# Chapter - 7

# System of Particles and Rotational Motion

## One mark Questions

1. ***What is a Rigid body?***

   A Rigid body is one for which the distances between different particles of the body do not change even though there are forces on them.

2. ***What type of motion a rigid body may have when it is fixed along an axis.***

   Rotational motion.

3. ***When does a rigid body said to have rotational motion?***

   A Rigid body is said to have rotational motion about a fixed axis if every particle of the body moves in a circle which lies in a plane perpendicular to the axis and has its centre on the axis.

4. ***What you mean precession of a spinning top?***

   The movement of the axis of the top around the vertical is called precession of the spinning top.

5. ***What is centre of mass of a system of particles?***

   Centre of mass of a system of particles is the point where the entire mass of the system can be assumed to be concentrated.

6. ***What is the location of the centre of mass of a lamina of triangular shape.***

   At the centroid of the triangle.

7. ***Give the location of centre of mass of sphere of uniform mass density?***

   At the geometric centre.

8. ***Give the location of centre of mass of cylinder of uniform mass density.***

   At the centre of its axis of symmetry

9. ***Give the location of centre of mass of ring of uniform mass density?***

   At the centre of ring.

***10. Give the location of centre mass of a cube of uniform mass density?***

At its geometrical centre.

***11. Does the centre of mass of a body necessarily lie inside the body?***

No, (it may lie outside the body also)

***12. Give the expression for moment of inertia about an axis passing through the centre perpendicular to its plane.***

$I=Mr^2$

***13. Give an example for a body whose centre of mass lies inside the body.***

Solid sphere or solid cube.

***14. Give an example for a body whose centre of mass lies outside the body***

Ring

***15. Give the expression for moment of Inertia of a thin rod about an axis perpendicular to the rod and passing through its mid point.***

$$I = \frac{ML^2}{12}$$

***16. Write the expression for the moment of inertia of a circular disc of radius R about an axis perpendicular to it and passing through its centre.***

$$I = \frac{MR^2}{2}$$

***17. Write the expression for the moment of inertia of a circular disc of radius R about its diameter.***

$$I = \frac{MR^2}{4}$$

***18. Give the expression for moment inertia of a hollow cylinder of radius R about its axis.***

$$I = MR^2$$

***19. Give the expression for moment of inertia of a solid cylinder of radius R about its axis.***

$$I = \frac{MR^2}{2}$$

***20. Give and expression for the moment of inertia of a solid sphere of radius R about its diameter.***

$$\mathrm{I} = \frac{2\mathrm{MR}^2}{5}$$

***21. Define linear momentum of a system of particles.***

Total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

***22. What is the total external force on a system of particles when its total momentum is constant.***

Zero

***23. What will be the nature of motion of centre of mass of a system when total external force acting on the system is zero.***

Moves uniformly in a straight line.

***24. A moving Radium nucleus decays into Radon and an α particle. The two particles produced during decay move in different directions. What is the direction of motion of the centre of mass after decay?***

The centre of mass moves along the original path.

***25. Mention any one rule to find the direction of vector product of two vectors.***

Rule of Right handed screw or Rule of the right hand.

***26. What is the vector product of two parallel vectors?***

Zero.

***27. What is the angle between a × b and b × a ?***

$180^0$

***28. Write the relation between linear velocity and angular velocity?***

$v = \omega r$

***29. Write the S.I. unit of angular velocity***

$Rad\ S^{-1}$

***30. Write the dimension of angular velocity***

$\left[M^0\ L^0\ T^{-1}\right]$

***31. Define angular acceleration***

The time rate of change of angular velocity $\left(\alpha = \frac{d\omega}{dt}\right)$.

***32. Is moment of force a vector or a scalar?***

Vector.

***33. What is the mechanical advantage of a lever.***

Using small effort one can lift large load.

***34. What is meant by a mechanical advantage of a lever?***

The Mechanical advantage of a lever is the ratio of load to the effort.

