#### Chapter 14

#### **OSCILLATIONS**

#### **Periodic Motion:-**

A motion which repeats at regular intervals of time is called a periodic motion.

#### Examples:-

- (i) The motion of any planet around the sun.
- (ii) The motion of moon around earth.
- (iii) The motion of the hands of a clock.

#### Oscillation or Harmonic motion.

To and fro motion of a body about a mean position is called oscillation or harmonic motion.

Oscillations with high frequency are usually called vibrations.

#### Examples of Oscillation or Vibration:-

- (i) Oscillation of a simple pendulum.
- (ii) To and fro motion of the piston of an automobile engine.
- (iii) Vibrations of an excited turning fork.

## **Difference between Periodic and Oscillatiotory motions:**

Every oscillatory motion is necessarily periodic. But every periodic motion need not be oscillatory. For example, The motion of earth around the sun is periodic but it is not oscillatory.

### Period(T)

The smallest time interval after which a periodic motion is repeated is called the time period (T).

If a particle oscillates N times in a time't' seconds, its time period.

$$T = \frac{t}{N}$$

#### Frequency(v)

The number of repetitions per second is called the frequency.

If a particle oscillates N times in a time't' seconds, its frequency.

$$\mathbf{V} = \frac{N}{t} = \frac{1}{T}$$

S I unit is S<sup>-1</sup> or Hz (hertz)

#### Angular Frequency(ω)

$$\omega = \frac{2\pi}{T} = 2\pi v$$

**SI Unit:**- radian per second (rad  $s^{-1}$ )

**Problem1**:- On an average a human heart is found to beat **72** times in a minute. Calculate period and frequency of the of heart beat.

#### Soln:

$$N = 72$$
,  $t = 1$  min. = 60sec.,  $T = ?$ ,  $v = ?$ 

$$T = \frac{t}{N} = \frac{60}{72} = 0.83s$$
$$V = \frac{1}{T} = \frac{1}{0.83} = 1.2 \text{ Hz}$$

#### **Displacement Variable**

The physical quantity which changes with time in a Periodic motion is called displacement Variable or displacement.

Displacement variable can be physical quantities such as position, angle, voltage, pressure, electric field, magnetic field etc.

#### Examples:-

 $\rightarrow$  When a body attached at the end of a spring vibrates, the displacement variable is the **position vector** from its equilibrium position.

 $\rightarrow$  In the case of the oscillation of a simple pendulum, the displacement variable is the **angle** from the vertical.

 $\rightarrow$  In the study of **a.c**, the displacement variable is the **voltage** or current.

 $\rightarrow$  For sound waves travelling through air, the displacement variable is the **pressure**.

 $\rightarrow$  For the propagation of electromagnetic waves the displacement variable is the electric and magnetic field vectors.

#### Mathematical Representation of Periodic motion

A periodic motion can be represented using a sine function, a cosine function or a linear combination of sine and cosine function.

 $f(t) = A \cos \omega t$ 

Or

 $f(t) = A \sin \omega t$ 

Or

 $f(t) = A \sin \omega t + B \cos \omega t$ 

A periodic function has the property f(t+T) = f(t), where T is the time period of the function.

#### Problem2:-

**S.** T the function  $f(t) = A \sin \omega t$  is periodic.

## Solution:-

$$f(t) = A \sin \omega t$$

$$f(t+T) = A \sin \omega(t+T)$$

= A sin (
$$\omega t + \omega T$$
)

But 
$$\omega T = \frac{2\pi}{T}T = 2\pi$$

$$f(t+T) = A \sin(\omega t + 2\pi)$$

$$= A \sin \omega t = f(t)$$

 $\Rightarrow f(t+T)=f(t)$ 

That is f(t) is a periodic function.

## Fourier theorem

This theorem states that any periodic function can be expressed as a linear combination of **sine** and **cosine** functions of different time periods with suitable coefficients.

