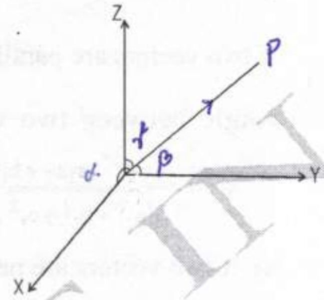
11. THREE DIMENSIONAL GEOMETRY

Direction Ratios

Let $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$ be the position vector of a point then a,b,c are called direction ratios of that point.

E.g.: If $\vec{r} = 2\hat{i} + 3\hat{j} + 5\hat{k}$, then dr's are 2, 3, 5

Direction Cosines



If α, β, γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive directions of the coordinate axes OX, OY and OZ respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosines of $\overrightarrow{OP}$ and are denoted by l, m, n respectively. i.e., $\ell = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$

Let $P(x, y, z)$ be a point in space such that $\vec{r}\left(= \overrightarrow{OP}\right) = a\hat{i} + b\hat{j} + c\hat{k}$ and their direction cosines ℓ, m, n then

a. $\ell = \dfrac{a}{|\vec{r}|}, m = \dfrac{b}{|\vec{r}|}, n = \dfrac{c}{|\vec{r}|}$

b. Hence the dr's are a, b, c

c. $\ell^2 + m^2 + n^2 = 1$

d. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

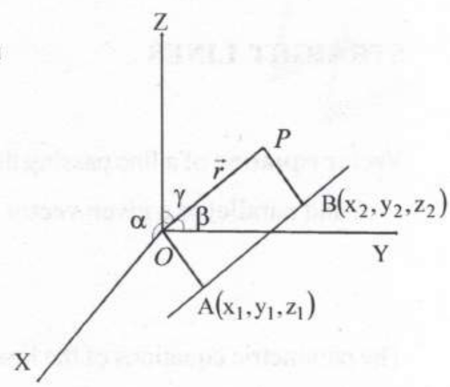
e. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Note:

Direction ratios are abbreviated by **dr's and direction cosines by dc's.**

The angles α, β, γ are called direction angles and lie between $[0, \pi]$.

dc's of x-axis	$\cos 0, \cos\frac{\pi}{2}, \cos\frac{\pi}{2}$	1,0,0
dc's of y-axis	$\cos\frac{\pi}{2}, \cos 0, \cos\frac{\pi}{2}$	0,1,0
dc's of z-axis	$\cos\frac{\pi}{2}, \cos\frac{\pi}{2}, \cos 0$	0,0,1



Let $P(x, y, z)$ be a point in space such that $\vec{r}\left(= \overrightarrow{OP}\right) = a\hat{i} + b\hat{j} + c\hat{k}$ and their direction cosines ℓ, m, n then

a. $\ell = \dfrac{a}{|\vec{r}|}, m = \dfrac{b}{|\vec{r}|}, n = \dfrac{c}{|\vec{r}|}$

b. Hence the dr's are a, b, c

c. $\ell^2 + m^2 + n^2 = 1$

d. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

e. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Direction ratios of the line joining two given points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points then

a. dr's of $\overrightarrow{AB}$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

Note: For finding the angle between the vectors, we can use the formula, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$.

Angle between two vectors in terms of their direction cosines ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 respectively, then

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

If two vectors are perpendicular, $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

If two vectors are parallel, $\dfrac{\ell_1}{\ell_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

Angle between two vectors in terms of their direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

a) If two vectors are perpendicular, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

b) If two vectors are parallel, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

c) dc's of $\overrightarrow{AB}$ are $\dfrac{x_2 - x_1}{|\overrightarrow{PQ}|}, \dfrac{y_2 - y_1}{|\overrightarrow{PQ}|}, \dfrac{z_2 - z_1}{|\overrightarrow{PQ}|}$

d) If two vectors are parallel, then its dr's and dc's are equal.

e) Projection of $\vec{r}$ on x-axis = $\ell|\vec{r}|$. Projection of $\vec{r}$ on y-axis = $m|\vec{r}|$ and Projection of $\vec{r}$ on z-axis = $n|\vec{r}|$.

The parametric equations of the line are $x = x_1 + \lambda a; y = y_1 + \lambda b; z = z_1 + \lambda c$, where λ is any parameter.

