5 CONTINUITY AND DIFFERENTIABILITY

CONTINUITY

- A function f is said to be continuous at a point c, a < c < b, if

$$\lim_{x\to c} f(x) = f(c)$$

- A function f is said to be continuous from the left and right:

$$\lim_{x\to c^+} f(x) = f(c)$$
$$\lim_{x\to c^-} f(x) = f(c)$$

DERIVATIVES

Derivatives :

$$\frac{dy}{dx} = f'(x) = y'$$

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

If u and v are functions of x,then

- $(cu)' = cu'$
- $(u+v)' = u' + v'$
- $(u-v)' = u' - v'$
- $(uv)' = vu' + uv' = uv\left(\frac{u'}{u} + \frac{v'}{v}\right)$
- $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

Derivatives :

- $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}\left(e^x\right) = e^x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Derivatives :

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
- $\frac{d}{dx}\left(a^x\right) = a^x \log a$
- $\frac{d}{dx}\left(x^{-n}\right) = -\frac{n}{x^{n+1}}$
- $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
- $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$
- $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$

MODEL QUESTIONS

Question 1 :

Check the continuity of the function $f(x) = 3x + 5$ at $x = 1$

Solution :

$$f(1) = 3(1) + 5 = 8$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(2x + 5) = 8$$

$$\therefore \quad \lim_{x\to 1} f(x) = f(1) = 8$$

Hence $f(x)$ is continous

Question 2 :

Check the continuity of the function,

$$f(x) = \begin{cases} x + 3, & x < 2 \\ 3 - x, & x \geq 2 \end{cases} \quad \text{at } x \in R$$

Solution :

$$f(x) = \begin{cases} x + 3, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$$

$c < 2$:

$$f(c) = c + 3$$

$$\lim_{x\to c} f(x) = \lim_{x\to c} x + 3$$
$$= c + 3$$

$$\therefore \quad \lim_{x\to c} f(x) = f(c)$$

Hence $f(x)$ is continous

$c = 2$:

$$f(2) = 2 + 3 = 5$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x + 3$$
$$= 2 + 3 = 5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 - x$$
$$= 3 - 2 = 1$$

$$\therefore \quad \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

Hence $f(x)$ is discontinous

$c > 2$:

$$f(c) = 3 - c$$

$$\lim_{x\to c} f(x) = \lim_{x\to c} 3 - x$$
$$= 3 - c$$

$$\therefore \quad \lim_{x\to c} f(x) = f(c)$$

Hence $f(x)$ is continous

Question 3 :

Check the continuity of the function,

$$f(x)=\begin{cases} x+1, & x>2 \\ 0, & x=2 \\ 1-x, & x<2 \end{cases} \quad \text{at } x\in R$$

Solution :

$$f(x)=\begin{cases} 1-x, & x<2 \\ 0, & x=2 \\ x+1, & x>2 \end{cases}$$

$c<2:$

$$f(c)=1-c$$

$$\lim_{x\to c} f(x)=\lim_{x\to c} 1-x$$

$$=1-c$$

$$\therefore \quad \lim_{x\to c} f(x)=f(c)$$

Hence $f(x)$ is continous

$c=2:$

$$f(2)=0$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 1-x$$

$$=1-1=0$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} x+1$$

$$=1+1=2$$

$$\therefore \quad \lim_{x\to 2^-} f(x)\neq \lim_{x\to 2^+} f(x)$$

Hence $f(x)$ is discontinous

$c>2:$

$$f(c)=c+1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c} x+1$$

$$=c+1$$

$$\therefore \quad \lim_{x\to c} f(x)=f(c)$$

Hence $f(x)$ is continous

Question 4 :

Check the continuity of the function,

$$f(x)=\begin{cases} \dfrac{\sin 2x}{x}, & x\neq 0 \\ 1, & x=0 \end{cases} \quad \text{at } x=0$$

Solution :

$$f(x)=\begin{cases} \dfrac{\sin 2x}{x}, & x\neq 0 \\ 1, & x=0 \end{cases}$$

$$f(0)=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sin 2x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2$$

$$\therefore \quad \lim_{x\to 0} f(x) \neq f(0)$$

Hence $f(x)$ is discontinous

Question 5 :

Check the continuity of the function $f(x) = |x|$ at $x = 1$

Solution :

At $x = 1$

$$f(x) = |x|$$

$$f(x) = \begin{cases} -x \ , \ x < 0 \\ x \ , \ x \geq 0 \end{cases}$$

$$f(0) = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x) = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x) = 0$$

$$\therefore \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 0$$

Hence $f(x)$ is continous

Question 6 :

Discuss the continuity of the function $f(x) = x^3 + x^2 - 3$

Solution :

$$f(x) = x^3 + x^2 - 3$$

If $x = c$

$$f(c) = c^3 + c^2 - 3$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^3 + x^2 - 3\right) = c^3 + c^2 - 3$$

$$\therefore \quad \lim_{x\to c} f(x) = f(c) = c^3 + c^2 - 3$$

Hence $f(x)$ is continous

Question 7 :

Prove that the function $10x - 6$ is continous at $x = 0$, $x = -3$ and $x = -3$

Solution :

$$f(x) = 10x - 6$$

At $x = 0$

$$f(0) = 10(0) - 6 = -6$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(10x-6) = \underline{\underline{-6}}$$

At $x = 3$

$$f(3) = 10(3) - 6 = 24$$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(10x-6) = (10(3)-6) = \underline{\underline{24}}$$

At $x = -3$

$$f(-3) = 10(-3) - 6 = -36$$

$$\lim_{x\to -3} f(x) = \lim_{x\to -3}(10x-6) = (10(-3)-6) = \underline{\underline{-36}}$$

$\therefore f(x)$ is continous at $x = 0, x = 3, x = -3$

Question 8 :

$$f(x) = \begin{cases} \dfrac{\sin x}{|x|} & , x \neq 0 \\ k & , x = 0 \end{cases} \quad \text{at } x = 0$$

Can you find k so that $f(x)$ is continous at $x = 0$

Solution :

$$f(x) = \begin{cases} \dfrac{\sin x}{|x|} & , x \neq 0 \\ k & , x = 0 \end{cases}$$

$c = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{|x|} = \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{|x|} = \frac{\sin x}{-x} = -1$$

$$\therefore \quad \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

$\therefore$ $f(x)$ cannot continous for $x = 0$, for any value of $k \in R$

Question 9 :

Discuss the continuity of sine function.