 $f(t) = A \sin \omega t + B \cos \omega t$  is a periodic function.

# Problem3:-

Which of the following functions of time represent (a)

Periodic and (b) non-periodic motion? Give the time period for each case (i)  $\sin\omega t + \cos\omega t$ (ii)  $\sin\omega t + \cos 2\omega t + \sin 4\omega t$ (iii)  $e^{-\omega t}$ (iv)  $\log(\omega t)$ Solution:- $\sin \omega t + \cos \omega t$  is a periodic (i) function, with time period T=  $2\pi$ Fαx ω (ii)  $\sin \omega t + \cos 2\omega t + \sin 4\omega t$  is a or periodic function Time period of sin  $\omega t$ , T<sub>o</sub>= $\frac{2\pi}{\omega}$ Time period of cos2 $\omega$ t, T=  $\frac{2\pi}{2\omega}$ F = ma $=\frac{\pi}{\omega}=\frac{T_{o}}{2}$ Time period of sin 4 $\omega$ t, T= $\frac{2\pi}{4\omega}$  $=\frac{\left(\frac{2\pi}{\omega}\right)}{1}=\frac{T\circ}{1}$ (i) (ii) ... The time period of the given (iii) function is  $T_0 = \frac{2\pi}{\omega}$  (common time (iv) period or LCM) (v) The function  $e^{-\omega t}$  is not (iii)

- periodic. It decreases continuously with increase in time. When  $t \rightarrow \infty$ ,  $e^{-\omega t} \rightarrow 0$  and thus never repeats its value.
- The function  $log(\omega t)$  is not (iv) periodic, it increases with time 't'. When  $t \rightarrow \infty$ , log  $\omega t \rightarrow \infty$ and thus never repeats its value.

#### Simple Harmonic Motion (SHM)

**Definition:**- An oscillating particle is said to execute SHM if the restoring force on the particle at any instant of time is directly proportional to its displacement from the mean position and is always directed towards the mean position.

Restoring force  $\alpha$  Displacement

F= -kx

 $k \rightarrow force constant$ 

By Newton's 2<sup>nd</sup> Law

$$\rightarrow$$
 ma = -kx

$$\Rightarrow a = \frac{-k}{m}x$$

i.e., a α x

## **Examples of SHM:**

- Oscillations of a loaded spring
- Vibrations of a tuning fork
- Vibrations of balance wheel of a watch.
- Oscillations of a simple pendulum
- Oscillations of a freely suspended magnet in a uniform magnetic field.

## **Differential equation of SHM**

We have  $\mathbf{F} = -kx$  for SHM.

By Newton's  $2^{nd}$  law F = ma,

$$a = \frac{d(v)}{dt} = \frac{d}{dt} \left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\therefore m \frac{d^2 x}{dt^2} = -kx$$
  
or  $\frac{d^2 x}{dt^2} = \frac{-k}{m}x$   
Put  $\frac{k}{m} = \omega^2$ , then  $\frac{d^2 x}{dt^2} = -\omega^2 x$   
I.e.,  $\frac{d^2 x}{dt^2} + \omega^2 x = 0$ , which is the  
differential equation of SHM

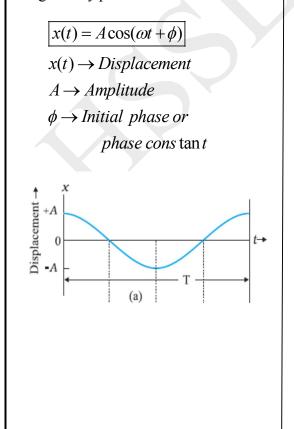
The solution of this Differential equation is of the form

 $x = A\cos(\omega t + \phi)$ 

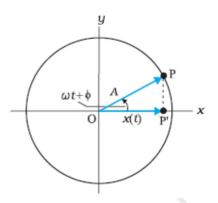
#### **Displacement in SHM**

Consider a particle vibrating back and forth about the origin of an x-axis between the limits -A and +Aas shown in figure.

