DIFFERENTIATION

Continuity and differentiability of a function

If a function is differentiable at a point, it is necessarily continuous at that point. But its converse it not necessarily true. E.g.: the function f(x) = |x| is continuous at $x = 0$, but it is not differentiable at $x = 0$.

Differentiability at a point

Let f be a real valued function defined in the open interval (a, b) and let $c \in (a,b)$. Then $f(x)$ is said to be differentiable or derivable at $x = c$ iff $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exits finitely. This limit is called derivative or differential coefficient of the function f(x) at $x = c$ and is denoted by $f'(c)$.

Derivative of a function

A function $f(x)$ is said to be derivable or differentiable if it is derivable at every points in its domain.

Suppose $f(x) = \frac{1}{x}$. Domain of the function is $R - \{0\}$

$f(x)$ is derivable at every point in R except 0.

Derivability of a function on an interval

i. A function $f(x)$ is said to be a derivable function on the open interval (a,b), it is derivable at every points in the open interval (a,b).

ii. A function $f(x)$ is said to be a derivable function on the closed interval $[a,b]$,
   a. it is derivable at every points in the open interval (a,b),
   b. it is derivable at $x = a$ from right
   c. it is derivable at $x = b$ from left

Standard results on differentiability

1. Every polynomial function is differentiable at each x∈R.
2. Every constant function is differentiable at each x∈R.
3. Every exponential function is differentiable at each x∈R.
4. Every logarithmic function is differentiable at each point in its domain.
5. Trigonometric and inverse T-functions are differentiable in their domains.
6. The sum, difference, product and quotient two differentiable functions is differentiable.
7. The composition of differentiable functions is a differentiable function.

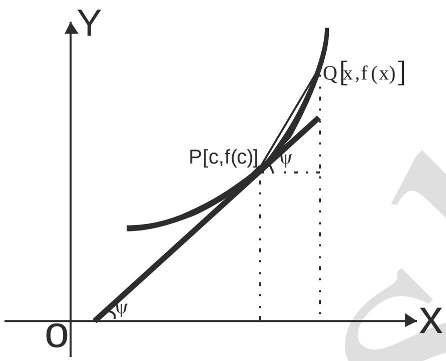
Differentiation

Let $f(x)$ be a differentiable function on $[a,b]$. Then corresponding to each point $x \in [a,b]$, we get a unique real number equal to the derivative of $f'(x)$ and are denoted by $f'(x)$ or $\frac{dy}{dx}$ or Dy y_1 or y', etc..

i.e., $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ (or) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x-h)-f(x)}{-h}$. The process of obtaining the derivative of a function is called differentiation.

Geometrical meaning of the derivative at a point

Consider the curve $y = f(x)$. Let $f(x)$ is differentiable at $x = c$. Let $P[c, f(c)]$ be a point on the curve and let Q be a neighbouring point on the curve. Then slope of the chord $PQ = \frac{f(x)-f(c)}{x-c}$. Taking limit as $Q \to P$ i.e., $x \to c$, we get $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. As $Q \to P$, the chord PQ becomes tangent at P.



Note: derivative of y w.r.t. x $= \frac{d}{dx}(y) = \frac{dy}{dx}$

derivative of y w.r.t. t $= \frac{d}{dt}(y) = \frac{dy}{dt}$

derivative of x w.r.t. t $= \frac{d}{dt}(x) = \frac{dx}{dt}$, etc.

Derivative of a function

Let $y = f(x)$ is a finite, single valued function of x. Let Δx be a small increment in x and Δy be the corresponding increment in y respectively.

Then $y + \Delta y = f(x + \Delta x)$

$\Delta y = f(x + \Delta x) - f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

taking limits we have,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\boxed{\frac{dy}{dx} = f'(x)}$$

i.e., $\frac{d}{dx}[f(x)] = f'(x)$.This is called derivative of y w.r.t x or differential coefficient of y w.r.t x. This method is called **first principles** or **delta (** Δ *or* δ **) method** or **differentiation by definition** or **ab initio**.

