APPLICATIONS OF DERIVATIVES

i. Rate of Change of a Function

Let $y = f(x)$ is a function of x. Let Δx be a small increment in x and be the corresponding increment in y respectively. Then the ratio $\frac{\Delta y}{\Delta x}$ is called average rate of change of y w.r.t. x and $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, is called instantaneous rate of change of y w.r.t. x. If x and y are differentiable functions varying with t (a parameter), then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

i.e., the rate of change of one variable can be calculated if the rate of change of other variable is known.

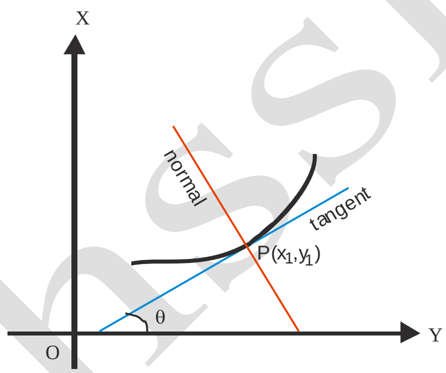
Note1: If the rate of change of one variable is positive, then the value of the variable increases with the increase in the value of the other variable.

Note2: If the rate of change of one variable is negative, then the value of the variable increases with the increase in the value of the other variable.

ii. Tangents and Normals

Let y = f(x) be a continuous curve and let P(x_1, y_1) be a point on it. Then

Slope of the tangent at P, $m = \left(\frac{dy}{dx}\right)_{at(x_1, y_1)}$

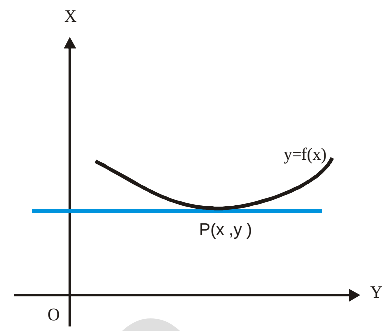


Slope of the normal at P, $m' = \frac{-1}{\left(\frac{dy}{dx}\right)_{at(x_1, y_1)}}$

Equation of the tangent at P is $y - y_1 = m(x - x_1)$

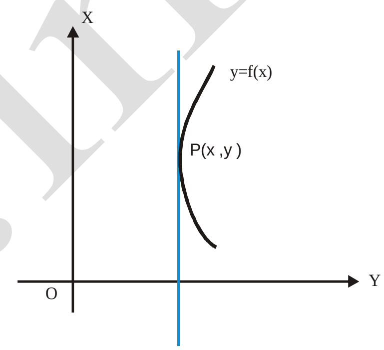
Equation of the normal at P is $y - y_1 = m'(x - x_1)$

Note1: If the tangent is parallel to the x-axis, slope, $\frac{dy}{dx} at (x_1, y_1) = 0$ and the equation of the tangent is $y - y_1 = 0$



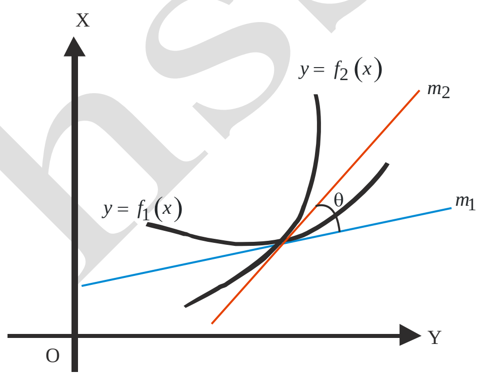
Note2: If the tangent is perpendicular to x axis , slope cannot be defined.

i.e., $\frac{dy}{dx} at (x_1, y_1) = \infty$. Then denominator of the fraction is equal to zero.



iii. Angle of Intersection of Curves

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection. Let $y = f_1(x)$ and $y = f_2(x)$ be any two curves. Let PT_1 and PT_2 be tangents to the curve at their common point of intersection. Then angle between PT_1 and PT_2 is the angle of intersection of the two curves. Let and be the angles made by PT_1 and PT_2 with the positive direction of x-axis. Then the angle of intersection of two curves is obtained by using the formula, $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$, *where* $m_1 = \tan\theta_1$ *and* $m_2 = \tan\theta_2$.



iv. Orthogonal Curves

If the angle of intersection of two curves is one right angle, then the curves are called orthogonal curves.