The displacement x(t) of the particle is given by phase



### <u>Simple Harmonic Motion and</u> <u>Uniform Circular Motion</u>



Consider the motion of a reference particle **P** executing uniform circular motion on a reference circle of radius A. At any time the angular position of the particle is  $\omega t + \phi$ 

where  $\phi$  is its angular position at t=0.

The projection of the point  $\mathbf{P}$  on the x -axis is the point  $\mathbf{P'}$ .

The projection of position vector of the reference particle **P** on the *x*-axis gives the location x(t) of **P'**. Thus we have,

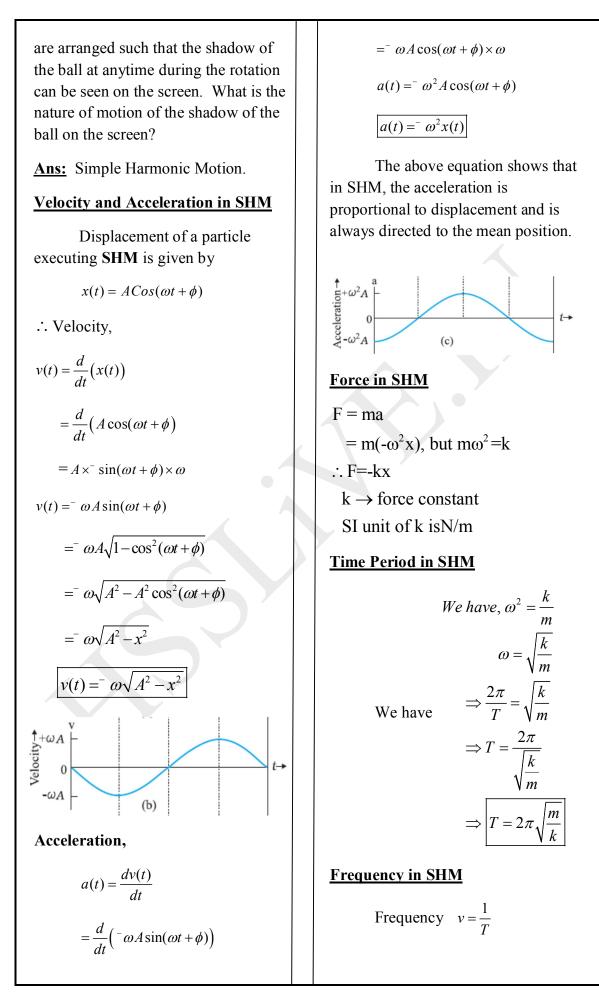
 $x(t) = A\cos(\omega t + \phi)$ 

This shows that if the reference particle **P** moves in a **uniform circular motion**; its projection particle **P'** executes a **simple harmonic motion** along a diameter of the circle.

Thus a simple harmonic motion can be defined as the projection of uniform circular motion on a diameter of the circle.

## <u>Question1:</u>

A ball is fixed on the edge of a turn table. A light source and a screen



$$= \frac{1}{2\pi\sqrt{\frac{k}{m}}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

# $V = \frac{1}{2\pi}\sqrt{m}$

# Energy in SHM

A particle executing **SHM has kinetic and potential energies**, both varying between the limits, zero and maximum.

Kinetic energy,

$$K = \frac{1}{2}mv^{2}$$
$$= \frac{1}{2}m[-\omega A\sin(\omega t + \phi)]^{2}$$
$$= \frac{1}{2}m\omega^{2}A^{2}\sin^{2}(\omega t + \phi)$$
$$K = \frac{1}{2}kA^{2}\sin^{2}(\omega t + \phi)$$

Potential Energy,

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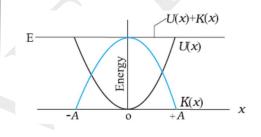
$$U = \frac{1}{2}kx^{2}$$
$$= \frac{1}{2}k(A\cos(\omega t + \phi))^{2}$$
$$U = \frac{1}{2}kA^{2}\cos^{2}(\omega t + \phi)$$

