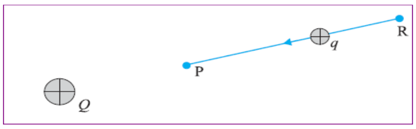
Chapter Two

ELECTROSTATIC POTENTIAL AND CAPACITANCE

Potential energy difference



- Electric potential energy difference between two points is the work required to be done by an external force in moving charge q from one point to another.

$$\Delta U = U_P - U_R = W_{RP}$$

Electrostatic potential energy

- Potential energy of charge q at a point is the work done by the external force in bringing the charge q from infinity to that point.

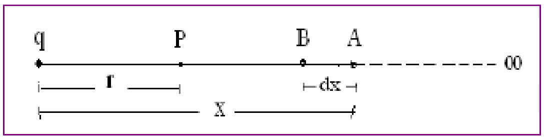
ELECTROSTATIC POTENTIAL

- The electrostatic potential (*V*) at any point is the work done in bringing a unit positive charge from infinity to that point.

$V = \dfrac{W}{q}$, W – work done, q – charge.

- Also $W = qV$
- It is a scalar quantity.
- Unit is J/C or volt (V)

POTENTIAL DUE TO A POINT CHARGE



- The force acting on a unit positive charge (+1 C) at A , is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q\times 1}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}$$

- Thus the work done to move a unit positive charge from A to B through a displacement dx is

$$dW = -\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}dx$$

- The negative sign shows that the work is done against electrostatic force.
- Thus the total work done to bring unit charge from infinity to the point P is

$$W = \int_{\infty}^{r} dW = \int_{\infty}^{r}\left[-\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}dx\right]$$

$$W = -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r}\left[\frac{1}{x^2}dx\right]$$

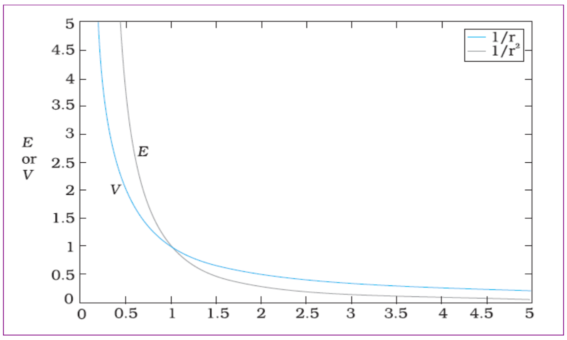
- Integrating

$$W = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{1}{\infty}\right] = \frac{q}{4\pi\varepsilon_0 r}$$

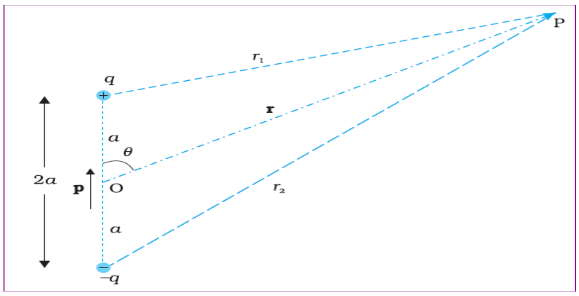
- Therefore electrostatic potential is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Variation of potential *V with r*



POTENTIAL DUE TO AN ELECTRIC DIPOLE



The potential due to the dipole at P is the sum of potentials due to the charges q and –q

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1}-\frac{q}{r_2}\right)$$

- Using cosine law

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta$$
$$r_2^2 = r^2 + a^2 + 2ar\cos\theta$$

- For r >> a

$$r_1^2 = r^2\left(1 - \frac{2a\cos\theta}{r} + \frac{a^2}{r^2}\right)$$

- Neglecting the higher order terms we get

$$r_1^2 \cong r^2\left(1 - \frac{2a\cos\theta}{r}\right)$$

- Similarly $r_2^2 \cong r^2\left(1 + \frac{2a\cos\theta}{r}\right)$

- Thus $\frac{1}{r_1} \cong \frac{1}{r}\left(1 - \frac{2a\cos\theta}{r}\right)^{-\frac{1}{2}}$ and

$$\frac{1}{r_2} \cong \frac{1}{r}\left(1 + \frac{2a\cos\theta}{r}\right)^{-\frac{1}{2}}$$