v. Rolle's Theorem

Let *f* be a real function defined on the closed interval [a,b] such that

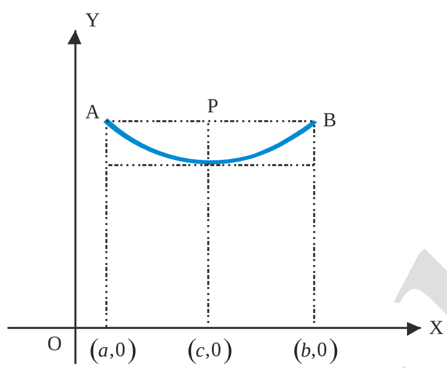
i. It is continuous on the closed interval [a,b],

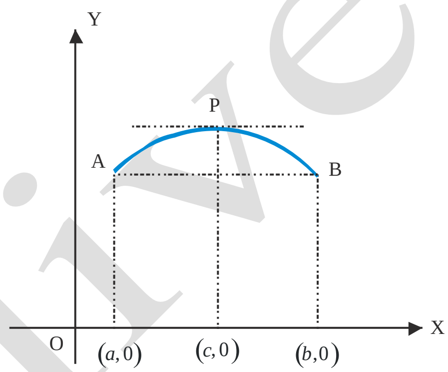
ii. It is differentiable on the open interval (a,b),

iii. $f(a) = f(b)$.

Then there exists a constant 'c' such that $f'(c) = 0$.

Geometrical Meaning





i. Since f(x) is continuous in the interval $a \le x \le b$, the graph is a continuous curve between a and b.

ii. Since f(x) is derivable in the interval $a < x < b$, the graph has unique tangent at every point in between A and B.

iii. Since f(a)=f(b), the ordinates of A and B are equal.

vi. Lagrange's Mean Value Theorem (LMV Theorem)

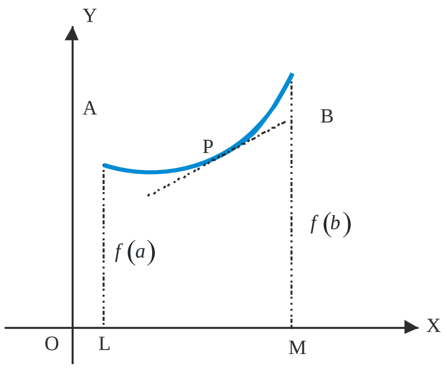
Let *f* be a real function defined on the closed interval [a,b] such that

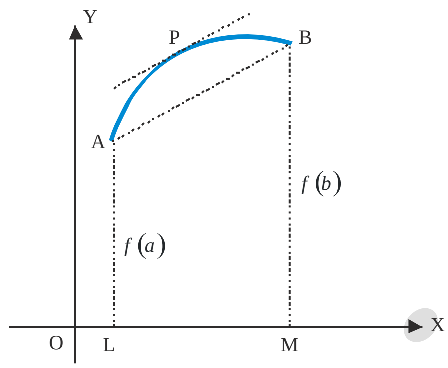
i. It is continuous on the closed interval [a,b],

ii. It is differentiable on the open interval (a,b),

Then there exists a constant 'c' such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Geometrical Meaning:





Let $y = f(x)$ be a function of x. Let A and B be the points on the graph. Therefore, co-ordinates of the points A and B are $[a, f(a)]$ and $[b, f(b)]$ respectively.

$\therefore$ slope of chord AB $= \dfrac{\textit{difference of ordinates}}{\textit{difference of abscissae}} = \dfrac{f(b)-f(a)}{b-a}$. Now,

i. $f(x)$ is continuous in the closed interval $[a,b]$.

ii. $f(x)$ is derivable the open interval (a,b). $\therefore$ it possesses a unique tangent at every point between A and B.

It is clear from the figure that there is at least one point P in between A and B, the tangent at which is parallel to the chord AB. If 'c' be the abscissa of this point, then slope of the tangent is $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