Solution :

$f(x) = \sin x$

$f(c) = \sin c$

Put $x = c + h$. If $x \to c$ we know that $h \to 0$

$$\therefore \quad \lim_{x\to c} f(x) = \lim_{x\to c} \sin x$$
$$= \lim_{h\to 0} \sin(c+h) = \lim_{h\to 0} \sin c \ \cos h + \cos c \ \sin h$$
$$= \lim_{h\to 0} \sin c \ \cos h + \lim_{h\to 0} \cos c \ \sin h = \sin c + 0 = \sin c$$

$\lim_{x\to c} f(x) = f(c)$ and hence f is a continuous function.

Question 10 :

Find the value of k, so that the function f defined below,is continous at $x = 0$,where

$$f(x) = \begin{cases} \dfrac{1-\text{co}4x}{8x^2} & , \text{ if } x \neq 0 \\ k & , \text{ if } x = 0 \end{cases}$$

Solution :

$$f(x) = \begin{cases} \dfrac{1-\text{co}4x}{8x^2} & , \text{ if } x \neq 0 \\ k & , \text{ if } x = 0 \end{cases}$$

Given, $f(x)$ is continous at $x = 0$

$$\therefore \quad (\text{LHL})_{x=0} = (\text{LHL})_{x=0} = f(0)$$

$$\text{LHL} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} \frac{1-\text{co}4x}{8x^2} = \lim_{h\to 0} \frac{1-\text{co}(-4h)}{8h^2}$$

$$\{\because \text{put } x = 0 - h = -h, \text{when } x \to 0, h \to 0\}$$

$$\text{LHL} = \lim_{h\to 0} \frac{1-\text{co}4h}{8h^2}$$

$$= \lim_{h\to 0} \frac{2\sin^2 2h}{8h^2}$$

$$= \lim_{h \to 0} \frac{\sin^2 2h}{4h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h} \right)^2 = 1$$

$\text{At} x = 0,\ f(0) = k$

$\text{We have LHL} = f(0) = k = 1,\ f(x)$ is continous at $x = 0$

Question 11 :

Find the derivative of x^6 w.r.t x.

Solution :

$$y = x^6$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^6\right) = 6x^{6-1} = \underline{\underline{6x^5}}$$

Question 12 :

Find the derivative of $\sin 4x$ w.r.t x.

Solution :

$$y = \sin 4x$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 4x) = \cos 4x \frac{d}{dx}(4x) = \cos 4x \times 4 = \underline{\underline{4\cos 4x}}$$

Question 13 :

Find the derivative of e^{4x} w.r.t x.

Solution :

$$y = e^{4x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{4x}\right) = e^{4x} \frac{d}{dx}(4x) = e^{4x} \times 4 = \underline{\underline{4e^{4x}}}$$

Question 14 :

Find the derivative of $\sqrt{5x}$ w.r.t x.

Solution :

$$y=\sqrt{5x}$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{5x}\right)=\frac{1}{2\sqrt{5x}}\frac{d}{dx}(5x)=\frac{1}{2\sqrt{5x}}\times 5=\frac{5}{2\sqrt{5x}}$$

Question 15 :

Find the derivative of $\log 7x$ w.r.t x.

Solution :

$$y=\log 7x$$

$$\frac{dy}{dx}=\frac{d}{dx}(\log 7x)=\frac{1}{\log 7x}\frac{d}{dx}(7x)=\frac{7}{\log 7x}$$

Question 16 :

Find the derivative of $8x^5$ w.r.t x.

Solution :

$$y=8x^5$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(8x^5\right)=8\frac{d}{dx}\left(x^5\right)=8\times 5x^4=40x^4$$

Question 17 :

Find the derivative of $2\cos^4 5x$ w.r.t x.

Solution :

$$y=2\cos^4 5x$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(2\cos^4 5x\right)=2\frac{d}{dx}\left(\cos^4 5x\right)$$

$$=2\times 4\cos^3 5x\frac{d}{dx}(5x)=2\times 4\cos^3 5x\times 5$$

$$=40\cos^3 5x$$

Question 18 :

Find the derivative of $\sec\sqrt{4x}$ w.r.t x.

Solution :

$$y = \sec\sqrt{4x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sec\sqrt{4x}\right) = \sec\sqrt{4x}\tan\sqrt{4x}\frac{d}{dx}\left(\sqrt{4x}\right)$$

$$= \sec\sqrt{4x}\tan\sqrt{4x} \times \frac{1}{2\sqrt{4x}}\frac{d}{dx}(4x)$$

$$= \sec\sqrt{4x}\tan\sqrt{4x} \times \frac{1}{2\sqrt{4x}} \times 4 = \frac{2\sec\sqrt{4x}\tan\sqrt{4x}}{\sqrt{4x}}$$

Question 19 :

Find the derivative of $\log\left(x^2+x+1\right)$ w.r.t x.

Solution :

$$y = \log\left(x^2+x+1\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\left(x^2+x+1\right)\right)$$

$$= \frac{1}{\log\left(x^2+x+1\right)}\frac{d}{dx}\left(x^2+x+1\right)$$

$$= \frac{1}{\log\left(x^2+x+1\right)}\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)\right]$$

$$= \frac{1}{\log\left(x^2+x+1\right)}[2x+1+0] = \frac{2x+1}{\log\left(x^2+x+1\right)}$$

Question 20 :

Find the derivative of $\sqrt{e^{3x}}$ w.r.t x.

Solution :

$$y = \sqrt{e^{3x}}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{e^{3x}}\right) = \frac{1}{2\sqrt{e^{3x}}}\frac{d}{dx}\left(e^{3x}\right) = \frac{1}{2\sqrt{e^{3x}}} \times e^{3x}\frac{d}{dx}(3x) = \frac{3e^{3x}}{2\sqrt{e^{3x}}}$$

Question 21 :

Find the derivative of $(3x+2)^2$ w.r.t x.

Solution :

$$y=(3x+2)^2$$

$$\frac{dy}{dx}=\frac{d}{dx}\left((3x+2)^2\right)$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(9x^2+12x+4\right)=\frac{d}{dx}\left(9x^2\right)+\frac{d}{dx}(12x)+\frac{d}{dx}(4)$$

$$=9\frac{d}{dx}\left(x^2\right)+12\frac{d}{dx}(x)+\frac{d}{dx}(4)=9\times 2x+12\times 1+0$$

$$=\underline{18x+12}$$

Question 22 :

Find the derivative of $(2x+2)^3$ w.r.t x.