- Using the Binomial theorem and retaining terms up to the first order in *a/r*,

$$\frac{1}{r_1} \cong \frac{1}{r}\left(1 - \frac{2a\cos\theta}{r}\right)^{-1/2} \cong \frac{1}{r}\left(1 + \frac{a}{r}\cos\theta\right)$$

$$\frac{1}{r_2} \cong \frac{1}{r}\left(1 + \frac{2a\cos\theta}{r}\right)^{-1/2} \cong \frac{1}{r}\left(1 - \frac{a}{r}\cos\theta\right)$$

- Thus the potential is

$$V = \frac{q}{4\pi\varepsilon_0}\frac{2a\cos\theta}{r^2}$$

- Using p=q x2a, we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

Special cases

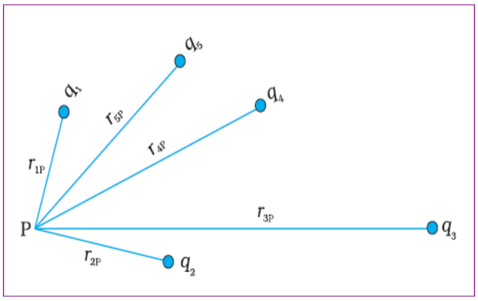
- **Potential at point on the axial line**
  At the axial point θ=0, therefore

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}$$

- **Potential at point on the equatorial line**
  At the equatorial line θ=90^0, thus , V=0.

POTENTIAL DUE TO A SYSTEM OF CHARGES

- By the superposition principle, the potential *at a point due* to a system of charges is the algebraic sum of the potentials due to the individual charges.



- Thus $V = V_1 + V_2 + ...V_n$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}} + \frac{q_2}{r_{2P}} + ... + \frac{q_n}{r_{nP}}\right)$$

Potential due to a uniformly charged spherical shell

- For a uniformly charged spherical shell, the electric field outside the shell is as if the entire charge is concentrated at the centre
- Thus potential at a distance r, from the shell is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

- Where $r \geq R$, radius of the shell
- Inside the shell, the potential is a constant and has the same value as on its surface.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

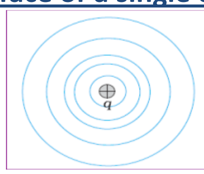
Equipotential surface

- Surface with constant value of potential at all points on the surface.

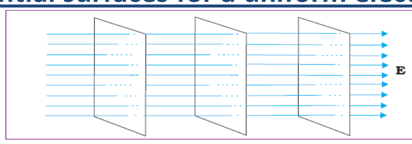
Properties of Equipotential surface

- **Work done** to move a charge on an equipotential surface is **zero**.
- Electric field is perpendicular to the surface.
- Two equipotential surfaces never intersect.

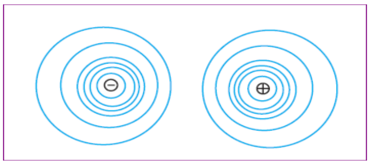
Equipotential surface of a single charge



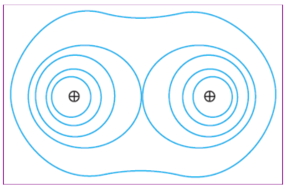
Equipotential surfaces for a uniform electric field



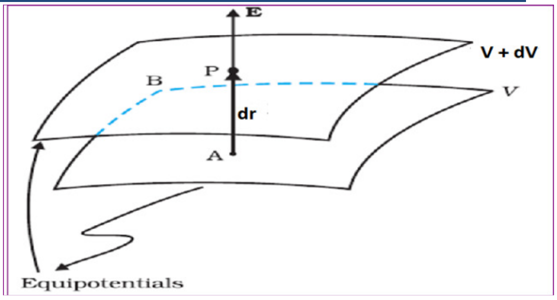
Equipotential surfaces for a dipole



Equipotential surfaces for two identical positive charges



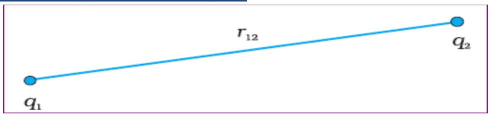
Relation between electric field and potential



- The work done to move a unit positive charge from B to A is
  $$Work = Edr$$
- This work equals the potential difference $V_A - V_B$.
- Thus $\quad Edr = V - (V + dV) = -dV$
- That is $$E = -\frac{dV}{dr}$$