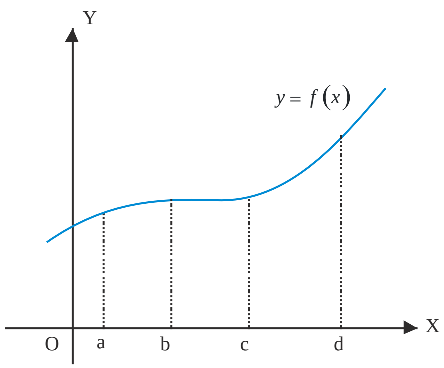
vii. Continuous Function

1. Every constant function defined by $f(x) = C$, where C is any constant, is everywhere continuous.
2. An identity function defined by $f(x) = x,\ x \in R$, is everywhere continuous.
3. The modulus function $f(x)$ defined by $f(x) = |x| = \begin{cases} x, if\ x > 0 \\ -x, if\ x < 0 \end{cases}$, is everywhere continuous, $x \in R$.
4. The exponential function $f(x)$ defined by $f(x) = a^x, x \in R^+ \text{ and } a \in R^+ - \{1\}$ is everywhere continuous.
5. The logarithmic function $f(x)$ defined by $f(x) = \log_a x,\ x \in R \text{ and } a \in R^+ - \{1\}$ is continuous and differentiable in its domain is everywhere continuous.
6. If $p(x)$ and $q(x)$ be any two polynomials, then the function of the form $f(x) = \dfrac{p(x)}{q(x)}, q(x) \neq 0$ is a rational function and is everywhere continuous except at the point where $q(x) = 0$.
7. All trigonometric functions viz. $\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$ and $\cot x$ are continuous at each point of their respective domains.

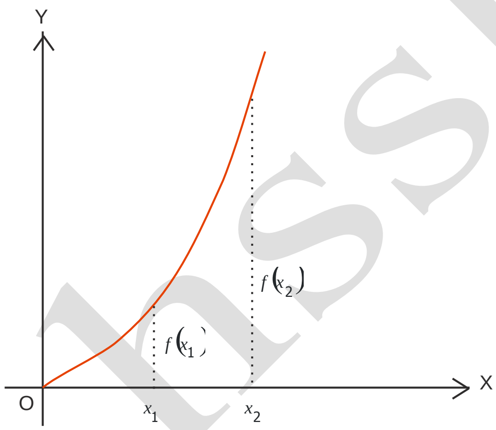
8. All inverse trigonometric functions viz. $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ are continuous at each point of their respective domains.
9. The sum, difference, product and quotient of two continuous functions is also continuous at its domain.

viii. Increasing and Decreasing Functions

A function f(x) said to be an increasing function on an open interval I, if $x_1 < x_2$ on an open interval I, then $f(x_1) \le f(x_2)\ \forall\ x_1, x_2 \in I$.

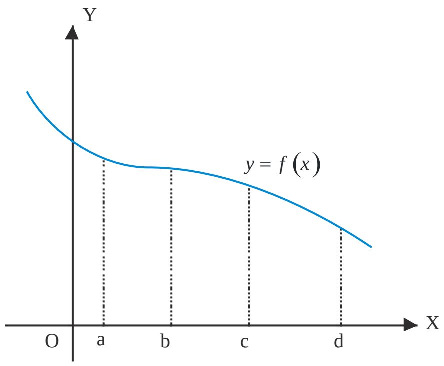


Strictly Increasing Functions: A function f(x) said to be a strictly increasing function on an open interval I, if $x_1 < x_2$ on an open interval I, then $f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in I$.

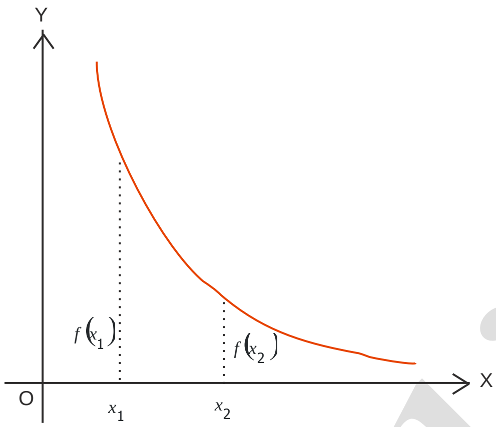


Theorem: A function $f(x)$ is strictly increasing if its derivative is positive. i.e., $f'(x) > 0$.
A function $f(x)$ is an increasing function if its derivative is positive. i.e., $f'(x) \ge 0$.

Decreasing function: A function f(x) is said to be a decreasing function on an open interval I, if $x_1 < x_2$ on I, then $f(x_1) \ge f(x_2)\ \forall\ x_1, x_2 \in I$.