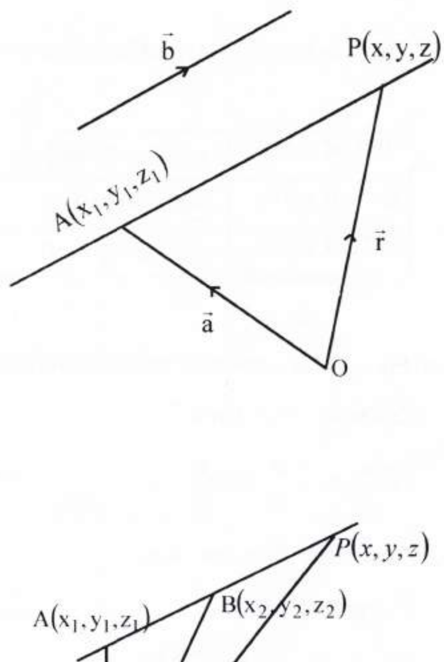
Vector equation of a line passing through two points with position vectors and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Cartesian equation of a line passing through two points $A(x_1, y_1, z_1)$ and

$B(x_2, y_2, z_2)$ is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

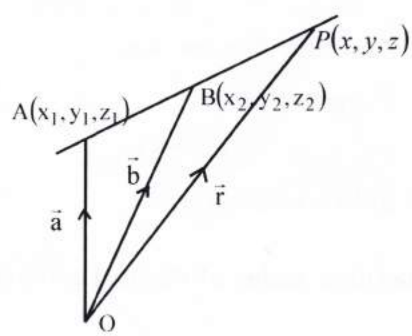
Cartesian equation of a line passing through a fixed point $A(x_1, y_1, z_1)$ and

having dr's a, b and c is $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$



STRAIGHT LINES

Vector equation of a line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$



The parametric equations of the line are $x = x_1 + \lambda a; y = y_1 + \lambda b; z = z_1 + \lambda c$, where λ is any parameter.

Vector equation of a line passing through two points with position vectors and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Cartesian equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

Cartesian equation of a line passing through a fixed point $A(x_1, y_1, z_1)$ and having dr's a, b and c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Angle between two lines (Vector form): Let $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ be two lines. Then angle between the lines

is $\cos\theta = \frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$

$$\theta = \cos^{-1}\left(\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right)$$

If two lines are perpendicular, $\vec{b}_1.\vec{b}_2 = 0$

If two lines are parallel, $\vec{b}_1 = \lambda\vec{b}_2$, where λ is any scalar.

If two lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ 32.

If two lines are parallel, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Angle between two lines (Cartesian form): Let the equations of the two lines be $l_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ *and*

$l_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is

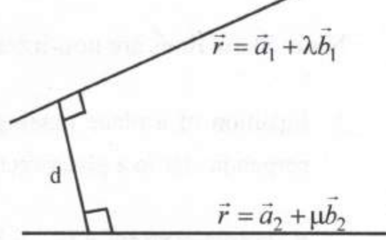
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\theta = \cos\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

If two lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

If two lines are parallel, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Coplanarity of lines Let $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{b}$ be any two lines. Let $\vec{a}$ and $\vec{c}$ be the position vectors of the fixed points A and C respectively.



If $\overrightarrow{AC}, \vec{b}$ and $\vec{d}$ are coplanar, $[\vec{c}-\vec{a}\ \vec{b}\ \vec{d}] = 0$ or $(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = 0$

If two lines intersecting or parallel, then $[\vec{c}-\vec{a}\ \vec{b}\ \vec{d}] = 0$

Shortest distance between the lines (Vector Equation)

Let the equations of two lines be $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

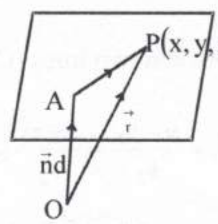
Cartesian Equation: The shortest distance between the lines

$l_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $l_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\left| \frac{\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

Distance between parallel lines:

Let $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ be any two paralle lines. Then the distance between the parallel lines is



$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

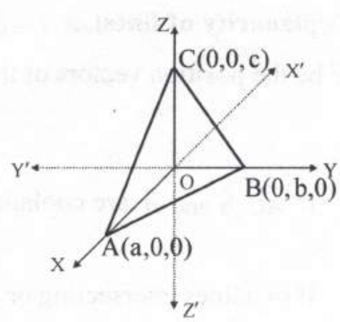
PLANE

If the line joining any two arbitrary points on the locus is the part of the locus, then the locus is called a plane.