Note: Other forms of $\frac{dy}{dx}$ **are** $f'(x),\ y',\ y_1,\ Dy,$ **etc..**

Derivative of the functions using the first principles:

1. Let $y = x^2$

   Let Δ x be a small increment in x and Δ y be the corresponding increment in y respectively.

$$y + \Delta y = (x+\Delta x)^2$$

$$\Delta y = (x+\Delta x)^2 - y = (x+\Delta x)^2 - x^2$$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$\frac{dy}{dx} = 2x + 0 = 2x$$

$$\frac{d}{dx}\left(x^2\right) = 2x$$

STANDARD RESULTS

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cos}ecx$	$-\operatorname{cos}ecx \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cos}ec^2 x$
x^n	nx^{n-1}
e^x	e^x
e^{-x}	$-e^x$
x^x	$x^x(1+\log x)$
x^a	$a.x^{a-1}$
a^x	$a^x.\log a$
a^a	0
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\log x$	$\frac{1}{x}$
x	1
x^2	$2x$
$\mathbf{\frac{1}{x^n}}$	$\mathbf{-\frac{1}{x^{n+1}}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{x^2}$	$-\frac{2}{x^3}$
xy	$x\frac{dy}{dx}+y$

y	$\frac{dy}{dx}$
y^2	$2y\frac{dy}{dx}$
$\sqrt{a^2-x^2}$	$\frac{-x}{\sqrt{a^2-x^2}}$
$\sqrt{a^2+x^2}$	$\frac{x}{\sqrt{a^2-x^2}}$
$\sqrt{x^2+a^2}$	$\frac{x}{\sqrt{x^2+a^2}}$
$\sqrt{x^2-a^2}$	$\frac{x}{\sqrt{x^2-a^2}}$

Note: Derivative of any trigonometric function starting with 'co' is negative.

FUNDAMENTAL RESULTS OF DIFFERENTIATION

1. **Differential coefficient of a constant is zero**. i.e., $\boldsymbol{\frac{d}{dx}(c)=0}$, where c is a constant.

   E.g.: $\frac{d}{dx}(5)=0$, $\frac{d}{dx}(-10)=0$, etc.

2. If c is a constant and u is a function of x then $\mathbf{\frac{d}{dx}(cu)=c\frac{d}{dx}(u)}$

3. If u and v are functions of x, then $\boldsymbol{\frac{d}{dx}(u\pm v)=\frac{d}{dx}(u)\pm\frac{d}{dx}(v)}$

   $$\frac{d}{dx}(5\sin x+\log x)=\frac{d}{dx}(5\sin x)+\frac{d}{dx}(\log x)=5\frac{d}{dx}(\sin x)+\frac{d}{dx}(\log x)=5\cos x+\frac{1}{x}$$

   $$\frac{d}{dx}\left(2e^x-\tan x\right)=\frac{d}{dx}\left(2e^x\right)-\frac{d}{dx}(\tan x)=2\frac{d}{dx}\left(e^x\right)-\frac{d}{dx}(\tan x)=2e^x-\sec^2 x$$

4. **Product rule**: If u and v are functions of x, then derivative of the product of two functions is equal to *first function x derivative of the second function + (plus) second function x derivative of the first function*.

   i.e., $\boldsymbol{\frac{d}{dx}(uv)=u.\frac{d}{dx}(v)+v.\frac{d}{dx}(u)}$

E.g.: i. $y = e^{3x}\sin 4x$

$$\frac{dy}{dx} = e^{3x}\frac{d}{dx}(\sin 4x)+\sin 4x.\frac{d}{dx}\left(e^{3x}\right) = e^{3x}.\cos 4x.\ 4+\sin 4x.e^{3x}.3 = e^{3x}(4\cos 4x+3\sin 4x)$$

ii. $y = x^2 \tan x$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}\left(x^2\right)$$

$$= x^2\sec^2 x+\tan x.2x = x^2\sec^2 x+2x\tan x$$

Corollary of product rule:

If u, v and w are functions of x, then $\boldsymbol{\frac{d}{dx}(uvw) = uv.\frac{d}{dx}(w)+vw.\frac{d}{dx}(u)+uw.\frac{d}{dx}(v)}$