Total energy,

$$E = K + U$$
$$= \frac{1}{2}kA^{2}\sin^{2}(\omega t + \phi)$$

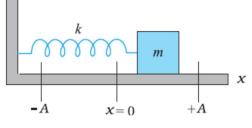
$$+\frac{1}{2}kA^{2}\cos^{2}(\omega t + \phi)$$
$$=\frac{1}{2}kA^{2}\left[\sin^{2}(\omega t + \phi) + \cos^{2}(\omega t + \phi)\right]$$
$$=\frac{1}{2}kA^{2}$$
$$\therefore \boxed{E = \frac{1}{2}kA^{2}}$$

Total energy of a harmonic oscillation is independent of time, for any conservative force.



# Some Systems Executing SHM.

1. Horizontal Oscillations of a block of mass attached to a spring.



A block of mass is placed on a horizontal frictionless surface. If the block is pulled on one side and is released, it then execute to and fro motion about the mean position x=0.

The restoring force acting on the block,

$$F = -kx$$

#### 'k' is called spring constant.

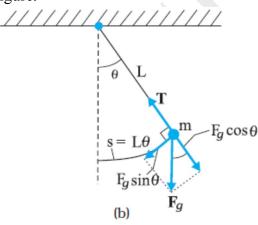
The above equation is same as the force law for SHM and therefore the system execute SHM.

Here  $k = m\omega^2$ 

$$\omega^{2} = \frac{k}{m} \Longrightarrow \omega = \sqrt{\frac{k}{m}}$$
$$But, \omega = \frac{2\pi}{T}$$
$$\therefore \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$
$$T = 2\pi \sqrt{\frac{m}{k}}$$

#### 2. The Simple Pendulum

The forces acting on the both are force '**T**', tension in the string and the gravitational force '**mg**' as shown in figure.



 $F_{g}cos\theta$  cancels with the tension T in the string.

 $F_g sin \theta$  acts as the restoring force.

Restoring torque,

- $\tau = Force \times \perp distance$
- $=-F_{g}\sin\theta \times L$

= -mg sin  $\theta \times L$ 

Negative sign shows that force acts to reduce  $\theta$ .

But we have  $\tau = I \alpha$ 

$$\therefore I\alpha = -mgL\sin\theta$$
$$\alpha = \frac{-mgL}{L}\sin\theta$$

If 
$$\boldsymbol{\theta}$$
 is small,  $\sin \boldsymbol{\theta} \approx \boldsymbol{\theta} (\boldsymbol{\theta} \text{ in radian})$ 

$$\therefore \alpha = \frac{mgL}{I}\theta$$

But the moment of inertia of the bob is,  $I = mL^2$ 

$$\therefore \alpha = \frac{mgL}{mL^2}\theta = \frac{g}{L}\theta$$

The above equation shows that simple pendulum swinging through small angles is SHM.

Comparing the above equation to the general equation of SHM

 $[\alpha = -\omega^2 \theta]$ , rotational analogue of  $a = -\omega^2 x$ ], we get  $\omega^2 = \frac{g}{L}$ 

$$\omega = \sqrt{\frac{g}{L}}$$
$$\frac{2\pi}{T} = \sqrt{\frac{g}{L}}$$

$$\Rightarrow T = 2\pi \sqrt{\frac{L}{g}}$$

This is the expression for time period of a simple pendulum.

Frequency,

$$v = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{g}{R}}$$

$$v = \frac{1}{2\pi} \sqrt{\frac{g}{L}}$$

## <u>The time period of a simple</u> <u>pendulum depends on:</u>

(1) Length of the pendulum (L) and(2) Acceleration due to gravity (g).

## Time period is independent of:

(1) Mass of the bob (**m**)

(2) Amplitude of  $oscillations(\theta)$ 

## Second's pendulum

Pendulum having time period T=2s is called a seconds pendulum.