***35. Write the expression for the torque in terms of position vector and force.***

$\tau = r \times F$

***36. What is the S.I. unit of Torque.***

$Nm$

***37. Write the dimensions of torque***

$M^1 L^2\, T^{-2}$

***38. Write the expression for the angular momentum in terms of linear momentum and position vector.***

$l = r \times p$

***39. Write the expression for angular momentum in terms of moment of inertia and angular velocity***

$L = I\ \omega$

***40. Define angular momentum.***

Moment of momentum $\left(L = \sum_{i=1}^{n} r_i \times p_i\right)$

***41. The time rate of change of angular momentum of a particle is equal to the torque acting on it. Is it true or false?***

TRUE

***42. What is the torque acting on a system when total angular momentum of a system is constant.***

ZERO

***43. Define a couple.***

A pair of equal and opposite forces with different lines of action is known as a couple.

***44. Define moment of a couple.***

The moment of a couple is equal to the sum of the moments of the two forces making the couple.

***45. The mechanical advantage of a lever is greater than one what does it mean?***

A small effort is enough to lift a large load.

***46. Write the expression for moment of inertia.***

$\left(I = \sum_{i=1}^{n} m_i r_i^2 \right)$

***47. Is moment inertia a vector or a scalar?***

Scalar

***48. Write the SI unit of moment of inertia.***

$Kgm^2$

***49. Give the dimensions of moment of inertia.***

$M^1L^2T^0$

***50. Define the term Radius of gyration***

The radius of gyration of a body about an axis may be defined as the distance from the axis of a mass point where mass is equal to the moss of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

***51. How is Torque related to the angular momentum.***

$\tau = \frac{dL}{dt}$ or Torque is proportional to the time rate of change of angular momentum.

***52. What is the magnitude of torque acting in a body rotating with a constant angular momentum.***

ZERO

***53. Name the physical quantity which is equal to the time rate of change of angular momentum.***

Torque.

***54. Does the moment of inertia of a thin rod change with change of the axis of rotation?***

YES

***55. Three bodies, a ring, a solid cylinder and a solid sphere roll down the same in lined plane without slipping. They start from rest. The radius of the bodies are identical. Which body has the greatest rotational kinetic energy while reaching the bottom of the inclined plane?***

Sphere has the greatest rotational kinetic energy while reaching the bottom of the inclined plane.

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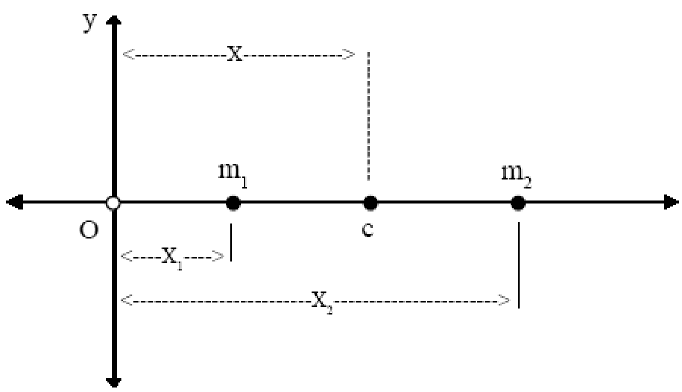
## **Two marks Questions**

1. Two particles of equal mass are at a distances of $X_1$ and $X_2$ from the origin of a co-ordinate system. Find the distance of their centre mass from the origin.




Let 'C' be the centre of mass of the system which is at a distance X from the origin O.