1. The Cartesian equation of a plane is
   $(l\hat{i} + m\hat{j} + n\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = d$
   $\Rightarrow lx + my + nz = d$
   is the general equation of the plane.

2. Equation of a plane in the normal form is $\vec{n}.\vec{r} = d$

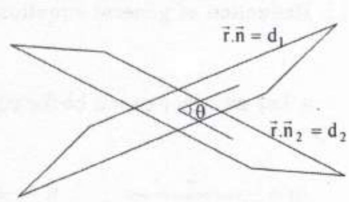
Note: Skew lines are non-intersecting and non parallel.



3. Eqaution of a plane passing through a fixed point whose position vector $\vec{r}_1$ and perpendicular to a given vector $\vec{n}$.

   a. Vector Equation: $\vec{n}.(\vec{r} - \vec{r}_1) = 0$

   b. Cartesian equation:



$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ is the cartesian equation.

4. Equation of a plane passing through three non collinear points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$.

   Let $P(x,y,z)$ be a general point on it. Then

   a. Vector equation is $\left[\overrightarrow{AP}\ \overrightarrow{AB}\ \overrightarrow{AC}\right]=0$

   b. Cartesian equation is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

5. **Intercept form** of a plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

6. **Angle between the planes**

a. Vector Equation: Let the equation of the planes be $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$.

   Then vector equation is $\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$ and $\theta=\cos^{-1}\left[\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right]$

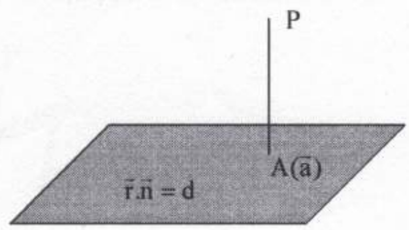
b. Cartesian Equation: Angle between the planes $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$ is

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \text{ and}$$

$$\theta=\cos^{-1}\left[\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right]$$

Note: If two planes are perpendicular, $a_1a_2+b_1b_2+c_1c_2=0$

If two planes are parallel, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.



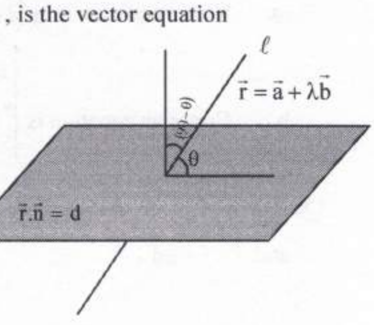
9. a) Vector equation of a plane parallel to a given plane $\vec{r}.\vec{n}=d$ is $\vec{r}.\vec{n}=k$, where k be any scalar.

   b) Cartesian equation of a plane parallel to the given plane $ax+by+cz=d$ is $ax+by+cz=k$

10. Equation of the planes bisecting the angle between the planes $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$ is

$$\frac{a_1x+b_1y+c_1z-d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z-d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

11. Length of the perpendicular from a point with position vector $\vec{a}$ to the plane $\vec{r}.\vec{n}=d$ is $\left|\frac{\vec{a}.\vec{n}-d}{|\vec{n}|}\right|$, is the vector equation



Cartesian equation: $PL=\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$

12. **Reduction of general equation to normal form**

a. Let $ax + by + cz = d$ be the equation of the plane. Let its normal be $lx + my + nz = p$, then $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$,

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$, $\quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$ and $p = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$.

Note: If d is positive, l, m, n and p are +ve and
if d is negative, l, m, n and p are negative.