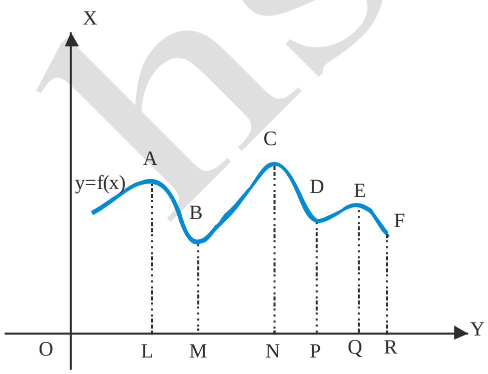
Strictly decreasing functions: A function f(x) said to be a strictly decreasing function on an open interval I, if $f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in I$



Theorem: A function is decreasing, if its derivative is negative. i.e., $f'(x) < 0$

A function is strictly decreasing, if its derivative is negative. i.e., $f'(x) \le 0$

ix. Maxima and minima of a function



A function $f(x)$ is said to have attained its maximum value for $x = a$ if the function ceases to increase and begins to decrease at $x = a$.

The function is said to attained its minimum value for $x = b$ if the function ceases to decrease and begins to increase at $x = b$.

Here A, C and E are the maximum points and B, D and F are the minimum points. And AL, CN and EQ are the maximum values of $f(x)$ at A, C and E and BM, DP and FR are the minimum values of the function $f(x)$ at M, P and R respectively.

- Thus a function has several maximum and minimum values and they are occurring alternately.
- The minimum value of the function is greater than the maximum value.
- Maximum and minimum values do not mean the greatest and least values of the function.

x. Stationary Points or Turning Points

The maximum or minimum point of a function is called turning point or stationary value of the function.

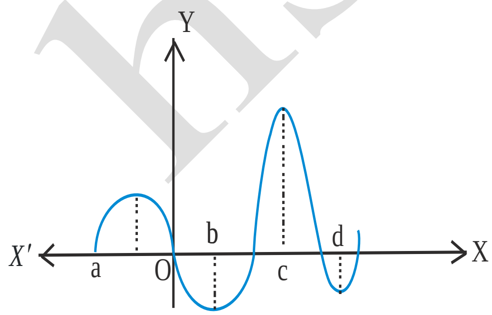
Note: The points at which the function attains either the local maximum or local minimum values are known as the extreme points or turning points and both these values are called extreme values of the function *f*(x).

Condition for Extreme Values

The necessary condition for *f*(a) to be an extreme value of a function *f*(x) is that $f'(x) = 0$. i.e., if the derivative exists, it must be zero at the extreme points.

xi. Critical Value of the Function

The values for x for which $f'(x) = 0$ are called stationary values or critical values of x and the corresponding values of *f*(x) are called stationary or turning values of *f*(x).



xii. First Derivative Test

Let $f(x)$ be a function differentiable at $x = a$, then

i. $x = a$ is a point of local maximum of *f*(x) if

a) $f'(a)=0$ and

b) $f'(x)$ changes from positive to negative as x passes through a. i.e., $f'(x)>0$ at every point in the left neighbourhood $(a-\delta,a)$ of a and $f'(x)<0$ at every point in the right neighbourhood $(a,a+\delta)$ of a.

ii. $x=a$ is a point of local minimum of *f*(x) if

i. $f'(a)=0$ and

ii. $f'(x)$ changes from negative to positive as x passes through a. i.e., $f'(x)<0$ at every point in the left neighbourhood $(a-\delta,a)$ of a and $f'(x)>0$ at every point in the right neighbourhood $(a,a+\delta)$ of a.

iii. For a particular critical value x = a, the sign of $f'(x)$ does not changes i.e., $f'(a)$ has the same sign in the complete neighbourhood of a. Therefore, f(x) has neither a local maximum value nor a local minimum value.

Steps to be followed to find the local maximum and minimum values of a function
(First Derivative Test)

1. Let $y=f(x)$
2. Find $\frac{dy}{dx}$
3. Treat $\frac{dy}{dx}=0$ and solve this equation for x. Let c_1, c_2, c_3, ... are stationary values of x.
4. Consider $x=c_1$. If $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_1, then the function attains a maximum value at $x=c_1$.
5. If $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_1, then the function attains a minimum value at $x=c_1$.
6. If $\frac{dy}{dx}$ does not change its sign as x increases through c_1, then the function is neither maximum nor minimum. In this case $x=c_1$ is called point of inflexion.

xiii. 2^{nd} derivative test (very important)

To find the maximum and minimum value of a function using 2^{nd} derivative test, perform the following steps:

1. Let $y=f(x)$ be a given function.
2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
3. For turning point, treat $\frac{dy}{dx}$ =0, and obtain the value(s) of x.
4. Substituting each value of x in $\frac{d^2y}{dx^2}$.

5. If $\frac{d^2y}{dx^2} < 0$ or negative, the function is maximum and the maximum value of the function is obtained by substituting that value of x in $y = f(x)$.