We have $X = \frac{M_1X_1 + M_2X_2}{M_1 + M_2}$

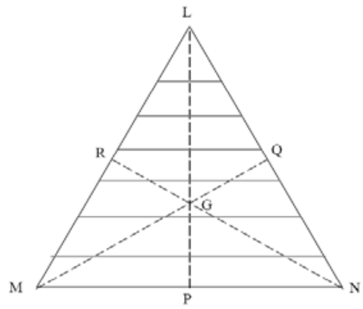
Since the two particles have the same mass $m_1 = m_2 = m$

$\therefore \; X = \frac{mx_1 + mx_2}{\text{m+m}} = \frac{x_1 + x_2}{2}$

Thus for two particles of equal mass the centre of mass lies exactly midway between them.

2. ***How do you find the centre of mass of a triangular lamina.***

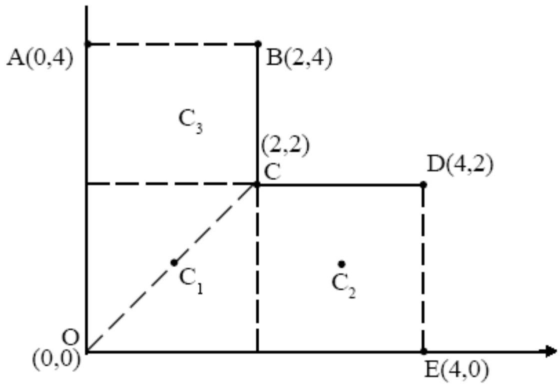
Subdivide the lamina (LMN) into narrow strips each parallel to the base MN as shown in the figure.




By symmetry each strip has its centre of mass at mid point. Join the midpoint of all the strips, we get a median LP. Therefore the centre of mass of the triangle as a whole must lie on the median LP. Similarly it must lie on the median MQ and NR. This means that the centre of mass lies on the point of concurrence of the median, i.e. on the centroid G of the triangle. Thus centroid of the triangle itself is the centre of mass of the triangular lamina.

3. ***Find the centre of mass of a L- shaped uniform lamina of mass 3 kg?***

We can think of the L – shape to consist of 3 squares each of length 2m. The mass of each square is 1kg, since the lamina is uniform. The centre of mass c, $c_1$ and $c_3$ of the squares are by symmetry, Their geometric centers and have coordinates (1,1), (3,1), 1,3) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of these points.




Hence,

$$x = \frac{[1(1) + 1\ (3) + 1\ (1)]kgm}{1+1+1} = \frac{5}{3}\ m\ =\ 1.66m$$

$$y = \frac{[1(1) + 1\ (1) + 1\ (3)]kgm}{1+1+1} = \frac{5}{3}\ m\ =\ 1.66m$$

Thus centre of mass of the L – Shape lies on the line OC.

4. ***Write the expression for the position vector of the centre of mass of a system consisting of three objects in terms of their masses and position vectors.***

$$\text{R} = \frac{\sum m_i r_i}{M}.$$

$$\therefore R = \frac{m_1 r_1 + m_2 r_1 + m_3 r_3}{m_1 + m_2 + m_3}$$

5. ***Name two examples for vector product.***
    a. ***Moment of a force***
    b. ***Angular momentum.***
6. ***Define vector product of two vectors.***
    a. ***A vector product of two vector a and b is a vector C such that,***

magnitude of $C$=$c\ =\ absin\theta$. Where a and b are magnitudes of a & b and $\theta$ is the angle between the two vectors.

C is perpendicular to the plane containing a and b.

7. ***Vector product is not commutative why?***
    a. The magnitude of both $a \times b$ and $b \times a$ is the same ($absin\theta$.); also, both of them are perpendicular to the plane of a and b. But the rotation of the right handed screw in case of $a \times b$ is from $a$ to $b$, Where in case of $b \times a$ it is from b to a.
    b. This means the two vectors are in opposite directions.
    **c.** We have $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
8. $\boldsymbol{a} \times \boldsymbol{b}$ does not change sign under reflection. Explain why?
    a. Under reflection we have
    b. $x \to -x$, $y \to -y$, $z \to -z$
    c. As a result all the components of a vector changes sign and then, $a \to -a$, $b \to -b$

    $\therefore a \times b \to (-a) \times (-b) = a \times b$

    Thus, $a \times b$ does not change sign under reflection.

9. Distinguish between vector product and scalar product of two vectors

| <u>*Vector product*</u> | <u>*Scalar product*</u> |
|---|---|
| Vector product of two vectors is a vector | Scalar product of 2 vector is a scalar. |
| It is not commutative | It is commutative |
| eg: angular momentum | eg: work |

10. ***Find the scalar product and vector product of two vectors***

    $a = (3\hat{i} - 4\hat{j} + 5\hat{k})$ and $b = (-2\hat{i} + \hat{j} - 3\hat{k})$

    $a.b = (3\hat{i} - 4\hat{j} + 5\hat{k}).(-2\hat{i} + \hat{j} - 3\hat{k}) = -6 - 4 - 15 = -25$

$$a \times b = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{bmatrix} = (-7\hat{i} - \hat{j} - 5\,\hat{k})$$

***11. A particle is moving in a circular path with a uniform speed, What is the direction of (i)*** $\omega$ and (ii) $V$ ?

i. $\omega$ is directed along the fixed axis of rotation.

ii. $V$ is perpendicular to both $\omega$ and $r$ and is directed along the tangent to the circle described by the particle.

**12.** ***Define torque. Is it a vector or a scalar?***

The moment of a force or torque acting on the particle with respect to the origin is defined on the vector product of position vector and the force acting on the particle $\tau = r \times F$

It is a vector.

**13.** ***Write the dimensions and SI unit of torque.***

a. Torque has dimension $ML^2T^{-2}$

b. Its SI unit is newtonmetre *(Nm)*

**14.** ***Define angular momentum. Write the expression for it.***

The angular momentum $\boldsymbol{l}$ of the particle with respect to the origin O is the vector product of position vector and the linear momentum of the rotating particle.

$\boldsymbol{l = r \times p}$

15. ***Is angular momentum a scalar or a vector ? Write the dimension of*** $\boldsymbol{l}$**.**

Angular momentum is a vector.

It has dimensions $M^1L^2T^{-1}$

**16.** ***Show that the total gravitational torque about the centre of gravity of the body is Zero.***

We have $\tau = r \times F$

Since position vector (r) = 0

(gravitational force acts at the centre of gravity of the body r = 0 )

$\therefore \tau = o$ (total gravitational torque about the C.G is zero)

**17.** ***Mention the two factors on which torque of a rotating body depends.***

a. *Magnitude of the force.*

b. *Perpendicular distance of the point of application of the force from the origin or axis of rotation.*

**18.** ***Write the relation between angular momentum and torque. What is the torque acting on a body rotating with constant angular momentum.?***

a. $\frac{dL}{dt} = \tau_{ext}$

b. Zero

**19. Find the torque of a force $7\hat{i} + 3\hat{j} - 5\hat{k}$ about the origin. The force acts on a particle whose position vector is $\hat{i} - \hat{j} + \hat{k}$.**

Here $r = \hat{i} - \hat{j} + \hat{k}$ and $F = 7\hat{i} + 3\hat{j} - 5\hat{k}$

We shall use the determinant rule to find the torque

$\tau = r \times F$

$$\tau = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{bmatrix} = (5-3)\,\hat{i} - (-5-7)\,\hat{j} + [3-(-7)]\,\hat{k}$$

$= 2\hat{i} + 12\hat{j} + 10\hat{k}$

**20.** ***State and explain the principle of conservation of angular momentum.***

We know that the time rate of the total angular momentum of as system of particles about a point is equal to the sum of the external torques acting on the system taken about the same point.

i.e: $\frac{dL}{dt} = \tau_{ext}$

*if* $\tau_{ext} = 0 = \frac{dL}{dt} = 0$

Or L = constant

Thus if the total external torque on a system of particles is zero, the total angular momentum of the system is conserved. i.e. remains constant.

**21.** Give the general conditions of equilibrium of a rigid body.

A rigid body is said to be in mechanical equilibrium if both its linear momentum and angular momentum are not changing with time or equivalently, the body has neither linear acceleration nor angular acceleration This means

The total force. i.e: the vector sum of the force on the rigid body is zero

a. $F_1 + F_2 + F_3 + \ldots + F_n = \sum_{i=1}^{n} F_i = 0$ (Condition for translational equilibrium.)

(2) The total torque i.e., the vector sum of the torques on the rigid body is Zero

$\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n = \sum_{i=1}^{n} \tau_i = 0$ (¨condition for rotational equilibrium)

22. Write the expression for work done by a torque and explain the terms.

$dw = \tau d\theta$ $d\theta$ is the angular displacement of the particle, $\tau$ is the external torque.

23. What are the factors on which the moment of inertia of a body depend?

a. The moment of inertia of a rigid body depends on the mass of the body, its shape and size; distribution of mass about the axis of rotation and the position and orientation of the axis of rotation.

24. Why a fly wheel is used in a engine of a train (Vehicle)?

i. A fly wheel has large moment of inertia. Because of its large moment of inertia, it resist the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for passengers on the vehicle.

## Three marks Questions

1. Write three Kinematic equations of rotational motion of a body with a uniform angular acceleration and explain the terms.

$\omega = \omega_0 + \propto t$

$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \propto t^2$

$\omega^2 = \omega_0^2 + 2 \propto (\theta - \theta_0)$

$\omega_0$ is the initial angular velocity, $\omega_0$ −angular velocity after `t' seconds, ∝- angular acceleration, $\theta_0$ - initial angular displacement, $\theta$ - angular displacement in 't' seconds.

2. The angular speed of a motor wheel is increased from 1200rpm to 3120 rpm in 16 seconds. What is its angular acceleration, assuming the acceleration to be uniform.?

We shall use $\omega = \omega_0 + \propto t$

$\omega_0$ initial angular speed in $rad/s$

$= 2\pi \times$ angular speed in $rev/s$

$= \frac{2\pi \times \text{angular speed in rev/min}}{\text{i.}60 s/min}$

$= \frac{2\pi \times 1200}{60} = rad/s = 40\pi \, rad/s$

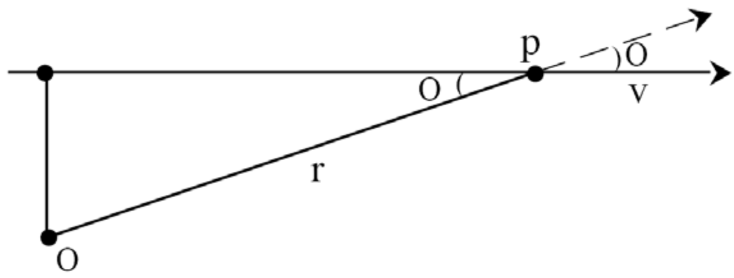
Similarly $\omega =$ final angular speed in $rad/s$

$= \frac{2\pi \times 3120}{60} \, rad/s = 2\pi \times 52 rad/s$

ii.$= 104\pi \, rad/s$

∴ angular acceleration $\propto = \frac{\omega - \omega_0}{t} = \frac{104\pi - 40\pi}{16} = \frac{64\pi}{16} = 4\pi \, rad/s^2$

3. ***Show the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.***




Consider a particle of mass m moving with velocity V at P.

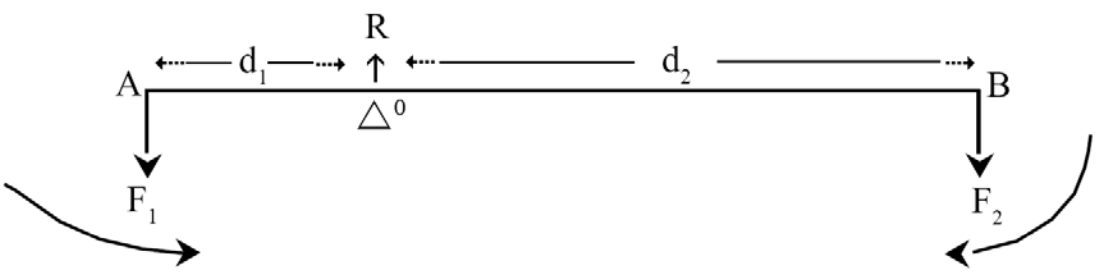
O be an arbitrary point about which angular momentum has to be calculated.

We have angular momentum $l = r \times mv$

Its magnitude is $mvr\ sin\theta$ where $\theta$ is the angle between $r$ & $v$. Although the particle changes position with time, the line of direction of $\boldsymbol{v}$ remains the same and hence $OM = rsin\theta$ is a constant.

Further, the direction of $\boldsymbol{l}$ is perpendicular to the plane of $\boldsymbol{r}$ and $\boldsymbol{v}$. This direction does not change with time. Thus, $\boldsymbol{l}$ remains the same in magnitude and direction and is therefore conserved.

4. ***Explain the principles of moments for a lever.***




Consider an ideal lever as shown in the figure.

Two forces $F_1$ and $F_2$ , parallel to each other and usually perpendicular to the lever, as shown in the figure, act on the lever at distances. $d_1$ and $d_2$ respectively from the fulcrum.

If R is the reaction of the support at fulcrum

There for translational equilibrium $R-F_1-F_2=0$ and for rotational equilibrium the sum of the moments $d_1F_1-d_2F_2 = 0$ …….. (1)

In the case of the lever force $F_1$ is usually some weight to be lifted. It is called the load and its distance from the fulcrum $d_1$ is called the load arm. Force $F_2$ is the effort applied to lift the load. distance $d_2$ of the effort arm from the fulcrum is the effort arm.

From equation (1) : $d_1F_1 = d_2 F_2$

Or

Load arm × load = effort arm × effort

This is the principle of moments for a lever.

The ratio $\frac{F_1}{F_2}$ is called the mechanical advantage.

M.A. $= \frac{F_1}{F_2} = \frac{D_2}{D_1}$

If the effort arm $d_2$ is larger than the load arm $d_1$, the mechanical advantage is greater than one.

i.e.: a small effort can be used to lift a large load.

5. Starting from the definition of moment of inertia obtain an expression for moment of inertia of a thin ring.

Consider a thin ring of radius R and Mass M, rotating in its own plane around its centre with angular velocity ω

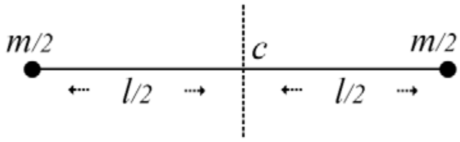
Each mass element of the ring is at a distance R form the axis and moves with a speed R ω, The Kinetic energy is therefore

$k = \frac{1}{2}Mv^2 = \frac{1}{2}MR^2\omega^2$ ……… (1)

But $k = I\omega^2$ ………(2)

Comparing equation (1) and (2) we get $I = \frac{1}{2}MR^2$

6. Obtain an expression for M.I. of a rotating pair of small masses attached to the two ends of a rigid mass less rod of length $l$ rotating about and axis through the centre of mass perpendicular to the rod.




From the figure each mass $\frac{m}{2}$ is at distance $\frac{l}{2}$ from the axis. The momentum is therefore $\left(\frac{m}{2}\right)\left(\frac{l}{2}\right)^2+\left(\frac{m}{2}\right)\left(\frac{l}{2}\right)^2$

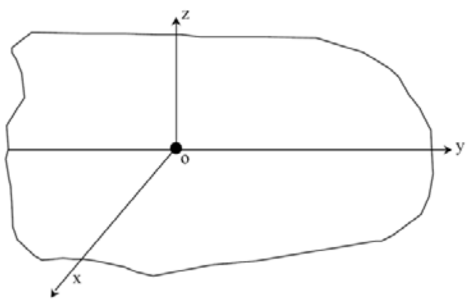
Therefore for the pair of masses, rotating about the axis through the centre of mass perpendicular to the rod $I = \frac{ml^2}{4}$

## Four marks questions

1. State and explain

perpendicular axis theorem and parallel axis theorem.

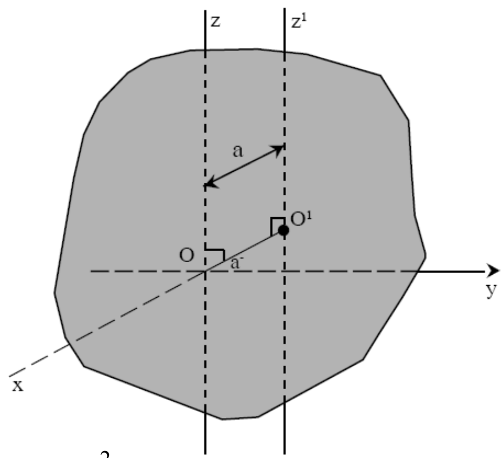
a. Perpendicular axis theorem : It states that the moment of inertia of a planer body (lamina) about an axis perpendicular to its plane is equal to the sum of its moment of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.




The figure shows a planar body An axis perpendicular to the body through a point O is taken as the Z axis. Two mutually perpendicular axis lying in the plane of the body and concurrent with Z axis. i.e., passing through O, are taken as the $\boldsymbol{x}$ and $\boldsymbol{y}$ axes.

The theorem states that $I_z = I_x + I_Y$.

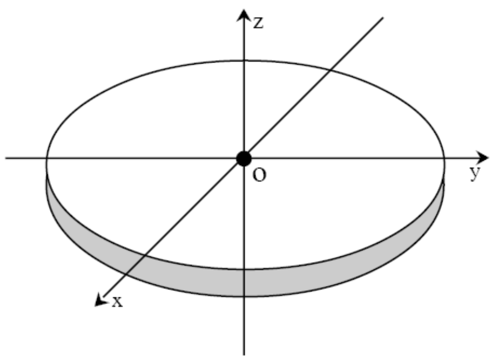
b. Theorem of parallel axis : The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.




Z and $Z^1$ are two parallel axes separated by a distance $\boldsymbol{a}$. The $\boldsymbol{z}$ axis passes through the centre of mass $\boldsymbol{O}$ of the rigid body. Then according to the theorem of parallel axis $I_z^1 = I_Z + Ma^2$

Where $I_Z$ and $I_z^1$ are the moments of inertia of the body about the Z and Z axes respectively, M is the total mass of the body and $\boldsymbol{a}$ is the perpendicular distance between the two parallel axes.

2. Using perpendicular axis theorem obtain the expression for moment of inertia of a disc about its diameter. Assume the expression for moment of inertia about a perpendicular axis passing through the centre.




Moment of inertia of the disc about an axis perpendicular to it and through its centre $I = \frac{MR^2}{2}$

Where M is the mass of the disc and R is the radius. The disc can be considered to be a planar body. Hence the theorem of perpendicular axis is applicable to it.

Consider three concurrent axis through the centre of the disc, O as the $\boldsymbol{x, y, z}$ axes.

$\boldsymbol{x}$ and $\boldsymbol{y}$ axes lie in the plane of the disc and z is perpendicular to it.

By the theorem of perpendicular axis

$I_z = I_X + I_y$

Here ***x*** and ***y*** axes are along two diameters of the disc and by symmetry the moment of inertia of the disc is same about any diameter.