Solution :

$$y=(2x+2)^3=8x^3+24x^2+24x+8$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(8x^3+24x^2+24x+8\right)$$

$$=\frac{d}{dx}\left(8x^3\right)+\frac{d}{dx}\left(24x^2\right)+\frac{d}{dx}(24x)+\frac{d}{dx}(8)$$

$$=\underline{24x^2+48x+24}$$

Question 23 :

Find the derivative of $\sec\left(\tan\sqrt{2x}\right)$ w.r.t x.

Solution :

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sec\left(\tan\sqrt{2x}\right)\right)$$

$$=\sec\left(\tan\sqrt{2x}\right)\tan\left(\tan\sqrt{2x}\right)\frac{d}{dx}\left(\tan\sqrt{2x}\right)$$

$$=\sec\left(\tan\sqrt{2x}\right)\tan\left(\tan\sqrt{2x}\right)\sec^2\sqrt{2x}\,\frac{d}{dx}\left(\sqrt{2x}\right)$$

$$= \sec\left(\tan\sqrt{2x}\right)\tan\left(\tan\sqrt{2x}\right)\sec^2\sqrt{2x}\times\frac{1}{2\sqrt{2x}}\frac{d}{dx}(2x)$$

$$= \sec\left(\tan\sqrt{2x}\right)\tan\left(\tan\sqrt{2x}\right)\sec^2\sqrt{2x}\times\frac{1}{2\sqrt{2x}}\times 2$$

$$= \frac{\sec\left(\tan\sqrt{2x}\right)\tan\left(\tan\sqrt{2x}\right)\sec^2\sqrt{2x}}{\sqrt{2x}}$$

Question 24 :

Find the derivative of $3x^5+3x+5$ w.r.t x.

Solution :

$$y = 3x^5+3x+5$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^5+3x+5\right) = \frac{d}{dx}\left(3x^5\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(5)$$

$$= 3.5x^4+3.1+0 = 15x^4+3$$

Question 25 :

Find the derivative of $2x^5+3\sec 5x+5\tan\sqrt{x}$ w.r.t x.

Solution :

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^5+3\sec 5x+5\tan\sqrt{x}\right)$$

$$= \frac{d}{dx}\left(2x^5\right)+\frac{d}{dx}(3\sec 5x)+\frac{d}{dx}\left(5\tan\sqrt{x}\right)$$

$$= 2.5x^4+3.3\sec 5x.\tan 5x\frac{d}{dx}(5x)+5\sec^2\sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= 10x^4+45\sec 5x.\tan 5x+\frac{5}{2\sqrt{x}}\sec^2\sqrt{x}$$

Question 26 :

Find the derivative of $\sin 4x\cos^2 x+3\sin x$ w.r.t x.

Solution :

$$y = \sin 4x\cos^2 x+3\sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin 4x\cos^2 x+3\sin x\right)$$

$$= \frac{d}{dx}\left(\sin 4x \cos^2 x\right) + \frac{d}{dx}(3\sin x)$$

$$= \sin 4x \frac{d}{dx}\left(\cos^2 x\right) + \cos^2 x \frac{d}{dx}(\sin 4x) + 3\frac{d}{dx}(\sin x)$$

$$= \sin 4x.2\cos x.(-\sin x) + \cos^2 x.4\cos 4x + 3\cos x$$

$$= -2\sin 4x.\cos x.\sin x + 4\cos^2 x.\cos 4x + 3\cos x$$

Question 27 :

Find the derivative of $\sin^3 x\cos^5 x$ w.r.t x.

Solution :

$$y = \sin^3 x\cos^5 x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^3 x\cos^5 x\right)$$

$$= \sin^3 x \frac{d}{dx}\left(\cos^5 x\right) + \cos^5 x \frac{d}{dx}\left(\sin^3 x\right)$$

$$= \sin^3 x.5\cos^4 x.(-\sin x) + \cos^5 x.3\sin^2 x.\cos x$$

$$= -5\sin^4 x.\cos^4 x + 3\cos^6 x.\sin^2 x$$

Question 28 :

Find the derivative of $\dfrac{\sin 4x}{\cos^2 2x}$ w.r.t x.

Solution :

$$y = \frac{\sin 4x}{\cos^2 2x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin 4x}{\cos^2 2x}\right)$$

$$= \cos^2 2x \frac{d}{dx}(\sin 4x) + \sin 4x \frac{d}{dx}\left(\cos^2 2x\right)$$

$$= \cos^2 2x.\cos 4x \times 4 + \sin 4x.2\cos 2x.(-\sin 2x).2$$

$$= 4\cos^2 2x.\cos 4x - 4\sin 4x.\cos 2x.\sin 2x$$

Question 29 :

Find the derivative of $5\sec^4 2x + 3\tan 3x$ w.r.t x.

Solution :

$$y = 5\sec^4 2x + 3\tan 3x$$

$$\frac{dy}{dx} = 5\frac{d}{dx}\left(\sec^4 2x\right) + 3\frac{d}{dx}(\tan 3x)$$

$$= 5.4\sec^3 2x\frac{d}{dx}(\sec 2x) + 3\sec^2 3x\frac{d}{dx}(3x)$$

$$= 5.4\sec^3 2x.\sec 2x.\tan 2x + 3\sec^2 3x.3$$

$$= 20\sec^3 2x.\sec 2x.\tan 2x + 9\sec^2 3x$$

Question 30 :

Find the derivative of $3\sec^3\sqrt{5x} + 5\tan(\log x)$ w.r.t x.

Solution :

$$y = 3\sec^3\sqrt{5x} + 5\tan(\log x)$$

$$\frac{dy}{dx} = 3\frac{d}{dx}\left(\sec^3\sqrt{5x}\right) + 5\frac{d}{dx}\left[\tan(\log x)\right].$$

$$= 3.3\sec^2\sqrt{5x}\frac{d}{dx}\left(\sec\sqrt{5x}\right) + 5.\sec^2(\log x)\frac{d}{dx}(\log x).$$

$$= 5.4\sec^3 2x.\sec 2x.\tan 2x + 3\sec^2 3x.3$$

$$= 20\sec^3 2x.\sec 2x.\tan 2x + 9\sec^2 3x$$

Question 31 :

Find the derivative of $\sqrt{\sin x^3}$ w.r.t x.