6. If $\frac{d^2y}{dx^2} > 0$ or positive, the function is minimum and the minimum value of the function is obtained by substituting that value of x in $y = f(x)$.

7. If $\frac{d^2y}{dx^2} = 0$, then the function is neither maximum nor minimum. Then the point of the function is known as point of inflexion.

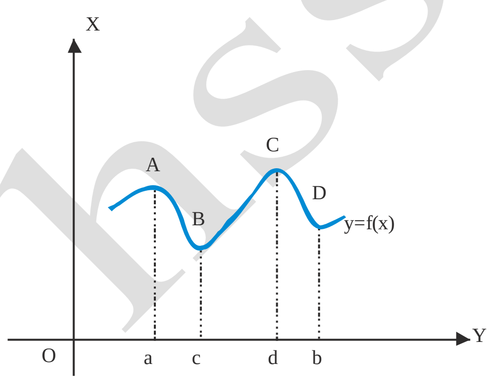
Note: A point of inflexion is the point on a curve where the slope is a maximum or minimum, i.e., has a stationary or turning value, is called a point of inflexion.

xiv. Absolute Maxima and Absolute minima

A function $f(x)$ is continuous function on the closed interval $[a,b]$, and differentiable in the open interval (a,b) then f takes on an absolute maximum value and an absolute minimum value on the closed interval $[a,b]$.

To find the absolute maximum and absolute minimum values of a function:

1. Find all the points where $f'(x) = 0$.
2. Take the end points of the interval.
3. Find the values of f at these points.
4. The greatest value of $f(x)$ is called the absolute maximum and the least value of $f(x)$ is called the absolute minimum value of $f(x)$.



Here of the maximum and minimum values of the function, AL is the greatest value and DP is the minimum value of the function, so they are the absolute maximum and absolute minimum value of the function in the closed interval $[a,b]$.

PRACTICE PROBLEMS

E.g.: 1. Find the absolute maximum and absolute minimum value of the function $f(x)=\sin x+\cos x,\ [0,\pi]$

Let $f(x)=\sin x+\cos x,\ [0,\pi]$

$f'(x)=\cos x-\sin x$

For turning points, $f'(x)=0\Rightarrow \cos x-\sin x=0\Rightarrow \cos x=\sin x\Rightarrow x=\frac{\pi}{4}$

The intervals be $0,\frac{\pi}{4},\pi$

$\therefore\ f(0)=\sin 0+\cos 0=1$

$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=2\times\frac{1}{\sqrt{2}}=\sqrt{2}$

$f(\pi)=\sin\pi+\cos\pi=0+-1=-1$

$\therefore$ The absolute minimum value $=-1$ and

The absolute maximum value $=\sqrt{2}$.

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2. The sum of three positive numbers is 26. The second number is thrice as large as the first. If the sum of the squares of these numbers is least, find the numbers.

Let the first number be x.
Second number is 3x
Since the sum of three numbers is 26, let the third number be 26–x–3x=26–4x
Let S be the sum of the squares of these numbers. Then

$S=x^2+(3x)^2+(26-4x)^2$

$=x^2+9x^2+676-208x+16x^2=26x^2-208x+676$

$\frac{dS}{dx}=52x-208$

$\frac{d^2S}{dx^2}=52<0$

$\therefore S$ is least.

For least S, $\frac{dS}{dx}=0\Rightarrow 52x-208=0\Rightarrow 52x-208\Rightarrow x=\frac{208}{52}=4$

$\therefore$ The first number = 4
The second number = 3(4)=12

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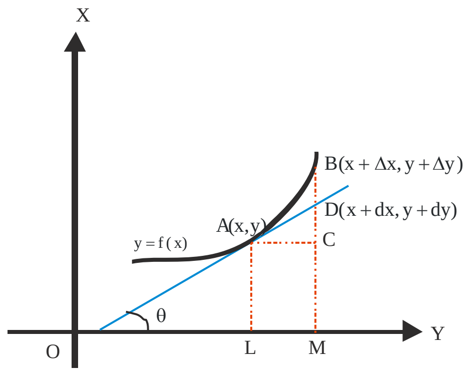
xv. Differentials and Approximations

Let $A(x,y)$ and $B(x+\Delta x,y+\Delta y)$ be any two points on the curve $y=f(x)$. Let dx and dy be the differentials of the variables x and y respectively. The differential dy and the increment Δx of the dependent variable y are unequal.