# Length of second's pendulum:-

$$T = 2\pi \sqrt{\frac{L}{g}}$$
$$T^{2} = 4\pi^{2} \frac{L}{g}$$
$$\Rightarrow 2^{2} = 4 \times 3.14^{2} \times \frac{L}{9.8}$$
$$\Rightarrow L = \frac{4 \times 9.8}{4 \times 3.14^{2}} \approx 1m$$

#### <u>Acceleration due to gravity using</u> <u>simple pendulum.</u>

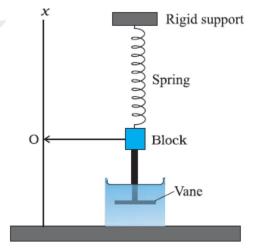
We have

$$T = 2\pi \sqrt{\frac{L}{g}}$$
$$\Rightarrow T^{2} = 4\pi^{2} \frac{L}{g}$$
$$\Rightarrow g = 4\pi^{2} \frac{L}{T^{2}}$$

By using the above equation, we can calculate the acceleration due to gravity at a place.

#### Damped Simple Harmonic Motion

The oscillations are said to be damped if the amplitude of oscillations continuously decreases due to dissipating forces like frictional force.



If the block of mass is set into vertical oscillations, due to the damping forces exerted by the viscous medium, the oscillations will be damped.

Here the damping force, F = -bv

 $b \rightarrow$  Damping constant.

The –ve sign shows that the damping force is opposite to the velocity of the block of mass.

The restoring force on the block due to the spring

F = -kx

Total force F = -kx - bv

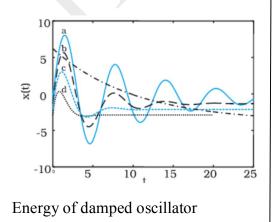
But F = ma  $\Rightarrow ma = -kx - bv$   $\Rightarrow ma + bv + kx = 0$  $\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$ 

This is a second order differential equation and the solution is of the form:  $x(t) = Ae^{-bt/2m} \cos(\omega' t + \phi)$ 

Here  $Ae^{-bt/2m}$  is the amplitude of oscillations and  $\omega'$  is the angular frequency of the damped oscillator.

$$w' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The amplitude of oscillations  $Ae^{-bt/2m}$  continuously **decreases** because of the factor  $e^{-bt/2m}$ .



$$E(t) = \frac{1}{2}k(ampitude)^{2}$$
$$= \frac{1}{2}k(A^{-bt/2m})^{2}$$
$$= \frac{1}{2}kA^{2}e^{-bt/m}$$

Energy also continuously **decreases** with time.

#### Forced Oscillations And Resonance

When a body oscillates under the influence of an external periodic force, not with its own natural frequency but with the frequency of the external periodic force, its oscillations are said to be forced oscillations or driven oscillations.

Eg:- When the stem of a vibrating tuning fork is pressed against a table, a loud sound is heard. This is because the particles of the table are forced to vibrate with the frequency of the tuning fork.

Suppose an external periodic force  $F_0 \cos \omega_d t$  applied to a damped oscillatior.

 $F_0 \rightarrow amplitude \text{ of the periodic}$ force,  $\omega_d \rightarrow$  frequency of the driving force.

Total force acting on the oscillator at any time,

$$F = -kx - bv + F_0 \cos \omega_d t$$
  

$$\Rightarrow ma = -kx - bv + F_0 \cos \omega_d t$$
  

$$\Rightarrow ma + bv + kx = F_0 \cos \omega_d t$$
  

$$\Rightarrow m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos \omega_d t$$

The displacement of the oscillator, after the natural oscillations die out and the oscillator oscillates with the frequency of the force, is obtained by solving the above differential equation.

$$x(t) = A' \cos(\omega_d t + \phi)$$

Here,

$$A' = \frac{F_0}{\left[m^2(\omega^2 - \omega_d^2)^2 + \omega_d^2 b^2\right]^{\frac{1}{2}}}$$

#### Special case 1

<u>Small damping</u>: Driving frequency far from natural frequency.