LINE AND A PLANE

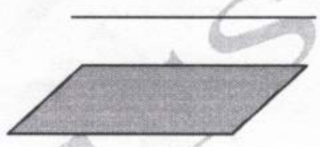
13. Angle between the line and a plane.

a. Let $\vec{r} = \vec{a} + \lambda\vec{b}$ be a line and $\vec{n}.\vec{r} = d$ a plane $\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$ and $\theta = \sin^{-1}\left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$

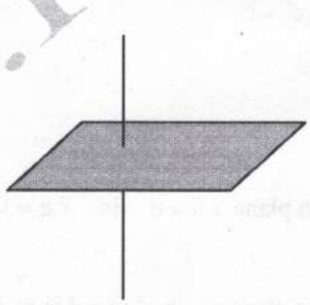
b. Let $\vec{a} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, $\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$ and $\vec{r} = r_1\vec{i} + r_2\vec{j} + r_3\vec{k}$ then

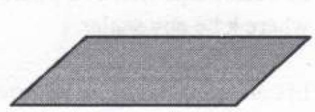
$$\sin\theta = \frac{b_1n_1 + b_2n_2 + b_3n_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Note1: If a line and a plane are parallel, $b_1n_1 + b_2n_2 + b_3n_3 = 0$



Note2: If a line and a plane are perpendicular, $\frac{b_1}{n_2} = \frac{b_2}{n_2} = \frac{b_3}{n_3}$.





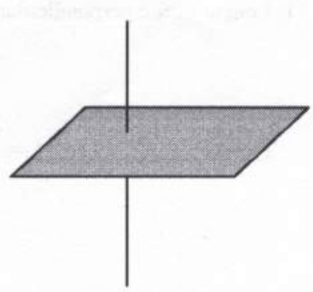
Note: The line $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$ lies entirely in the plane $n_1x + n_2y + n_3z = d$ is $b_1n_1 + b_2n_2 + b_3n_3 = 0$ and

$n_1a_1 + n_2a_2 + n_3a_3 = d$

14. Point of intersection of a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and a plane $\vec{n}.\vec{r} = d$

a. Vector equation is $\vec{n}.\left(\vec{a} + \lambda\vec{b}\right) = d$

b. Cartesian equation is $\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ x_1 - a_1 & y_1 - a_2 & z_1 - a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$



15. Equation of the plane passing through the point $A(x_1, y_1, z_1)$ and parallel to the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$.

b. Let $\vec{a} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, $\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$ and $\vec{r} = r_1\vec{i} + r_2\vec{j} + r_3\vec{k}$ then

$$\sin\theta = \frac{b_1n_1 + b_2n_2 + b_3n_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Note1: If a line and a plane are parallel, $b_1n_1 + b_2n_2 + b_3n_3 = 0$

Note2: If a line and a plane are perpendicular, $\frac{b_1}{n_2} = \frac{b_2}{n_2} = \frac{b_3}{n_3}$.

Note3: The line $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ lies entirely in the plane $n_1x + n_2y + n_3z = d$ is $b_1n_1 + b_2n_2 + b_3n_3 = 0$ and $n_1a_1 + n_2a_2 + n_3a_3 = d$

14. Point of intersection of a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and a plane $\vec{n}.\vec{r} = d$

   a. Vector equation is $\vec{n}.\left(\vec{a} + \lambda\vec{b}\right) = d$

   b. Cartesian equation is $\begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ x_1-a_1 & y_1-a_2 & z_1-a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

15. Equation of the plane passing through the point $A(x_1, y_1, z_1)$ and parallel to the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$.

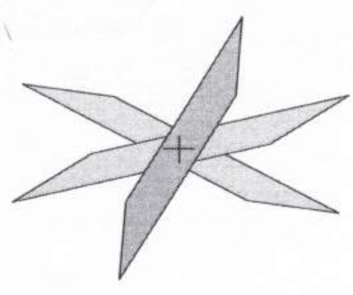
   Let $P(x, y, z)$ be a general point on it. Then

   a. Vector equation is $\left[\overrightarrow{AP}\ \vec{b}\ \vec{d}\right] = 0$

   b. Cartesian equation is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

16. To find the distance between two parallel planes, take any point on one plane and find the length of the perpendicular from that point to the plane.

17. Equation of a plane passing through the point of intersection of other two planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$.



a. Vector equation is $\vec{r}.\vec{n}_1 - d_1 + \lambda\left(\vec{r}.\vec{n}_2 - d_2\right) = 0$

b. Cartesian equation: Let $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ be the equations of two planes.
   Then equation is $a_1x + b_1y + c_1z - d_1 + \lambda\left(a_2x + b_2y + c_2z - d_2\right) = 0$.
   Find the value of λ by substituting the given points and then obtain the equation.