! \times r\,(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{(r-1)! \times r\,(n-r)!} + \frac{n!}{(r-1)!(n-r)! \times (n-r+1)}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$

$$= \frac{n!(n+1)}{(r-1)!r(n-r)!(n-r+1)}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!}$$

put $n+1 = m$

$$= \frac{m!}{r!(m-r)!} = {}^m C_r$$

replacing m by $(n+1)$

$$= {}^{(n+1)} C_r = \text{RHS}$$

Relation between ${}^nP_r$ and ${}^nC_r$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$r! \times {}^nC_r = \frac{n!}{(n-r)!}$$

$$\boxed{{}^nC_r \times r! = {}^n P_r}$$

Note:

1. $^{n}C_0 = \frac{n!}{0!\times(n-0)!} = \frac{n!}{1\times n!} = 1$
2. $^{n}C_1 = \frac{n!}{1!\times(n-1)!} = \frac{(n-1)!\times n}{(n-1)!} = n$
3. $^{n}C_n = \frac{n!}{n!\times(n-n)!} = \frac{n!}{n!\times 0!} = \frac{1}{1} = 1$
4. $^{n}C_x = {}^{n}C_y \Rightarrow x = y \text{ or } x + y = n$

E.g.:

i) If $^{n}C_8 = {}^{n}C_9$, then $n = 8 + 9 = 17$

ii) $^{n}C_{n-2} = 28$, find n?

$^{n}C_{n-2} = 28 \Rightarrow {}^{n}C_{n-(n-2)} = 28$

$\Rightarrow {}^{n}C_2 = 28 \Rightarrow \frac{n(n-1)}{1\times 2} = 28$

$\Rightarrow n^2 - n = 56$

$\Rightarrow n^2 - n - 56 = 0$

$\Rightarrow (n-8)(n+7) = 0$

$\Rightarrow n = 8 \text{ or } n = -7$

But n cannot be $-$ve.

$\therefore n = 8$

Q: How many diagonals does a polygon have?

Let the number of sides = n

Total number of lines that can be formed from n points = $^{n}C_2$

Number of diagonals $= {}^{n}C_2 - n = \frac{n(n-1)}{1\times 2} - n$

$$= \frac{n^2 - n - 2n}{2} = \frac{n^2 - 3n}{2} = \frac{n(n-3)}{2}$$

Note: Similarly, No. of sides of the polygon $= {}^{n}C_2 - n = \frac{n(n-1)}{2} - n$

$$= \frac{n^2 - n - 2n}{2} = \frac{n^2 - 3n}{2} = \frac{n(n-3)}{2}$$

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