E.g.: $y = x^2e^x\tan x$

$$\frac{dy}{dx} = x^2e^x\frac{d}{dx}(\tan x)+e^x\tan x\frac{d}{dx}\left(x^2\right)+x^2\tan x\frac{d}{dx}\left(e^x\right)$$

$$= x^2e^x\sec^2 x+e^x\tan x.2x+x^2\tan x.e^x$$

$$= xe^x\left(x\sec^2 x+2\tan x.+x\tan x.\right) = xe^x\left(x\sec^2 x+(2+x)\tan x.\right)$$

5. **Quotient formula**: If u and v are any two functions of x, then quotient of two functions is equal to (2nd function x derivative of the 1st function minus 1st function x derivative of the 2nd function) divided by square of the 2nd function.

i.e., $$\boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{d}{dx}(u)-u.\frac{d}{dx}(v)}{v^2}}$$

E.g.: $y = \dfrac{Sin\,x+Cos\,x}{Sin\,x-Cos\,x}$.

$$\frac{dy}{dx} = \frac{(\sin x-\cos x).\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$= \frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{(\sin x-\cos x).-(\sin x-\cos x)-(\sin x+\cos x)^2}{(\sin x-\cos x)^2} = \frac{(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= \frac{\sin^2 x - 2\sin x.\cos x + \cos^2 x - (\sin^2 x + 2\sin x.\cos x + \cos^2 x)}{(\sin x - \cos x)^2}$$

$$= \frac{\sin^2 x - 2\sin x.\cos x + \cos^2 x - \sin^2 x - 2\sin x.\cos x - \cos^2 x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2\sin x.\cos x - 2\sin x.\cos x}{(\sin x - \cos x)^2} = \frac{-2.2\sin x.\cos x}{(\sin x - \cos x)^2} = \frac{-2\sin 2x}{(\sin x - \cos x)^2}$$

Function of a function

Let $y = f(u)$, where $u = \phi(x)$, then the derivative or differential coefficient of y w.r.t x is $\mathbf{\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}}$

E.g.: $y = \sqrt{2x+3}$

put $u = 2x+3$

Then $y = \sqrt{u}$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{du}{dx} = 2\times 1 + 0 = 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2\sqrt{u}}\times 2 = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{2x+3}}$$

Short-cut method:

i. Let us assume that the inside function be x.

ii. Find the derivative of the function in the standard form.

iii. Replace the value of x.

iv. Multiply it with derivative of the inside function.

The above question will be done using the short-cut method:

$y = \sqrt{2x+3}$

i. Assume $2x+3$ as x

ii. Now the function becomes in the form $y = \sqrt{x}$.

iii. Find the derivative of $y = \sqrt{x}$. i.e., $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

iv. Replace x by $2x+3$. i.e., $\frac{1}{2\sqrt{2x+3}}$

v. Find the derivative of $2x+3$. i.e., $2\times1+0=2$

vi. Find the product of steps iii and iv. i.e., $\frac{dy}{dx} = \frac{1}{2\sqrt{2x+3}}\times 2 = \frac{1}{\sqrt{2x+3}}$

ii. $y = e^{-ax^2}$

$$\frac{dy}{dx} = e^{-ax^2}\times\frac{d}{dx}\left(-ax^2\right) = e^{-ax^2}\times -a\times 2x = -2axe^{-ax^2}$$

Note: If $y = f\left[\phi(x)\right]$,then $\frac{dy}{dx} = f'\left[\phi(x)\right]\times\phi'(x)$

Chain rule

Function of a function can be extended to more than two functions is called chain rule. If $y = f(u)$, where $u=\phi(v)$ and $v=\phi(x)$ then the derivative or differential coefficient of y w.r.t x is $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