In this case,  $w_d^2 b^2 \ll m^2 (\omega^2 - \omega_d^2)^2$ and we can neglect that term.

#### Special case 2

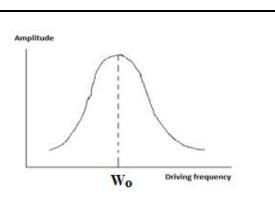
Driving frequency close to the natural frequency:

If  $\omega_d$  is very close to  $\omega$ , then  $m^2(\omega^2 - \omega_d^2)^2 << w_d^2 b^2$ 

Therefore, 
$$A' = \frac{F_0}{\omega_d t}$$

In this case, if the damping also is small, the amplitude may increase to a very large value.

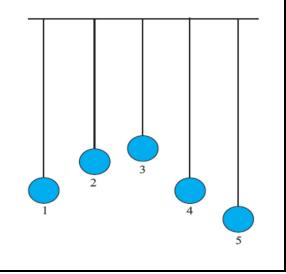
The phenomenon of increase in amplitude of oscillations when the driving frequency is close to the natural frequency is called resonance.



#### **Examples of resonance.**

- 1. Resonance of air column in resonance column apparatus.
- 2. Resonance of **sonometer** wire.
- 3. The rattling of window glass during thunder.
- 4. Soldiers marching along a road are asked to break up their steps while reaching a bridge. This is to avoid resonance between the stamping of soldier and vibrations of the bridge and consequent breakdown of the bridge.
- Tacoma Bridge in Washington was destroyed by resonance produced by wind.
- 6. During Earthquakes, short and tall structures remain unaffected while the medium height structures fall down.

Experimental illustration of resonance.



Let us set **pendulum 1** into motion. Then the pendulums 2,3,4 and 5 start oscillating with their natural frequencies and their frequencies of oscillations gradually change and finally they oscillate with the frequency of **pendulum 1** but with different amplitudes.

**Pendulum 4** oscillates with the same frequency of pendulum 1 and its amplitude gradually increases to a large value. **Pendulum 4 has resonance.** 

Free oscillation is for : Pendulum 1.

**Forced oscillations** : Pendulum 2,3,4 and 5.

**Resonance** : Pendulum 4 (since its frequency is same as that of pendulum 1)

## Important Questions and their Answers

 Write down the equations related to SHM parallel to (i) X axis (ii) Y axis.

Ans:- Parallel to X axis

$$x = A\cos(\omega t + \phi$$

$$v_x = -\omega A \sin \omega t$$

$$1 \int d^2$$

$$T = \frac{2\pi}{\omega}$$

## Parallel to Y axis

$$y = A\sin(\omega t + \phi)$$

$$v_{y} = \omega A(\cos \omega t + \phi)$$
$$v_{y} = \omega \sqrt{A^{2} - y^{2}}$$
$$a_{y} = \omega^{2} y$$
$$T = \frac{2\pi}{\omega}$$

2. A girl is oscillating in a swing with a time period T. What will happen to the period, it she stands up?

**Ans:**- 
$$T = 2\pi \sqrt{\frac{l}{g}}$$

L is the distance of suspension to the centre of gravity. When she stands up, the centre of gravity will get raised. Therefore  $\ell$  decreases and T decreases.

3. The bob of a simple pendulum is a ball filled with water. As it is oscillating water flows out through a hole at the bottom. What happens to the period?

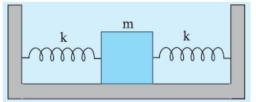
Ans:- The centre of gravity is originally at the centre. When water flows out the centre of gravity gets lowered, reaches a certain lower point and then rises to the original level when all the water flows out. Therefore ' $\ell$ ' will first increase, reach a maximum and then decrease to the original value. Therefore period will first increase, reach a maximum and then decrease to the original value.