Solution :

$$y = \sqrt{\sin x^3}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\sin x^3}\right) = \frac{1}{2\sqrt{\sin x^3}}\frac{d}{dx}\left(\sin x^3\right)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \cos x^3 \frac{d}{dx}\left(x^3\right) = \frac{1}{2\sqrt{\sin x^3}} \cos x^3 .3x^2$$

$$= \frac{3}{2} \frac{x^2 .\cos x^3}{\sqrt{\sin x^3}}$$

Question 32 :

Find the derivative of $2\cos\left(\tan\left(x^5\right)\right) + 4e^{3x} + 5e^x$ w.r.t x.

Solution :

$$y = 2\cos\left(\tan\left(x^5\right)\right) + 4e^{3x} + 5e^x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\cos\left(\tan\left(x^5\right)\right) + 4e^{3x} + 5e^x\right)$$

$$= \frac{d}{dx}\left(2\cos\left(\tan\left(x^5\right)\right)\right) + \frac{d}{dx}\left(4e^{3x}\right) + \frac{d}{dx}\left(5e^x\right)$$

$$= 2\frac{d}{dx}\left(\cos\left(\tan\left(x^5\right)\right)\right) + 4\frac{d}{dx}\left(e^{3x}\right) + 5\frac{d}{dx}\left(e^x\right)$$

$$= 2\left(-\sin\left(\tan\left(x^5\right)\right).\sec^2\left(x^5\right).5x^4\right) + 4.3.e^{3x} + 5.e^x$$

$$= -10x^4.\sin\left(\tan\left(x^5\right)\right).\sec^2\left(x^5\right) + 12e^{3x} + 5e^x$$

Question 33 :

Find the derivative of $5\sec x + 3\tan x$

Solution :

$$y = 5\sec x + 3\tan x$$

$$\frac{dy}{dx} = \frac{d}{dx}(5\sec x + 3\tan x)$$

$$= \frac{d}{dx}(5\sec x) + \frac{d}{dx}(3\tan x)$$

$$= 5\frac{d}{dx}(\sec x) + 3\frac{d}{dx}(\tan x)$$

$$= 5\sec x.\tan x + 3\sec^2 x$$

Question 34 :

Find the derivative of sinx cosx

Solution :

$y = \sin x \cos x$

By Leibnitz product rule

$$\frac{d}{dx}(\sin x \cos x) = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$
$$= \sin x(-\sin x) + \cos x \cos x$$
$$= -\sin^2 x + \cos^2 x = \cos^2 x - \sin^2 x$$
$$= \cos 2x$$

Question 35 :

Find the derivative $\dfrac{\sin x}{\cos x}$

Solution :

$$y = \frac{\sin x}{\cos x}$$

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Question 36 :

Find the derivative of $y + \sin y = \cos x$ w.r.t x.

Solution :

$$y + \sin y = \cos x$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(y) + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos 2x)$$

$$\frac{dy}{dx}+\cos y\frac{dy}{dx}=-2\sin 2x$$

$$(1+\cos y)\frac{dy}{dx}=-2\sin 2x$$

$$\frac{dy}{dx}=\frac{-2\sin 2x}{(1+\cos y)}$$

Solution :(Trick)

$$y+\sin y-\cos 2x=0$$

Differentiating

$$\frac{dy}{dx}=\frac{-(\text{Taking } y \text{ as constant})}{(\text{Taking } x \text{ as constant})}$$

$$\frac{dy}{dx}=\frac{-(0+0-(-2\sin 2x))}{(1+\cos y-0)}=\frac{-2\sin 2x}{1+\cos y}$$

Question 37 :

Find $\frac{dy}{dx}$,if $y=\cos^{-1}(4x^3-3x), 0<x<1$

Solution :

$y=\cos^{-1}(4x^3-3x)$

put $x=\cos\theta$

$$y=\cos^{-1}(4\cos^3\theta-3\cos\theta)=\cos^{-1}(\cos 3\theta)=3\theta=3\cos^{-1}x$$

$$\frac{dy}{dx}=\frac{d}{dx}(3\cos^{-1}x)=3\frac{-1}{\sqrt{1-x^2}}=\frac{-3}{\sqrt{1-x^2}}$$

Question 38 :

If $x^3+y^3=3axy$. Find $\frac{dy}{dx}$

Solution :

$x^3+y^3=3axy$

$$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(3axy)$$

$$3x^2+3y^2\frac{dy}{dx}=3a\left(x\frac{dy}{dx}+y.1\right)$$

$$3x^2+3y^2\frac{dy}{dx}=3ax\frac{dy}{dx}+3ay$$

$$3y^2\frac{dy}{dx}-3ax\frac{dy}{dx}=3ay-3x^2$$

$$\frac{dy}{dx}\left(3y^2-3ax\right)=3ay-3x^2$$

$$\frac{dy}{dx}=\frac{3ay-3x^2}{\left(3y^2-3ax\right)}=\frac{ay-x^2}{\left(y^2-ax\right)}$$

Solution : (Trick)

$$x^3+y^3-3axy=0$$

Differentiating,

$$\frac{dy}{dx}=\frac{-(\text{Taking } y \text{ as constant})}{(\text{Taking } x \text{ as constant})}$$

$$\frac{dy}{dx}=\frac{-\left(3x^2-3ay\right)}{\left(3y^2-3ax\right)}=\frac{ay-x^2}{y^2-ax}$$

Question 39 :

Find $\frac{dy}{dx}$, if $x = a\cos\theta$, $y = b\sin\theta$

Solution :

$$x=a\cos\theta, \qquad y=b\sin\theta$$

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos\theta)=-a\sin\theta\ , \quad \frac{dy}{d\theta}=\frac{d}{d\theta}(b\sin\theta)=b\cos\theta$$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{b\cos\theta}{-a\sin\theta}$$

Question 40 :

Find $\frac{dy}{dx}$, if $x = a\cos2\theta$, $y = b\sin3\theta$

Solution :

$$x = a\cos 2\theta \ , \quad y = b\sin 3\theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos 2\theta) = -2a\sin 2\theta, \ \frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin 3\theta) = 3b\cos 3\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3b\cos 3\theta}{-2a\sin 2\theta}$$

Question 41 :