POTENTIAL ENERGY OF A SYSTEM OF CHARGES

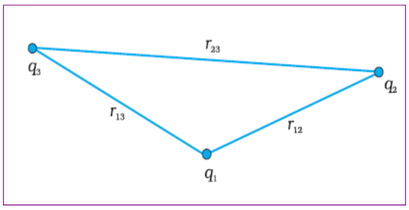
For a system of two charges



- As there is no external field work done in bringing q_1 from infinity to $\mathbf{r_1}$ **is zero.**
- The potential due to the charge q_1 is
  $$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{1P}}$$
- The work done in bringing charge q_2 *from* infinity to the point $\mathbf{r_2}$ **is**
  $$\text{work done on } q_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$
- This work gets stored in the form of potential energy of the system.
- Thus, the potential energy of a system of two charges q_1 *and* q_2 *is*

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

For a system of three charges



- The work done to bring q_1 from infinity to the point is zero.
- The work done to bring q_2 is
  $$q_2 V_1(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$
- The charges q_1 *and* q_2 *produce a potential, which at any point P is* given by
  $$V_{1,2} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}} + \frac{q_2}{r_{2P}}\right)$$
- Work done in bringing q_3 *from infinity to the point* $\mathbf{r_3}$ ***is***
  $$q_3 V_{1,2}(\mathbf{r}_3) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$
- The total work done in assembling the charges
  $$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

POTENTIAL ENERGY IN AN EXTERNAL FIELD

Potential energy of a single charge

- Potential energy of *q at* ***r in an external field is***
  $$U = qV(r)$$
- where *V(r) is the external potential at the point r.*

Potential energy of a system of two charges in an external field

- Work done to bring q1 is
  $$q_1 V(\mathbf{r}_1)$$
- Work done on *q2 against the external field*
  $$= q_2 V(\mathbf{r}_2)$$
- Work done on *q2 against the field due to q1*

$$= \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$$

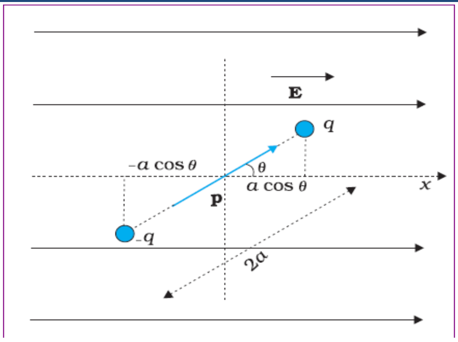
- Thus the total Work done in bringing q_2 *to* $\mathbf{r_2}$

$$= q_2 V(\mathbf{r}_2) + \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$$

- Thus, Potential energy of the system = the total work done in assembling the configuration

$$= q_1 V(\mathbf{r}_1) + q_2 V(\mathbf{r}_2) + \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}$$

Potential energy of a dipole in an external field



- The torque experienced by the dipole is

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$$

- *The amount of work done by the* external torque will be given by

$$W = \int_{\theta_0}^{\theta_1} \tau_{ext}(\theta) d\theta = \int_{\theta_0}^{\theta_1} pE \sin\theta \, d\theta$$

$$= pE(\cos\theta_0 - \cos\theta_1)$$

- This work is stored as the potential energy of the system.
- Thus

$$U(\theta) = pE\left(\cos\frac{\pi}{2} - \cos\theta\right) = -pE\cos\theta = -\mathbf{p}\cdot\mathbf{E}$$

- Therefore

$$U = -pE\cos\theta,$$

- Where p- dipole moment , E – electric field

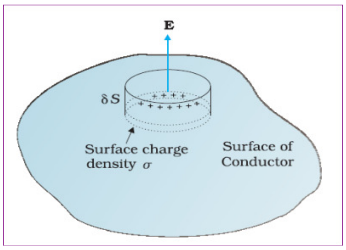
Properties of conductors

- **Inside a conductor, electrostatic field is zero**
- A conductor has free electrons
- In the static situation, the free charges have so distributed themselves that the electric field is zero everywhere inside
- **At the surface electric field is normal.**
- If E were not normal to the surface, it would have some non-zero component along the surface.
- Free charges on the surface of the conductor would then experience force and move.
- In the static situation, therefore, E should have no tangential component.
- ***The interior of a conductor can have no excess charge in the static situation***
- A neutral conductor has equal amounts of positive and negative charges in every small volume or surface element.
- When the conductor is charged, the excess charge can reside only on the surface in the static situation.
- **Electric potential is constant throughout the volume of the conductor**.
- Since **E = 0 inside the conductor** and has no tangential component on the surface, no work is done in moving a small test charge within the conductor and on its surface.
- That is, there is no potential difference between any two points inside or on the surface of the conductor.
- ***Electric field at the surface of a charged conductor***

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

- where σ is the surface charge density and ˆn is a unit vector normal to the surface in the outward direction

Derivation

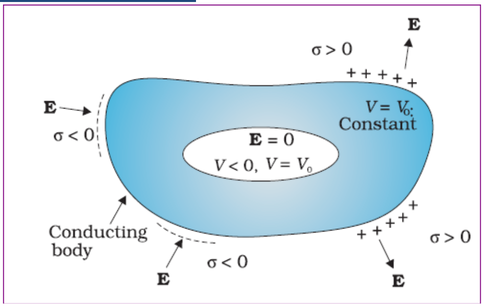


- choose a pill box (a short cylinder) as the Gaussian surface about any point P on the surface.
- The pill box is partly inside and partly outside the surface of the conductor.
- It has a small area of cross section δS and negligible height.
- The contribution to the total flux through the pill box comes only from the outside (circular) cross-section of the pill box.
- The charge enclosed by the pill box is σδ*S.*
- By Gauss's law

$$E\delta S = \frac{|\sigma|\delta S}{\varepsilon_0}$$

$$E = \frac{|\sigma|}{\varepsilon_0}$$

Electrostatic shielding



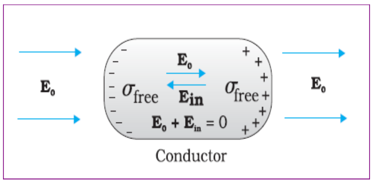
- The vanishing of electric field inside a charged conducting cavity is known as **electrostatic shielding.**
- The effect can be made use of in protecting sensitive instruments from outside electrical influence.
- **Why it is safer to be inside a car during lightning?**
- Due to Electrostatic shielding, E=0 inside the car.

DIELECTRICS

- Dielectrics are non-conducting substances.
- They have no (or negligible number of) charge carriers.

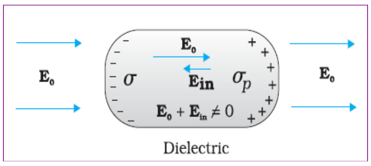
Conductor in an external field

- In an external field the free charge carriers in the conductor move and an electric field which is equal and opposite to the external field is induced inside the conductor.
- The two fields cancel each other and the net electrostatic field in the conductor is zero.



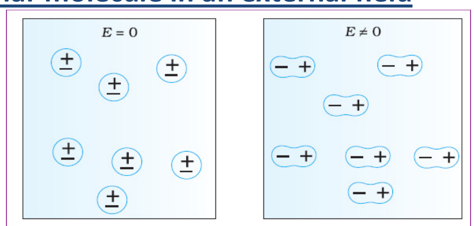
Dielectric in an external field

- In a dielectric, the external field induces dipole moment by **stretching or re-orienting** molecules of the dielectric.
- Thus a net electric field is induced inside the dielectric in the opposite direction.
- The induced field does not cancel the external field.



- Dielectric substances may be made of polar or non polar molecules.

Non polar molecule in an external field



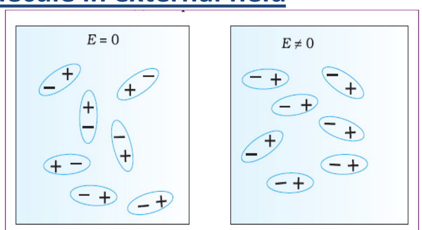
(a) Non-polar molecules

- In an external electric field, the positive and negative charges of a nonpolar molecule are displaced in opposite directions.
- The non-polar molecule thus develops an induced dipole moment.

Linear isotropic dielectrics

- When a dielectric substance is placed in an electric field, a net dipole moment is induced in it.
- When the induced dipole moment is in the direction of the field and is proportional to the field strength the substances are called ***linear isotropic dielectrics*.**

Polar molecule in external field



(b) Polar molecules

- In the absence of any external field, the different permanent dipoles are oriented randomly due to thermal agitation; so the total dipole moment is zero.
- When an external field is applied, the individual dipole moments tend to align with the field.
- A dielectric with polar molecules also develops a net dipole moment in an external field.

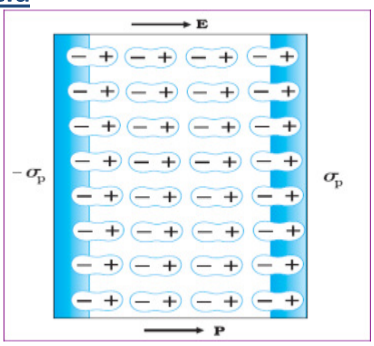
Polarisation

- The dipole moment per unit volume is called *polarisation and is denoted by **P.***
- For linear isotropic dielectrics,

$$\boxed{\mathbf{P} = \chi_e \, \mathbf{E}}$$

- where χ_e *is the **electric susceptibility** of the dielectric medium.*

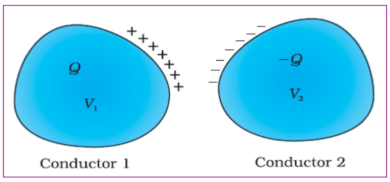
A rectangular dielectric slab placed in a uniform external field



- The polarised dielectric is equivalent to two charged surfaces with induced surface charge densities, say σ_p and $-\sigma_p$
- That is a uniformly polarised dielectric amounts to induced surface charge density, but no volume charge density.

Capacitor

- It is a charge storing device.
- A capacitor is a system of two conductors separated by an insulator.



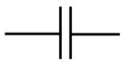
- A capacitor with large capacitance can hold large amount of charge *Q at a relatively small V.*

Capacitance

- The potential difference is proportional to the charge , Q.
- Thus $C = \dfrac{Q}{V}$
- The constant C is called the *capacitance of the capacitor. C is independent* of *Q or V.*
- The capacitance *C depends only on the* geometrical configuration (shape, size, separation) of the system of two conductors
- SI unit of capacitance is **farad.**
- Other units are, 1 μF = 10^{-6} F, 1 nF = 10^{-9} F, 1 pF = 10^{-12} F, etc.

Symbol of capacitor

Fixed capacitance



Variable capacitance

Dielectric strength

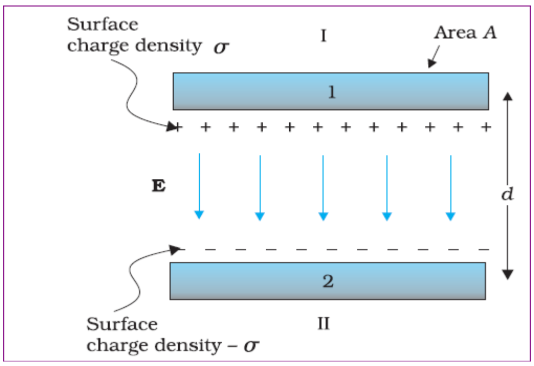
- The maximum electric field that a dielectric medium can withstand without break-down is called its dielectric strength.
- The dielectric strength of air is about $3 \times 10^{6}\ \mathrm{Vm}^{-1}$.

THE PARALLEL PLATE CAPACITOR

- A parallel plate capacitor consists of two large plane parallel conducting plates separated by a small distance

Capacitance of parallel plate capacitor

- Let *A be* the area of each plate and *d the separation between* them.
- The two plates have charges *Q and –Q.*
- Plate 1 has surface charge density σ = *Q/A and plate 2 has a surface* charge density –σ.



- At the region I and II, E=0

$$E=\frac{\sigma}{2\varepsilon_0}-\frac{\sigma}{2\varepsilon_0}=0$$

- At the inner region

$$E=\frac{\sigma}{2\varepsilon_0}+\frac{\sigma}{2\varepsilon_0}=\frac{\sigma}{\varepsilon_0}=\frac{Q}{\varepsilon_0 A}$$

- The direction of electric field is from the positive to the negative plate.
- For a uniform electric field the potential difference is

$$V=Ed=\frac{1}{\varepsilon_0}\frac{Qd}{A}$$

- The capacitance *C of the parallel plate capacitor is then*

$$C=\frac{Q}{V}==\frac{\varepsilon_0 A}{d}$$

- Thus $C=\dfrac{\varepsilon_0 A}{d}$

Effect of dielectric on capacitance

- When dielectric medium is placed capacitance **increases.**
- When there is vacuum between the plates,

$$E_0=\frac{\sigma}{\varepsilon_0} \qquad V_0=E_0 d$$

- The capacitance C_0 *in this case is*

$$C_0=\frac{Q}{V_0}=\varepsilon_0\frac{A}{d}$$

- When a dielectric is introduced

$$E=\frac{\sigma-\sigma_P}{\varepsilon_0}$$

- so that the potential difference across the plates is

$$V=Ed=\frac{\sigma-\sigma_P}{\varepsilon_0}d$$

- For linear dielectrics, σ_p *is proportional to* E_0*, and hence to* σ*.*
- Thus, $(\sigma-\sigma_p)$ *is proportional to* σ *and we can write*

$$\sigma-\sigma_P=\frac{\sigma}{K}$$

- where *K is the dielectric constant.*
- Thus

$$V=\frac{\sigma d}{\varepsilon_0 K}=\frac{Qd}{A\varepsilon_0 K}$$

- The capacitance C, with dielectric between the plates, is then

$$C=\frac{Q}{V}=\frac{\varepsilon_0 KA}{d}$$

- That is

$$C=\frac{\varepsilon_0 KA}{d}$$

- The product $\varepsilon_0 K$ *is called the permittivity of the medium and is* denoted by ε.

$$\varepsilon=\varepsilon_0 K$$

- For vacuum *K = 1 and* $\varepsilon=\varepsilon_0$*;* ε_0 *is called the permittivity of the vacuum.*
- Thus the ***dielectric constant*** *of the substance is*

$$K=\frac{\varepsilon}{\varepsilon_0}$$

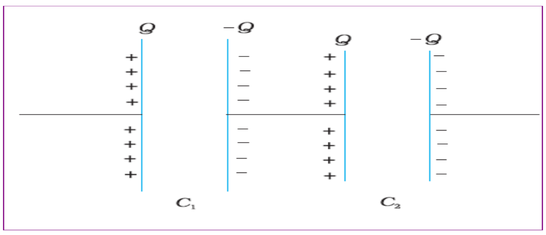
- Also

$$K=\frac{C}{C_0}$$

- C_0 – capacitance in vacuum, C- capacitance in dielectric medium.

Combination of capacitors

Capacitors in series



- In series charge is same and potential is different on each capacitors.
- The total potential drop *V* across the combination is

$$V=V_1+V_2$$

- Considering the combination as an effective capacitor with charge *Q and potential* difference *V, we get*

$$\frac{Q}{C}=\frac{Q}{C_1}+\frac{Q}{C_2}$$

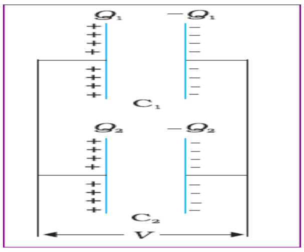
- Therefore effective capacitance is

$$\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}$$

- For n capacitors in series

$$\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}+.......+\frac{1}{C_n}$$

Capacitors in parallel



- In parallel the charge is different, potential is same on each capacitor.
- The charge on the equivalent capacitor is

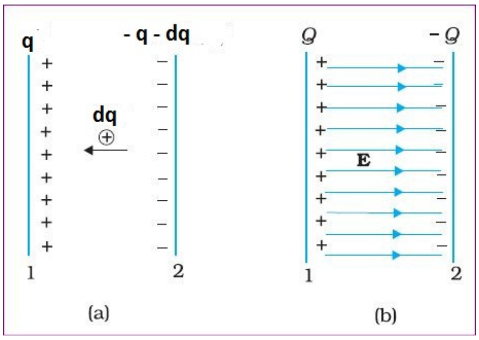
$$Q=Q_1+Q_2$$

- Thus $CV=C_1V+C_2V$
- Therefore $C=C_1+C_2$
- In general , for n capacitors

$$C=C_1+C_2+.............+C_n$$

Energy stored in a capacitor

- Energy stored in a capacitor is the **electric potential energy.**



- Charges are transferred from conductor 2 to conductor 1 bit by bit, so that at the end, conductor 1 gets charge Q.
- Work done to move a charge **dq** from conductor 2 to conductor 1, is

$$dW = Potential\times Charge$$

- That is $dW=\frac{q}{C}\times dq$
- Since potential at conductor 1 is , **q/C.**
- Thus the total work done to attain a charge Q on conductor 1, is

$$W=\int_0^Q dW=\int_0^Q \frac{q}{C}\times dq$$

- On integration we get,

$$W=\frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q=\frac{Q^2}{2C}$$

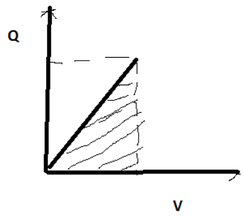
- This work is stored in the form of potential energy of the system.
- Thus energy stored in the capacitor is

$$U=\frac{Q^2}{2C}$$

- Also $U=\frac{1}{2}QV$ or $U=\frac{1}{2}CV^2$

Alternate method

- We have the Q – V graph of a capacitor,



- Energy = area under the graph
- Thus , $U=\frac{1}{2}\times Q\times V$
- Also $U=\frac{1}{2}CV^2$

Energy Density of a capacitor

- Energy density is the energy stored per unit volume.
- We have $U=\frac{Q^2}{2C}$
- But $Q=\sigma A$ and $C=\frac{\varepsilon_0 A}{d}$
- Thus we get $U=\frac{(\sigma A)^2}{2}\left(\frac{d}{\varepsilon_0 A}\right)$
- Using $E=\frac{\sigma}{\varepsilon_0}$, we get

$$U=\frac{1}{2}\varepsilon_0E^2\times Ad$$

- Thus energy per unit volume is given by

$$\frac{U}{Ad} = \frac{1}{2}\varepsilon_0 E^2$$

- That is the energy density of the capacitor is

$$u = \frac{1}{2}\varepsilon_0 E^2$$

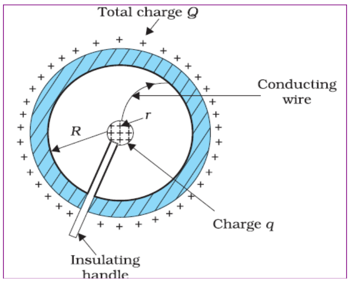
VAN DE GRAAFF GENERATOR

Uses

- Used to generate high potential about million volts and electric field close to 3×10^6 V/m.
- High potential generated is used to accelerate charged particles to high energies.

Principle

- When a small conducting sphere is kept inside a large sphere, potential is high at the small sphere.



- Potential due to small sphere of radius *r carrying charge q*

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \text{ at surface of small sphere}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \text{ at large shell of radius } R.$$

- Taking both charges *q and Q, we have*

$$V(R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} + \frac{q}{R}\right)$$

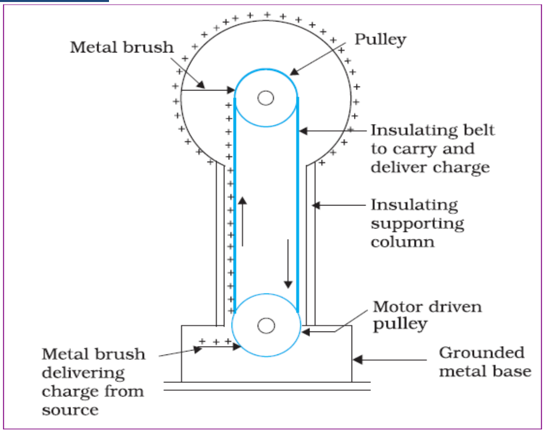
$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} + \frac{q}{r}\right)$$

$$V(r) - V(R) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)$$

- Thus the potential at the smaller sphere is higher than the larger sphere.

- If the two spheres are connected with a wire , charge will flow from smaller sphere to larger one.

Construction



- A large spherical conducting shell is supported at a height several meters above the ground on an insulating column.
- A long narrow endless belt insulating material, like rubber or silk, is wound around two pulleys – one at ground level, one at the centre of the shell.

Working

- The belt is kept continuously moving by a motor driving the lower pulley.
- It continuously carries positive charge, sprayed on to it by a brush at ground level, to the top.
- At the top, the belt transfers its positive charge to another conducting brush connected to the large shell.
- Thus positive charge is transferred to the shell, where it spreads out uniformly on the outer surface.
- In this way, voltage differences of as much as **6 or 8 million volts** (with respect to ground) can be built up.
- The voltage produced by an open-air Van de Graaff machine is limited by corona discharge to about 5 megavolts.
- Most modern industrial machines are enclosed in a pressurized tank of insulating gas these can achieve potentials of as much as about 25 megavolts.

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