E.g.: $y = \log\left(\tan\frac{x}{2}\right)$

Here $y = \log\left(\tan\frac{x}{2}\right)$, $u = \tan\frac{x}{2}$ and $v = \frac{x}{2}$

$$\frac{dy}{du} = \frac{1}{\tan\frac{x}{2}};\ \frac{du}{dv} = \sec^2\frac{x}{2};\ \frac{dv}{dx} = \frac{1}{2}$$

$$\therefore \frac{dy}{du} = \frac{1}{\tan\frac{x}{2}}.\sec^2\frac{x}{2}.\frac{1}{2} = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\frac{1}{\cos^2\frac{x}{2}}\times\frac{1}{2} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{\sin x} = \cos ecx$$

(or)

$$\frac{dy}{dx}=\frac{1}{\tan\frac{x}{2}}\frac{d}{dx}\left(\tan\frac{x}{2}\right)=\frac{1}{\tan\frac{x}{2}}\sec^2\frac{x}{2}\times\frac{1}{2}=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\frac{1}{\cos^2\frac{x}{2}}\times\frac{1}{2}=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{1}{\sin x}=\cos ec x$$

Inverse Trigonometric Functions

Consider a function $y=f(x)$. If it is possible to write x as a function of y, we say x is an inverse function of y, and is symbolically written as $x=f^{-1}(y)$. There are six inverse trigonometric functions viz. $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cos ec^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$, etc.. The principle value of $\sin^{-1}x$ lies between $\pm\frac{\pi}{2}$, the principal value of $\cos^{-1}x$ lies between 0 and π and the principal value of $\tan^{-1}x$ lies between $\pm\frac{\pi}{2}$.

1. $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$
4. $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x.\sqrt{x^2-1}}$
6. $\frac{d}{dx}(\cos ec^{-1}x)=-\frac{1}{x.\sqrt{x^2-1}}$

E.g.: Find $\frac{dy}{dx}$ if

1. $y=e^{a\cos^{-1}x}$

$$\frac{dy}{dx}=e^{a\cos^{-1}x}.a.\frac{-1}{\sqrt{1-x^2}}=-\frac{a\,e^{a\cos^{-1}x}}{\sqrt{1-x^2}}.$$

2. $y=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

put $x=\tan\theta$; $\theta = \tan^{-1}x$

$$y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x$$

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

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3. $y=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$put\ x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$y = Sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = Sin^{-1}\sin 2\theta = 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Implicit functions

When the two variables x and y are connected in a single relation such as $f(x,y)=0$, it is called an implicit function. If is often difficult to find y explicitly. To find the derivative of an implicit function, perform the following steps:

1. Differentiate the whole expression w.r.t. x
2. Keep $\dfrac{dy}{dx}$ terms to one side and all other terms to the other side
3. Then obtain $\dfrac{dy}{dx}$.

E.g.:

Find $\dfrac{dy}{dx}$ if

1. $x^2+y^2=a^2$

Given $x^2+y^2=a^2$

Diff. w.r.t. x

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

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2. $\cos(x+y) = y\sin x$

diff. w.r.t x

$$-\sin(x+y).\left[1+\frac{dy}{dx}\right] = y.\cos x + \sin x.\frac{dy}{dx}$$

$$-\sin(x+y).-\sin(x+y)\frac{dy}{dx} = y.\cos x + \sin x.\frac{dy}{dx}$$

$$-\sin(x+y)\frac{dy}{dx} - \sin x.\frac{dy}{dx} = y.\cos x + \sin(x+y)$$

$$-[\sin(x+y)+\sin x.]\frac{dy}{dx} = y.\cos x + \sin(x+y)$$

$$\frac{dy}{dx} = -\frac{[y.\cos x + \sin(x+y)]}{[\sin(x+y)+\sin x]}$$

Exponential functions

A function is of the form $y = e^x$ is known as an exponential function.

Derivative of e^x

Let $\quad e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+........$

Diff. w.r.t. x we have

$$\frac{d}{dx}\left(e^x\right) = \frac{d}{dx}(1)+\frac{d}{dx}\left(\frac{x}{1!}\right)+\frac{d}{dx}\left(\frac{x^2}{2!}\right)+\frac{d}{dx}\left(\frac{x^3}{3!}\right)+........$$

$$\frac{d}{dx}\left(e^x\right) = 0+\left(\frac{1}{1!}\right)+\left(\frac{2x}{2!}\right)+\left(\frac{3x^2}{3!}\right)+........$$

$$= 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}........ = e^x$$

Logarithmic functions

A function of the form $y = u^v$, both u and v are functions of x. Then follow the following steps;

Taking 'log' on both sides

$$\log y = \log u^v$$

$$\log y = v \log u$$

Diff. w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = v.\frac{d}{dx}(\log u) + \log u.\frac{d}{dx}(v)$$

$$\therefore \frac{dy}{dx} = y\left[v.\frac{d}{dx}(\log u) + \log u.\frac{d}{dx}(v)\right] = u^v\left[v.\frac{d}{dx}(\log u) + \log u.\frac{d}{dx}(v)\right]$$

E.g.: Find $\frac{dy}{dx}$ if

1. $y = x^{\sin x}$

$$\log y = \log x^{\sin x}$$

$$\log y = \sin x \log x$$

Diff w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = \sin x.\frac{1}{x} + \log x.\cos x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \log x.\cos x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \log x \ \cos x\right]$$

2. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Given $x^y = e^{x-y}$

Taking log on both sides,

$$\log x^y = \log e^{x-y} \Rightarrow y\log x = (x-y)\log e \Rightarrow y\log x = x - y \qquad (\because \log e = 1)$$

$$y + y\log x = x \Rightarrow y(1+\log x) = x \Rightarrow y = \frac{x}{1+\log x}$$

Differentiating w.r.t. x we have

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}(x) - x.\frac{d}{dx}(1+\log x)}{(1+\log x)^2} = \frac{(1+\log x).1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2} = \frac{1+\log x - x.\frac{1}{x}}{(1+\log x)^2} = \frac{1+\log x - 1}{(1+\log x)^2}$$

$= \frac{\log x}{(1+\log x)^2}$. Hence proved.

The following formulae will be found very useful in differentiation of logarithmic functions:

1. $\log ab = \log a + \log b$
2. $\log \frac{a}{b} = \log a - \log b$
3. $\log \frac{ab}{c} = \log a + \log b - \log c$
4. $\log m^n = n \log m$
5. $\log_n^m = \log_b^m \times \log_n^b$
6. $\log_n^m = \frac{\log_b^m}{\log_b^n}$
7. $\log_b^a = \frac{1}{\log_a^b}$
8. $\log_a^a = 1$
9. $-\log x = \log \frac{1}{x}$
10. $\frac{1}{2}\log x = \log \sqrt{x}$
11. $\log 1 = 0$

Note: i. $e^{\log x} = x$

ii. $\log e^x = x$

E.g.: Find $\frac{dy}{dx}$ *if* $y = \frac{x^2\sqrt{x+1}}{e^{3x}\tan x}$

$$y = \frac{x^2\sqrt{x+1}}{e^{3x}\tan x}$$

Taking log on both sides,

$$\log y = \log\left(\frac{x^2\sqrt{x+1}}{e^{3x}\tan x}\right) = \log x^2 + \log\sqrt{x+1} - \left(\log e^{3x} + \log\tan x\right)$$

$$\log y = \log\left(\frac{x^2\sqrt{x+1}}{e^{3x}\tan x}\right) = 2\log x + \frac{1}{2}\log(x+1) - 3x\log e - \log\tan x$$

diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x} + \frac{1}{2}\frac{1}{x+1} - 3x\times 1 - \frac{1}{\tan x}\sec^2 x$$

$$\frac{dy}{dx} = \frac{x^2\sqrt{x+1}}{e^{3x}\tan x}\left(\frac{2}{x} + \frac{1}{2(x+1)} - 3x - \frac{1}{\sin x\cos x}\right)$$

Parametric functions

When the variables x and y are given as functions of a third variable, known as parameter, say $x = f(t)$ and $y = \psi(t)$, is called parametric functions. To find the derivative of such functions:

$$\boldsymbol{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (or) \quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}}$$

E.g.: Find $\frac{dy}{dx}$ if

1. $x = \sin\theta;\ y = \cos\theta$

$$\frac{dx}{d\theta} = \cos\theta \qquad \frac{dy}{d\theta} = -\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$$

$$= -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

2. $x = ct;\ y = \frac{c}{t}$

$$\frac{dx}{dt} = c\times 1 = c \qquad \frac{dy}{dt} = c\times\frac{d}{dx}\left(\frac{1}{t}\right) = c\times\frac{-1}{t^2} = -\frac{c}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = -\frac{c}{t^2}\times\frac{1}{c} = -\frac{1}{t^2}$$

Successive differentiation

Let $y = f(x)$ is a function of x. Then $\frac{dy}{dx} = f'(x)$, is called first differential coefficient of y w.r.t. x. It we differentiate $\frac{dy}{dx}$ w.r.t x, we have $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f'(x)] \Rightarrow \frac{d^2y}{dx^2} = f''(x)$, is called second differential coefficient of y w.r.t x. If we differentiate again and again we have $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, ..., \frac{d^ny}{dx^n}$ are called 3rd derivative, 4th derivative,…, nth derivative of y w.r.t x. The process of obtaining the derivatives in succession is called Successive Differentiation.

Note:

1. $\frac{dy}{dx} = f'(x) = y_1 = y' = Dy$
2. $\frac{d^2y}{dx^2} = f''(x) = y_2 = y'' = D^2y$

E.g.: 1. Find $\frac{d^2y}{dx^2}$ if $x = a(1+\sin\theta); y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(0+\cos\theta) = a\cos\theta \quad ; \quad \frac{dy}{d\theta} = a(0 - -\sin\theta) = a.\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta) = \frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx} = \sec^2\theta.\frac{1}{a\cos\theta} = \frac{1}{a}\sec^2\theta.\sec\theta = \frac{1}{a}\sec^3\theta$$

2. $y = \sin\left(m\sin^{-1}x\right)$

$$\frac{dy}{dx} = \cos(m\sin^{-1}x).\frac{d}{dx}(m\sin^{-1}x) = \cos(m\sin^{-1}x).\,m.\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}.\frac{dy}{dx} = m\cos(m\sin^{-1}x)$$

Diff. again w.r.t x

$$\sqrt{1-x^2}\,.\frac{d^2y}{dx^2}+\frac{dy}{dx}.\frac{1}{2\sqrt{1-x^2}}\times -2x = m.-\sin(m\sin^{-1}x).m.\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,.\frac{d^2y}{dx^2} -.\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{-m^2.\sin(m\sin^{-1}x)}{\sqrt{1-x^2}}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2\sin(m\sin^{-1}x) = -\mathrm{m}^2\mathrm{y} \qquad \times ing \text{ by } \sqrt{1-x^2}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$ Hence proved.

Derivative of a function with another function.

If u and v are functions of then the derivative of u w.r.t. v is $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$.

E.g.: i. Find the Derivative of: $\sin x$ w.r.t. $\cos x$.

$Let\ u = \sin x$ and $v = \cos x$

$$\frac{du}{dv} = \frac{d(\sin x)}{d(\cos x)} = \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(\cos x)} = \frac{\cos x}{-\sin x} = -\cot x$$

ii. derivative of $\sin\left(x^2\right)$ $w.r.t.$ $\cos x$

$$\frac{du}{dv} = \frac{d\left[\sin\left(x^2\right)\right]}{d(\cos x)} = \frac{\cos\left(x^2\right)\times 2x}{-\sin x} = -\frac{2x\cos\left(x^2\right)}{\sin x}.$$

Note:

derivative of $\sin x$ w.r.t. x $= \frac{d}{dx}(\sin x) = \cos x$

derivative of $\sin y$ w.r.t. x $= \frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$

derivative of $\sin x$ w.r.t. y $= \frac{d}{dy}(\sin x) = \cos x\frac{dx}{dy}$

derivative of $\sin y$ w.r.t. y $= \frac{d}{dy}(\sin y) = \cos y$