In the diagram, $OL = x$; $OM = x + \Delta x$; $CM = AL = y$; $AC = dx \text{ or } \Delta x$. Then by definition,

$$dy = \frac{dy}{dx} \times dx \;.$$

Thus, differential of $y =$ *derivative of the function* $\times$ *differential of* x.

Then

i) differential (approximate error) of x,

ii) differential (approximate error) of y,

Note: $\dfrac{\text{differential of } y}{\text{differential of } x} = \dfrac{f'(x).\Delta x}{\Delta x} = f'(x)$ is the differential coefficient of y w.r.t. x.

PRACTICE PROBLEMS

1. Using differentials find the value of $\sqrt[4]{257}$

Let $y = \sqrt[4]{257} = (257)^{\frac{1}{4}}$

Let $y = x^{\frac{1}{4}}$ ....(1)

$y + \Delta y = (x + \Delta x)^{\frac{1}{4}} \Rightarrow \Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

Put $x = 256$ *and* $\Delta x = 1$

$\Delta y = (256 + 1)^{\frac{1}{4}} - 256^{\frac{1}{4}}$

$= 257^{\frac{1}{4}} - \left(4^4\right)^{\frac{1}{4}} = 257^{\frac{1}{4}} - 4$

$\Delta y + 4 = 257^{\frac{1}{4}} \Rightarrow 257^{\frac{1}{4}} = \Delta y + 4$ ....(2)

Diff. w.r.t. x

$\dfrac{dy}{dx} = \dfrac{1}{4} x^{\frac{1}{4} - 1} = \dfrac{1}{4} x^{-\frac{3}{4}} = \dfrac{1}{4}(256)^{-\frac{3}{4}} = \dfrac{1}{4} \times 4^{-3} = \dfrac{1}{4} \times \dfrac{1}{4^3} = \dfrac{1}{256}$

$dy = \dfrac{1}{256} dx \Rightarrow \Delta y = \dfrac{1}{256} \Delta x$

Substituting in (2) we have,

$$\therefore\ 257^{\frac{1}{4}} = 4 + \frac{1}{256} = 4.004\,(approx.)$$

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2. Find the approximate value of $\tan 46^0$, given that $1^0 = 0.01745$ radians.

$$y = \tan x$$
$$y + \Delta y = \tan(x + \Delta x) \Rightarrow \Delta y = \tan(x + \Delta x) - y = \tan(x + \Delta x) - \tan x$$
$$put\ x = 45^0\ \ and\ \ \Delta x = 1^0 = 1 \times \frac{\pi}{180} radians = \frac{22}{7 \times 180} = 0.0175$$
$$\therefore \Delta y = \tan(45 + 1) - \tan 45 = \tan 46^0 - 1 \Rightarrow \tan 46^0 = 1 + \Delta y \ \ldots\ldots\ldots.(1)$$
$$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x \Rightarrow dy = \sec^2 x dx \Rightarrow \Delta y = \sec^2 45 \times \Delta x = \left(\sqrt{2}\right)^2 \times 0.0175 = 2 \times 0.0175 = 0.035$$
$$in\ (1)\ we\ have\ \tan 46^0 = 1 + 0.035 = 1.035\,(approx)$$

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3. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

Let $y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x).dx \Rightarrow \Delta y = f'(x).\Delta x$, where $x = 5$ and $\Delta x = 0.001$

$$f'(x) \Rightarrow 3x^2 - 7(2x) + 0 = 3x^2 - 14x$$
$$f'(5) = 3(25) - 14(5) = 75 - 70 = 5$$
$$\therefore \Delta y = (5)(0.001) = 0.005$$
$$f(5) = \left(5^3\right) - 7\left(5^2\right) + 15 = 125 - 175 + 15 = -35$$
$$\text{Also } \Delta y = f(x + \Delta x) - f(x)$$
$$\Rightarrow f(x + \Delta x) = f(x) + \Delta y$$
$$\Rightarrow f(5.001) = f(5) + \Delta y$$
$$\therefore f(5.001) = -35 + 0.005 = -34.995(\text{approx})$$

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4. If the radius of a sphere is measured as 7 m with an error of 0.02m, then find the approximate error in calculating its volume.

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then, r = 7 m and Δr = 0.02 m

Now, the volume *V* of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 12\pi r^2$$
$$\therefore dV = 4\pi r^2.\Delta r = 4\pi \times 7^2 \times 0.02 = 4 \times 49 \times 0.02\pi = 3.92\pi\ \ m^3$$

Hence, the approximate error in calculating the volume is $3.92\pi\ m^3$.