Find $\frac{dy}{dx}$, if $x = at^3$, $y = 4at$

Solution :

$$x = at^3, \quad y = 4at$$

$$\frac{dx}{dt} = \frac{d}{dt}(at^4) = 4at^3 \ , \quad \frac{dy}{dt} = \frac{d}{dt}(4at) = 4a$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4a}{4at^3} = \frac{1}{t^3}$$

Question 42 :

Find $\frac{dy}{dx}$, $y = x^{\sin x}$

Solution :

$$y = x^{\sin x}$$

$$\log y = \sin x \log x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \ \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \ \frac{1}{x} + \log x.\cos x$$

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x.\cos x\right)$$

$$\frac{dy}{dx}=x^{\sin x}\left(\frac{\sin x+x\cos x.\log x}{x}\right)$$

$$\frac{dy}{dx}=x^{\sin x}.x^{-1}(\sin x+x\cos x.\log x)$$

$$\frac{dy}{dx}=x^{(\sin x)-1}(\sin x+x\cos x.\log x)$$

Question 43 :

Find $\frac{dy}{dx}$, $x^y+y^x+a^x+x^a=1$

Solution :

$$x^y+y^x+a^x+x^a=1$$

Let $u=x^y, v=y^x, w=a^x+x^a$

Now $\log u=y\log x$

$$\frac{1}{u}\frac{du}{dx}=\frac{y}{x}+\log x\frac{dy}{dx}$$

$$\frac{du}{dx}=u\left(\frac{y}{x}+\log x\frac{dy}{dx}\right) \Rightarrow \frac{du}{dx}=x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)$$

Now $v=y^x$

$$\log v=x\log y$$

$$\frac{1}{v}\frac{dv}{dx}=\frac{x}{y}\frac{dy}{dx}+\log y$$

$$\frac{dv}{dx}=v\left(\frac{x}{y}\frac{dy}{dx}+\log y\right) \Rightarrow \frac{dv}{dx}=y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)$$

Now $w=a^x+x^a$ or $w=a^x\log a+ax^{a-1}$

$$\frac{dw}{dx}=a^x\log a+ax^{a-1}$$

$$\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}=0$$

$$x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)+\left(a^x\log a+ax^{a-1}\right)=0$$

$$x^y\frac{y}{x}+x^y\log x\frac{dy}{dx}+y^x\frac{x}{y}\frac{dy}{dx}+y^x\log y+a^x\log a+ax^{a-1}=0$$

$$x^y\log x\frac{dy}{dx}+y^x\frac{x}{y}\frac{dy}{dx}=-\left(x^y\frac{y}{x}+y^x\log y+a^x\log a+ax^{a-1}\right)$$

$$\frac{dy}{dx}\left(x^y\log x+y^{x-1}x\right)=-\left(x^{y-1}y+y^x\log y+a^x\log a+ax^{a-1}\right)$$

$$\frac{dy}{dx}=-\left(\frac{x^{y-1}y+y^x\log y+a^x\log a+ax^{a-1}}{x^y\log x+y^{x-1}x}\right)$$

$$\frac{dy}{dx}=-\left(\frac{yx^{y-1}+y^x\log y+a^x\log a+ax^{a-1}}{x^y\log x+xy^{x-1}}\right)$$

Question 44 :

Write the derivative of $\sin x$ with respect to $\cos x$.

Solution :

Let $u=\sin x$

On differentiating u w.r.t x,

$$\frac{du}{dx}=\cos x$$

Let $v=\cos x$

On differentiating v w.r.t x,

$$\frac{dv}{dx}=-\sin x$$

$$\frac{du}{dv}=\frac{du}{dx}\times\frac{dx}{dv}=\cos x.\frac{1}{-\sin x}=-\cot x$$

Question 45 :

Find $\frac{dy}{dx}$,if $y=(\sin x)^x+\sin x^x$.

Solution :

$u=(\sin x)^x, v=\sin x^x$

$\log u = x\log\sin x$

$$\frac{1}{u}\frac{du}{dx} = x\ \frac{d}{dx}(\log\ \sin x) + \log\ \sin x\frac{d}{dx}(x)$$

$$\frac{1}{u}\frac{du}{dx} = x\ \frac{1}{\sin x}.\cos x + \log\ \sin x.1$$

$$\frac{du}{dx} = y\left(\frac{x\cos x}{\sin x} + \log\ \sin x\right)$$

$$\frac{du}{dx} = y\left(x\cot x + \log\ \sin x\right)$$

$$\frac{du}{dx} = (\sin x)^x\left(x\cot x + \log\ \sin x\right)$$

$v = \sin x^x$

$$\frac{dv}{dx} = \cos x^x\frac{d}{dx}\left(x^x\right)$$

$$\frac{dv}{dx} = \cos x^x.\frac{d}{dx}e^{x\log x} \qquad \left\langle \because x^x = e^{\log x^x} = e^{x\log x}\ \middle|\ x = e^{\log x}\right.$$

$$\frac{dv}{dx} = \cos x^x.e^{x\log x}\frac{d}{dx}(x\log x)$$

$$\frac{dv}{dx} = \cos x^x.e^{x\log x}(1 + \log x)$$

$$\frac{dv}{dx} = \cos x^x.x^x(1 + \log x)$$

$$\frac{dy}{dx} = (\sin x)^x\left(x\cot x + \log\ \sin x\right) + x^x.\cos x^x(1 + \log x)$$

Question 46:

If $\cos y = x\cos(x + y)$,where $\cos a \neq \pm 1$,prove that

$$\frac{dy}{dx} = \frac{\cos^2(a + y)}{\cos a}$$

Solution:

Given, $\cos y = x\cos(a + y)$ or $x = \dfrac{\cos y}{\cos(a + y)}$

On differentiating u w.r.t y,we get

$$\frac{dx}{dy}=\frac{\cos(a+y)\frac{d}{dy}(\cos y)-\cos y\frac{d}{dy}[\cos(a+y)]}{\cos^2(x+y)}$$

$$=\frac{\cos(a+y)(-\sin y)-\cos y.[-\sin(a+y)]}{\cos^2(x+y)}$$

$$=\frac{\sin(a+y).\cos y-\cos(a+y)(\sin y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\sin(a+y-y)}{\cos^2(a+y)}\{\because \sin(x-y)=\sin x\cos y-\cos x\sin y\}$$

$$\frac{dx}{dy}=\frac{\sin a}{\cos^2(a+y)} \quad \text{or} \quad \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