**4.** What is the formula for the effective force constant of two springs in parallel?

**Ans**:- 
$$k = k_1 + k_2$$

$$T=2\pi\sqrt{\frac{m}{k_1+k_2}}$$

5. Write the time period of oscillations of the mass in figure.



Ans:

$$F = -2kx$$
  

$$\omega^{2} = \frac{2k}{m}$$
  

$$\omega = \sqrt{\frac{2k}{m}}$$
  

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{2k}}$$

**6**. What is the formula for effective force constant of two springs in series?

Ans:-

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$
$$\Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$$
$$T = 2\pi \sqrt{\frac{m}{\frac{k_1 k_2}{k_1 + k_2}}}$$

7. Write the expressions f or **KE**, **PE** and total energy in SHM.

Ans:-

$$PE = \frac{1}{2}kx^{2}$$
$$KE = \frac{1}{2}mv^{2} = \frac{1}{2}m(\omega\sqrt{A^{2} - x^{2}})^{2}$$

$$=\frac{1}{2}m\omega^{2}(A^{2} - x^{2})$$

$$\overline{KE} = \frac{1}{2}k(A^{2} - x^{2})$$

$$TE = PE + KE$$

$$= \frac{1}{2}kx^{2} + \frac{1}{2}k(A^{2} - x^{2})$$

$$= \frac{1}{2}kx^{2} + \frac{1}{2}kA^{2} - \frac{1}{2}kx^{2}$$

$$= \frac{1}{2}kA^{2}$$

8. A spring with spring constant 1200 N/m is mounted on a horizontal table as shown. A mass of 3 Kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2 cm and released.

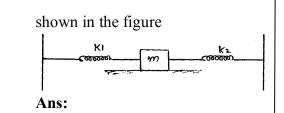
Determine:

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- (i) Frequency of oscillations.
- (ii) Maximum acceleration of the mass.

Ans:

**9.** Write the expression for the time period of oscillation of the system



**10.** a) Write the expression for the time period of a spring mass system?

b) If the spring is cut into two equal halves and one half of the spring is used to suspend the same mass then obtain an expression for the ratio of periods of oscillation in the two cases.

c) If this system is completely immersed in water then what happens to the oscillation? **Ans:** 

11. Ramu tied a spherical pot with a string and suspended on a clamp. He then filled it with water. Length of the string is 90 cm and radius of the pot is 10 cm. He then slightly displaced the pot to one side and made it to oscillate. Calculate the period of oscillation of the pot. Ans:

**12.** What is the frequency of oscillations of a simple pendulum mounted in a cabin which is freely falling under gravity?

Ans:

13. If 'T' is the period of oscillation of a simple pendulum. Show that the time taken by the bob to go directly from its mean position to half the amplitude is  $\frac{T}{12}$ .

Ans:

**14.** a) What is the shape of  $L-T^2$  graph of a simple pendulum? b) How would you calculate the value of acceleration from L- $T^2$  graph? c) Calculate the length of a simple pendulum of period  $\sqrt{3}$ seconds. Ans: 16. A body oscillates with S.H.M. is is given by  $x = 5\cos\left(2\pi t + \frac{\pi}{4}\right)$ . Calculate the displacement at time t=1.5s Ans: **15.** Explain what happens to the period of oscillations, if the point of suspension of the pendulum Moves vertically upwards i) with an acceleration 'a' Moves downwards ii) vertically with acceleration 17. A particle executes S.H.M. less than the acceleration according to the equation due to gravity.  $\mathbf{x} = 5\sin\left(\frac{2\pi}{3}\mathbf{t}\right).$ Falls freely under gravity. iii) Moves horizontally with an iv) a) Find the period of oscillation acceleration 'a' Is taken to a height equal to b) What is the minimum time v) the radius of earth required for the particle to move between two points Ans:

