1 RELATION AND FUNCTION

- ***Empty relation*** is the relation R in X given by $R = \phi \subset X \times X$.
- ***Universal relation*** is the relation R in X given by $R = X \times X$.
- ***Reflexive relation*** R in X is a relation with $(a,a) \in R \forall a \in X$.
- ***Symmetric relation*** R in X is a relation satisfying $(a,b) \in R$ implies $(b,a) \in R$.
- ***Transitive relation*** R in X is a relation satisfying $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$.
- ***Equivalence relation*** R in X is a relation which is reflexive, symmetric and transitive.
- ***Equivalence class*** $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f : X \to Y$ is *one-one* (or *injective*) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in X$.
- A function $f : X \to Y$ is *onto* (or *surjective*) if given any $y \in Y$, $\exists\ x \in X$ such that $f(x) = y$.
- A function $f : X \to Y$ is *one-one and onto* (or *bijective*), if f is both one-one and onto.

- The *composition* of functions $f : A \to B$ and $g : B \to C$ is the function $gof : A \to C$ given by

$$gof(x) = g(f(x)) \forall x \in A.$$

- A function $f : X \to Y$ is *invertible* if $\exists g : Y \to X$ such that $gof = IX$ and $fog = IY$.
- A function $f : X \to Y$ is ***invertible*** if and only if *f* is one-one and onto.
- Given a finite set X, a function $f : X \to X$ is one-one (respectively onto) if and only if *f* is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set
- A ***binary operation*** $*$ on a set A is a function $*$ from $A \times A$ to A.
- An element $e \in X$ is the ***identity*** element for binary operation $* : X \times X \to X$, if $a * e = a = e * a \forall a \in X$.
- An element $a \in X$ is ***invertible*** for binary operation $* : X \times X \to X$, if there exists $b \in X$ such that $a * b = e = b * a$ where, *e* is the identity for the binary operation $*$. The element b is called ***inverse*** of a and is denoted by a^{-1}.
- An operation $*$ on X is ***commutative*** if $a * b = b * a \forall a, b$ in X.
- An operation $*$ on X is ***associative*** if $(a * b) * c = a * (b * c) \forall a, b, c$ in X.

1 RELATION AND FUNCTION

Question :

If a relation R on the set $\{1,2,3\}$ be defined by $R=\{1,2\}$, then R is

Answer :

R on the set $\{1,2,3\}$ be defined by $R=\{1,2\}$

R is not reflexive,transitive and symmetric

Question :

Show that the relation R in the set R of real numbers, defined as $R=\{(a,b):a\le b^2\}$ is neither reflexive nor symmetric nor transitive.

Answer :

$R=\{(a,b):a\le b^2\}$

It can be observed that $\left(\frac{1}{2},\frac{1}{2}\right)\notin R$,since $\frac{1}{2}>\left(\frac{1}{2}\right)^2=\frac{1}{4}$

$\therefore$ R is not reflexive.

$(1,3)\in R$ as $1<3^2$

But 3 is not less than 1^2

$\therefore$ $(3,1)\notin R$

$\therefore$ R is not symmetric.

$(3,2),(2,1.5)\in R$ as $3<2^2=4$ and $2<(1.5)^2=2.25$

But $3>(1.5)^2$=2.25

$\therefore$ $(3,1.5)\notin R$

$\therefore$ R is not transitive.

Hence, R is neither reflexive nor symmetric nor transitive.

Question :

What is the range of the function $f(x)=\frac{|x-1|}{x-1},x\ne 1$

Answer :

Given, $f(x)=\frac{|x-1|}{x-1}$

$$f(x)=\begin{cases}\frac{x-1}{x-1}, & x>1\\ -\frac{x-1}{x-1}, & x<1\end{cases}=\begin{cases}1, & x>1\\ -1, & x<1\end{cases}$$

Range $=\{-1,\ 1\}$

Question :

Let $A=R-\{3\}$ and $B=R-\{3\}$. Consider the function $f:A\to B$ defined by $f(x)=\frac{x-2}{x-3}$. Is f one-one and onto? Justify your answer

Answer :

$A=R-\{3\}$ and $B=R-\{3\}$

$f:A\to B$ defined by $f(x)=\frac{x-2}{x-3}$

Let $x,y\in A$ such that $f(x)=f(y)$

$$\frac{x-2}{x-3}=\frac{y-2}{y-3}$$

$$(x-2)(y-3)=(y-2)(x-3)$$

$$xy-3x-2y+6=xy-3y-2x+6$$

$$-3x-2y=-3y-2x$$

$$3x-2x=3y-2y$$

$$x=y$$

$\therefore$ R is one-one.

Let $y\in B=R-\{1\}$. Then, $y\neq 1$.

The function f is onto if there exists $x\in A$ such that $f(x)=y$.

$$f(x)=y$$

$$\frac{x-2}{x-3}=y$$

$$x-2=y(x-3)$$

$$x-2=xy-3y$$

$$x-xy=2-3y$$

$$x(1-y)=2-3y$$

$$x=\frac{2-3y}{(1-y)}$$

Thus, for any $y\in B$, there exists $\frac{2-3y}{(1-y)}\in A$ such that

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}$$

$$=\frac{2-3y-2(1-y)}{2-3y-3(1-y)}$$

$$=\frac{2-3y-2+y}{2-3y-3+y}=y$$

$\therefore$ f is onto.

Hence the function is one-one and onto.

Question :

If $f(x)=27x^3$ and $g(x)=x^{1/3}$,then find $gof(x)$

Answer :

Given, $f(x)=27x^3$ and $g(x)=x^{1/3}$

$$gof(x)=g[f(x)]$$

$$=g[27x^3]$$

$$=(27x^3)^{1/3}$$

$$=(27)^{1/3}(x^3)^{1/3}$$

$$=\underline{3x}$$

Question :

If $f, g: R \to R$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in R$, respecively. Find gof.

Answer :

We have, $f(x) = 2x + 1$ and $g(x) = x^2 - 2, \forall x \in R$

$$\begin{aligned} \therefore \quad gof &= gf(x) \\ &= g(2x+1) \\ &= (2x+1)^2 - 2 \\ &= 4x^2 + 4x + 1 - 2 \\ &= \underline{\underline{4x^2 + 4x - 1}} \end{aligned}$$

Question :

Let $A = R - \{3\}$, $B = R - \{1\}$. If $f: A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3}, \forall x \in R$. Then show that f is bijective.

Answer :

Given $A = R - \{3\}, B = R - \{1\}$

$f: A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3}, \forall x \in A$

$$\therefore \quad f(x) = \frac{x-3+1}{x-3} = 1 + \frac{1}{x-3}$$

Let $f(x_1) = f(x_2)$

$$1 + \frac{1}{x_1 - 3} = 1 + \frac{1}{x_2 - 3}$$

$$\frac{1}{x_1 - 3} = \frac{1}{x_2 - 3}$$

$$x_1 = x_2$$

So, $f(x)$ is an injective function

Now let $y = \dfrac{x-2}{x-3}$

$$x - 2 = xy - 3y$$

$$x(1-y) = 2-3y$$

$$x = \frac{2-3y}{(1-y)}$$

$$y \in R-\{1\} = B$$

Hence $f(x)$ is onto or subjective.

Hence $f(x)$ is a bijective.

Question :

Consider f : $R_+ \to [-5, \infty]$ given by $f(x) = 9x^2 + 6x - 5$.

Show that f is invertible with $f^{-1}y = \left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$

Answer :

Let $\quad y = 9x^2 + 6x - 5$

$$y = (3x)^2 + 6x + 1 - 1 - 5$$

$$y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$$

$$(3x+1)^2 = y+6$$

$$3x+1 = \sqrt{y+6}$$

$$3x = \sqrt{y+6} - 1$$

$$x = \frac{\sqrt{y+6}-1}{3}$$

$\therefore$ f is onto, thereby range $f = [-5, \infty)$

Let us define g: $[-5, \infty) \to R_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$

Now

$$(gof)(x) = g(f(x)) = g(9x^2 + 6x - 5)$$

$$= g\left((3x+1)^2 - 6\right)$$

$$= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3} = \frac{3x+1-1}{3} = x$$

and $(fog)(y) = f(g(y))$

$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$$

$$= \left[\sqrt{y+6}-1+1\right]^2 - 6$$

$$= y+6-6 = y$$

$\therefore$ gof $= I_{R_+}$ and gof $= I_{[-5,\infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}y = g(y) = \frac{\left(\sqrt{y+6}\right)-1}{3}$$

Question :

Let $*$ be binary operation defined on R by $a*b = 1+ab$, $a,b \in R$. Then the operation $*$ is .....

Answer :

Given $a*b = 1+ab, a,b \in R$.

$\therefore \quad a*b = 1+ab = b*a$

So, $*$ is a commutative binary operation

Now $a*(b*c) = a*(1+bc) = 1+a(1+bc) = 1+a+abc$

Also $(a*b)*c = (1+ab)*c = 1+a(1+ab)c = 1+c+abc$

$\therefore$ $*$ is not associative.

Hence $*$ is commutative but not associative.

Question :

Let $*$ $R\times R \to R$ is defined as $a*b = 2a+b$.Find $(2*3)*4$.

Answer :

Given, $*$ $R\times R \to R$

$a*b = 2a+b$

Put $a = 2$ and $b = 3$

So, $2*3 = 2(2)+3 = 7$

$(2*3)*4 = 7*4 = 2(7)+4 = 14+4 = 18$

HOME WORK QUESTIONS

Question : (March 2018)

If $f(x)=\frac{x}{x-1}$, $x\neq 1$.

(a) Find $fof(x)$

(b) Find the inverse of f.

Hint or Answer :

(a) $fof(x)=x$ (b) $\frac{x}{x-1}$

Question : (March 2018)

Let $A=N\times N$ and '$*$' be a binary operation on A defined by $(a,b)*(c,d)=(a+c,b+d)$

(a) Find $(1,2)*(2,3)$

(b) Prove that '$*$' is commutative

(c) Prove that '$*$' is associative .

Hint or Answer :

(a) $(3,5)$

Question : (Imp 2017)

(a) If $(x+1,\ y-2)=(3,\ 1)$, write the values of x and y.

(b) Let $A=\{1,2,3,4,5\}$ and $B=\{4,6,9\}$.

Define a relation $R:A\to B$ by

$$R=\{(x,y): x-y \text{ is a positive integer}\}$$

Find $A\times B$ and write R in the roster form.

(c) Define the modulus function. What is its domain?

Draw a rough sketch.

Hint or Answer :

(a) $x=2,\ y=3$ (b) $R=\{(5,4)\}$ (c) ....

Question : (March 2017)

(a) Let R be a relation on $A = \{1,2,3\}$ by $R = \{(1,3),(3,1),(2,2)\}$
R is Reflexive, Symmetric, Transitive, Reflexive but not transitive

(b) Find fog and gof if $f(x) = |x+1|$ and $g(x) = 2x - 1$.

(c) Let $*$ be a binary operation defined on $N \times N$ by
$(a,b)*(c,d) = (a+c, b+d)$
Find the identity element for $*$ if it exists.

Hint or Answer :

(a) Symmetric

(b) $fog(x) = |2x|, gof(x) = |2x+1| - 1$

(c) identity element does not exist.

Question : (Imp 2016)

(a) If $f: R \to R$ and $g: R \to R$ defined by $f(x) = x^2$ and $g(x) = x + 1$, then $gof(x)$ is

(b) Consider the function $f: N \to N$ given by $f(x) = x^3$. Show that the function f is injective but not surjective.

(c) The given table shows an operation $*$ on $A = \{p, q\}$

*	p	q
p	p	q
q	p	q

(i) Is $*$ a binary operation on A?

(ii) Is $*$ commutative? Give reason.

Hint or Answer :

(a) $x^2 + 1$ (b) injective, not surjective (c) (i) Yes (ii) No

Question : (March 2016)

(a) The function $f:N \to N$, given by $f(x) = 2x$ is

(a) one-one and onto

(b) one-one but not onto

(c) not one-one and not onto

(d) onto but not one-one

Find $gof(x)$, if $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

(b) Let $*$ be a binary operation such that $a * b = \text{LCM of } a \text{ and } b$ defined on the set $A = \{1, 2, 3, 4, 5\}$. Is $*$ a binary operation? Justify your answer.

Answer :

(a) one-one and but not onto (b) 2x (c) No

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