Question 47 :

If $y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$, and $0< x <1$, then find $\frac{dy}{dx}$

Solution :

$$y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$$

Use the formula, $\sin^{-1}x+\sin^{-1}x=\sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right]$

$$y=\sin^{-1}\left\{x\sqrt{1-\left(\sqrt{x}\right)^2}-\sqrt{x}\sqrt{1-x^2}\right\}= \sin^{-1}x-\sin^{-1}\sqrt{x}$$

On differentiating w.r.t x,we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}x)-\frac{d}{dx}(\sin^{-1}\sqrt{x})=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-\sqrt{x}^2}}\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$$

Question 48 :

If $e^x+e^y=e^{x+y}$, prove that $\frac{dy}{dx}+e^{y-x}=0$

Solution :

$$\frac{e^x}{e^{x+y}}+\frac{e^y}{e^{x+y}}=\frac{e^{x+y}}{e^{x+y}}$$

$$e^{-y}+e^{-x}=1$$

On differentiating w.r.t x,we get

$$\frac{d}{dx}\left(e^{-y}+e^{-x}\right)=\frac{d}{dx}(1)$$

$$\frac{d}{dx}\left(e^{-y}\right)+\frac{d}{dx}\left(e^{-x}\right)=0$$

$$e^{-y}\left(-\frac{dy}{dx}\right)+e^{-x}(-1)=0$$

$$-e^{-y}\left(\frac{dy}{dx}\right)=e^{-x}$$

$$\frac{dy}{dx}=\frac{e^{-x}}{-e^{-y}}$$

$$\frac{dy}{dx}=-e^{-x+y}$$

$$\frac{dy}{dx}=-e^{y-x}$$

$$\frac{dy}{dx}+e^{y-x}=0$$

Question 49 :

Find the value of $\frac{dy}{dx}$ at $\theta=\frac{\pi}{4}$, if $x=ae^{\theta}(\sin\theta-\cos\theta)$ and $y=ae^{\theta}(\sin\theta+\cos\theta)$

Solution :

Given, $x=ae^{\theta}(\sin\theta-\cos\theta)=a\left[e^{\theta}\sin\theta-e^{\theta}\cos\theta\right]$

On differentiating w.r.t θ,we get

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}\left[e^{\theta}\sin\theta-e^{\theta}\cos\theta\right]=a\left[\frac{d}{d\theta}\left(e^{\theta}\sin\theta\right)-\frac{d}{d\theta}\left(e^{\theta}\cos\theta\right)\right]$$

$$=a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta)+\sin\theta\frac{d}{d\theta}\left(e^{\theta}\right)-e^{\theta}\frac{d}{d\theta}(\cos\theta)-\cos\theta\frac{d}{d\theta}\left(e^{\theta}\right)\right]$$

$$=a\left[e^{\theta}.\cos\theta+\sin\theta.e^{\theta}-e^{\theta}(-\sin\theta)-\cos\theta.e^{\theta}\right]$$

$$=a\left[e^{\theta}\cos\theta+e^{\theta}\sin\theta+e^{\theta}\sin\theta-e^{\theta}\cos\theta\right]=a\left[2e^{\theta}\sin\theta\right]$$

Given, $y=ae^{\theta}(\sin\theta+\cos\theta)=a\left[e^{\theta}\sin\theta+e^{\theta}\cos\theta\right]$

On differentiating w.r.t θ,we get

$$\frac{dy}{d\theta}=a\frac{d}{d\theta}\left[e^{\theta}\sin\theta+e^{\theta}\cos\theta\right]=a\left[\frac{d}{d\theta}\left(e^{\theta}\sin\theta\right)+\frac{d}{d\theta}\left(e^{\theta}\cos\theta\right)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta)+\sin\theta\frac{d}{d\theta}(e^{\theta})+e^{\theta}\frac{d}{d\theta}(\cos\theta)-\cos\theta\frac{d}{d\theta}(e^{\theta})\right]$$

$$= a\left[e^{\theta}.\cos\theta+\sin\theta.e^{\theta}+e^{\theta}(-\sin\theta)+\cos\theta.e^{\theta}\right]$$

$$= a\left[e^{\theta}\cos\theta+e^{\theta}\sin\theta-e^{\theta}\sin\theta+e^{\theta}\cos\theta\right]=a\left[2e^{\theta}\cos\theta\right]$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=a\left[2e^{\theta}\cos\theta\right]\times\frac{1}{a\left[2e^{\theta}\sin\theta\right]}$$

$$=\frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}=\underline{\underline{\cot\theta}}$$

Question 50 :

If $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$ and $y=a\sin t$. Find $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{3}$

Solution :

Given, $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$

On differentiating w.r.t t,we get

$$\frac{dx}{dt}=a\frac{d}{dt}\left(\cos t+\log\tan\frac{t}{2}\right)=a\left[\frac{d}{dt}(\cos t)+\frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\times\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]=a\left[-\sin t+\frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\times\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{\cot\frac{t}{2}}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right]=a\left[-\sin t+\frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}\times\cos^2\frac{t}{2}}\right]$$

$$=a\left[-\sin t+\frac{1}{2\sin\frac{t}{2}\times\cos\frac{t}{2}}\right]=a\left[-\sin t+\frac{1}{\sin t}\right]$$

$$=a\left[\frac{-\sin^2 t+1}{\sin t}\right]=a\left[\frac{1-\sin^2 t}{\sin t}\right]=\frac{a\cos^2 t}{\sin t}$$

Given, $y = a\sin t$

On differentiating w.r.t θ,we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin t) = a\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = a\cos t \times \frac{\sin t}{a\cos^2 t} = \frac{\sin t}{\cos t} = \underline{\underline{\tan t}}$$

On differentiating w.r.t x,we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t\frac{dt}{dx} = \sec^2 t \times \frac{\sin t}{a\cos^2 t} = \frac{\sin t\sec^4 t}{a}$$

$$\left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}\sec^4\frac{\pi}{3}}{a} = \frac{\frac{\sqrt{3}}{2} \times 2^4}{a} = \underline{\underline{\frac{8\sqrt{3}}{a}}}$$

Question 51:

If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$

Solution:

Given, $x^m y^n = (x+y)^{m+n}$

On taking log both sides, we get

$$\log(x^m) + \log(y^n) = (m+n)\log(x+y)$$

$$m\log x + n\log y = (m+n)\log(x+y)$$

On differentiating w.r.t x,we get

$$m\frac{d}{dx}(\log x) + n\frac{d}{dx}(\log y) = (m+n)\frac{d}{dx}\log(x+y)$$

$$m\left(\frac{1}{x}\right) + n\left(\frac{1}{y}\frac{dy}{dx}\right) = (m+n)\frac{1}{(x+y)}\frac{d}{dx}(x+y)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right) \quad \Rightarrow \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\frac{dy}{dx}$$

$$\frac{n}{y}\frac{dy}{dx} - \frac{m+n}{x+y}\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x} \quad \Rightarrow \frac{m+n}{x+y}\frac{dy}{dx} - \frac{n}{y}\frac{dy}{dx} = \frac{m}{x} - \frac{m+n}{x+y}$$

$$\frac{dy}{dx}\left(\frac{m+n}{x+y} - \frac{n}{y}\right) = \frac{m}{x} - \frac{m+n}{x+y}$$

$$\frac{dy}{dx}\left(\frac{y(m+n)-(x+y)n}{y(x+y)}\right)=\frac{m(x+y)-x(m+n)}{x(x+y)}$$

$$\frac{dy}{dx}=\frac{\left(\dfrac{\cancel{mx}+my-\cancel{mx}-nx}{x(x+y)}\right)}{\left(\dfrac{my+\cancel{ny}-nx-\cancel{ny}}{y(x+y)}\right)}=\frac{my-nx}{my-nx}\times\frac{y}{x}=\frac{y}{x}$$

Question 52 :

If $x=a\cos\theta+b\sin\theta$ and $y=a\sin\theta-b\cos\theta$.

Show that $y^2\dfrac{dy^2}{dx^2}-x\dfrac{dy}{dx}+y=0$

Solution :

Given, $x=a\cos\theta+b\sin\theta$

On differentiating w.r.t θ,we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos\theta+b\sin\theta)=-a\sin\theta+b\cos\theta=b\cos\theta-a\sin\theta$$

Given, $y=a\sin\theta-b\cos\theta.$

On differentiating w.r.t θ,we get

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(a\sin\theta-b\cos\theta)=a\cos\theta+b\sin\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{a\cos\theta+b\sin\theta}{-(a\sin\theta-b\cos\theta)}=\frac{x}{-y}$$

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{x}{y}\right)\Rightarrow\frac{d^2y}{dx^2}=-\left(\frac{y-x\dfrac{dy}{dx}}{y^2}\right)\Rightarrow y^2\frac{d^2y}{dx^2}=-y+x\frac{dy}{dx}$$

$$\Rightarrow y^2\frac{d^2y}{dx^2}+y-x\frac{dy}{dx}=0\Rightarrow y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$$

Question 53 :

Differentiate, $\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ w.r.t $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

Solution :

Let $u = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$

Put $x = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right) = \tan^{-1}\left(\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$u = \cos^{-1}x$$

On differentiating w.r.t x,we get

$$\frac{du}{dx} = \frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$$

Let $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}x$

$$v = \cos^{-1}\left(2\cos\theta\sqrt{1-\cos^2\theta}\right)$$

$$v = \cos^{-1}(2\cos\theta\sin\theta) = \cos^{-1}(\sin 2\theta)$$

$$v = \cos^{-1}\left(\cos\left(\frac{\pi}{2}-2\theta\right)\right) = \frac{\pi}{2}-2\theta = \frac{\pi}{2}-2\cos^{-1}x$$

On differentiating w.r.t x,we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-2\cos^{-1}x\right) = \frac{2}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{du}{d\theta}\times\frac{d\theta}{dv} = -\frac{1}{\sqrt{1-x^2}}\times\frac{\sqrt{1-x^2}}{2} = -\frac{1}{2}$$

Question 54 :

If $y = ae^{x} + be^{5x}$, prove that $\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} + 20y = 0$

Solution :

Given, $y = ae^{4x} + be^{5x}$

$$\frac{dy}{dx}=4ae^{4x}+5be^{5x}$$

$$\frac{d^2y}{dx^2}=4a.4e^{4x}+5b.5e^{5x}=16ae^{4x}+25be^{5x}$$

$$\frac{d^2y}{dx^2}-9\frac{dy}{dx}+20y$$

$$=16ae^{4x}+25be^{5x}-9\left(4ae^{4x}+5be^{5x}\right)+20\left(ae^{4x}+be^{5x}\right)$$

$$=16ae^{4x}+25be^{5x}-36ae^{4x}-45be^{5x}+20ae^{4x}+20be^{5x}$$

$$=45be^{5x}-20ae^{4x}-45be^{5x}+20ae^{4x}=\underline{\underline{0}}$$

Question 55 :

If $y=\cos^{-1}x$, prove that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$

Solution :

Given, $y=\cos^{-1}x$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=1$$

$$\left(1-x^2\right).\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2.\frac{d}{dx}\left(1-x^2\right)=0$$

$$\left(1-x^2\right)2\frac{dy}{dx}.\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2.(-2x)=0$$

$$\left(1-x^2\right)\cancel{2}\cancel{\frac{dy}{dx}}.\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^{\cancel{2}}.\cancel{2}x$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$$

Question 56 :

If $y=\tan^{-1}x$, prove that $\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0$

Solution :

Given, $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\left(1+x^2\right)\frac{dy}{dx} = 1$$

$$\left(1+x^2\right).\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}.\frac{d}{dx}\left(1+x^2\right) = 0$$

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$$

Question 57 :

If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\frac{dy}{dx} - \frac{y}{x} = 0$

Solution :

Given, $y = x^x$

On taking log both sides, we get

$$\log y = \log x^x$$

$$\log y = x \log x$$

On differentiating w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x) = 1 + \log x$$

$$\frac{dy}{dx} = y(1+\log x)$$

Again, differentiating w.r.t x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[y(1+\log x)\right] = y\frac{d}{dx}(1+\log x) + (1+\log x)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = y.\frac{1}{x} + (1+\log x)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} - \frac{y}{x} - \left(\frac{1}{y}\frac{dy}{dx}\right)\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$

Question 58 :

Verify Rolle's theorem for $f(x)=x^3-6x^2+11x-6$ in [1, 3].

Solution :

$f(x)=f(x) = x^3-6x^2+11x-6$

$f(x)$ is continous in $[1,3]$ and differentiable in $(1,3)$

$f(1)=0=f(3)$

$f'(x)=3x^2-12x+12$

$$x = \frac{12\pm\sqrt{144-132}}{6}=2\pm\frac{1}{\sqrt{3}}\in(1,3)$$

$\therefore \quad c=2\pm\frac{1}{\sqrt{3}}$

Question 59 :

Verify Mean Value Theorem for the function $f(x)=x^2$ in the interval [2, 4].

Solution :

The function $f(x)=x^2$ is continuous in [2, 4] and differentiable in (2, 4) as its derivative $f'(x)=2x$ is defined in (2, 4).

Now, $f(2)=4$ and $f(4)=16$

Hence $\quad \frac{f(b)-f(a)}{b-a}=\frac{16-4}{4-2}=6$

MVT states that there is a point $c\in(2, 4)$ such that $f'(x)=6$.

But $f'(x)=2x$ which implies $c=3$.

Thus at $c=3\in(2, 4)$, we have $f'(c)=6$

HOME WORK QUESTIONS

Question : (March2018)

(a) $\frac{d(a^x)}{dx} = \ldots\ldots\ldots$

$\left(a^x,\ \log(a^x),\ a^x \log a,\ xa^{x-1}\right)$

(b) find $\frac{dy}{dx}$ if $x^y = y^x$.

Hint or Answer :

(a) $a^x \log a$

(b) $\frac{dy}{dx} = \frac{\log y - \frac{y}{x}}{\log x - \frac{x}{y}}$

Question : (March2018)

(a) Prove that the function defined by $f(x) = \cos(x^2)$ is a continous function.

(b) (i) If $y = e^{a\cos^{-1}t}, -1 \le x \le 1$, show that $\frac{dy}{dx} = \frac{-ae^{a\cos^{-1}t}}{\sqrt{1-x^2}}$

(ii) Hence, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Hint or Answer :

(ii) $\frac{dy}{dx} = \frac{-ae^{a\cos^{-1}x}}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = -ay \quad \Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

$\Rightarrow (1-x^2).2\frac{dy}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \times (-2x) = 2a^2y\frac{dy}{dx}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Question :(Imp2017)

(a) Examine whether the function defined by

$$f(x)=\begin{cases} x+5 & , \ x \le 1 \\ x-5 & , \ x > 1 \end{cases}$$

is continous or not.

(b) If $x=a^{\sin^{-1}t}, y=a^{\cos^{-1}t}, a>0,$

show that $\dfrac{dy}{dx}=\dfrac{-y}{x}$

(c) If $y=\sin^{-1}x,$

show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}=x\dfrac{dy}{dx}$

Hint or Answer :

(a) $\lim\limits_{x\to 1^-} f(x)=6 \quad , \quad \lim\limits_{x\to 1^+} f(x)=-4$

$\lim\limits_{x\to 1^+} f(x)\neq \lim\limits_{x\to 1^-} f(x)$

$\therefore$ $f(x)$ is not continous.

(b) $\dfrac{dx}{dt}=\log a\dfrac{a^{\sin^{-1}t}}{\sqrt{1-t^2}}=\log a\dfrac{x}{\sqrt{1-t^2}}$

$\dfrac{dy}{dt}=-\log a\dfrac{a^{\cos^{-1}t}}{\sqrt{1-t^2}}=-\log a\dfrac{y}{\sqrt{1-t^2}}$

$\Rightarrow \quad \dfrac{dy}{dt}=\dfrac{-y}{x}$

(c) $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \sqrt{1-x^2}\dfrac{dy}{dx}=1$

$\sqrt{1-x^2}\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}\dfrac{1}{2\sqrt{1-x^2}}(-2x)=0$

$\left(1-x^2\right)\dfrac{d^2y}{dx^2}=x\dfrac{dy}{dx}$

Question :(M2017)

(a) Find the value of a and b such that the function

$$f(x)=\begin{cases}5a & ,x\le 0\\ a\sin x+\cos x & ,0<x<\frac{\pi}{2}\\ b-\frac{\pi}{2} & ,x\ge \frac{\pi}{2}\end{cases}\quad \text{is continous.}$$

(b) Find $\frac{dy}{dx}$ if $(\sin x)^{\cos y}+(\cos y)^{\sin x}$.

Hint or Answer :

(a) $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)=f(a)$

$$5a=1 \text{ or } a=\frac{1}{5}\ ;\ a=b-\frac{\pi}{2} \text{ or } b=a+\frac{\pi}{2}$$

(b) $\dfrac{\cos x.\log y-\cos y\cot x}{\sin x.\tan y-\sin y.\log\sin x}$

Question :(Imp2016)

(a) Find $\frac{dy}{dx}$ if $x=a\cos^2\theta,\ y=b\sin^2\theta$

(b) Find the second derivative of the function $y=e^x\sin x$

Hint or Answer : (a) $-\frac{b}{a}$ (b) $2e^x\cos x$

Question :(M2016)

(a) Find all point of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}2x+3 & ,\ x\le 2\\ 2x-3 & ,\ x>2\end{cases}$$

(b) If $e^{x-y}=x^y$, then prove that $\frac{dy}{dx}=\frac{\log x}{[\log e^x]^2}$

Hint or Answer :

(a) $LHL=7, RHL=1;$

$$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$$

$\therefore$ 2 is point od discontinuity.

(b) $(x-y)\log e = y\log x$

$$x-y = y\log x \;\Rightarrow\; x = y - y\log x \;\Rightarrow\; x = y(1+\log x)$$

$$y = \frac{x}{(1+\log x)}$$

$$\frac{dy}{dx} = \frac{1+\log x - x\frac{1}{x}}{[1+\log x]^2} = \frac{1+\log x - 1}{[\log e + \log x]^2} = \frac{\log x}{[\log ex]^2}$$

Question : (Imp2015)

(a) Find the relationship between a and b if the function f defined by,

$$f(x) = \begin{cases} ax+1 & , x \le 3 \\ bx+3 & , x > 3 \end{cases}$$ is continuous (NCERT)

(b) Find $\frac{dy}{dx}$ if $y^x + x^y$. (NCERT)

(c) If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ (NCERT)

Hint or Answer :

(a) $a = b + \frac{2}{3}$ (b) $\frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$

(c) $\frac{dy}{dx} = \frac{-1}{x+1}, \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(\frac{-1}{x+1}\right)^2 = \left(\frac{dy}{dx}\right)^2$