Hence,

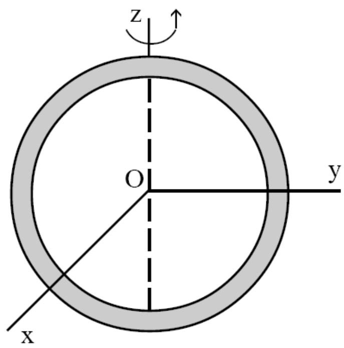
$I_x = I_y$

And $I_z = 2\ I_x$

$$\boldsymbol{I_z = \frac{MR^2}{2}}$$

$$\boldsymbol{I_x = \frac{I_z}{2} = \frac{MR^2}{4}}$$

3. Using perpendicular axis theorem obtain expression for moment of inertia of a ring about its diameter. Assume the expression for MI about a perpendicular axis passing through the centre.




Moment of inertia of a circular ring about an axis perpendicular to it and through its centre

$I = MR^2$

Where M is the mass of the disc and R is its radius.

The ring can be considered to be a planar body.

Hence the theorem of perpendicular axis is applicable to it.

Consider three concurrent axis through the centre of the ring **O** as the ***x, y, z*** axes. ***x*** and ***y*** axis lie in the plane of the disc and z is perpendicular to it.

By the theorem of perpendicular axes $I_Z = I_x + I_y$

Here ***x*** & ***y*** axes are along two diameters of the ring and by symmetry the M.I. of the ring is the same about any diameter.

Hence $I_x = I_y$

and $I_z = 2I_X$

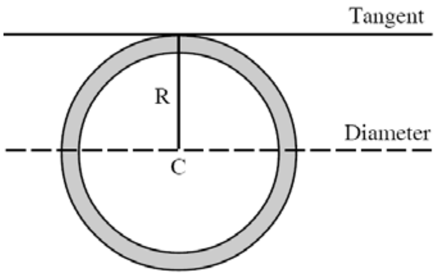
But $I_z = MR^2$

$I_x = \frac{Iz}{2}$

$I_x = \frac{MR^2}{2}$

4. Using parallel axis theorem obtain the expression for M.I. of a ring about a tangent to the circumference of the ring.

   Assume the expression for M.I. of a ring about its diameter.




The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring.

The distance between these two parallel axes is R, the radius of the ring.

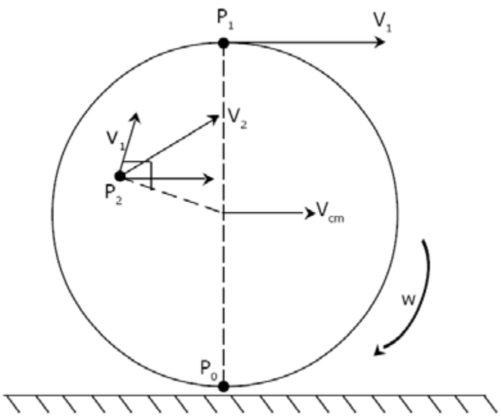
Using the parallel axis theorem

$$I_{tangent} = I_{diameter} + MR^2 = \frac{MR^2}{2} + MR^2$$

a. $= \frac{3}{2} MR^2$

5. Derive an expression for the kinetic energy of a rolling body.

   Consider a body of mass M rolling with a velocity $v.$ The total kinetic energy of a rolling body is the sum of the Kinetic energy of the body due to the motion of the centre of mass $\left(\frac{Mv^2}{2}\right)$ and Kinetic energy of rotational motion about the centre of mass of the system of particles ($K^1$)




Thus $K = K^1 + \frac{Mv^2}{2}$ ….. (1)

Here the kinetic energy of the centre

of mass, i.e., the K.E. of translation of the rolling body is $\frac{\boldsymbol{Mv^2}\text{cm}}{\boldsymbol{2}}$, where m is the mass of the body and $\boldsymbol{v}$ cm is the Velocity of the centre of mass.

Since the motion of the rolling body about the centre of mass is rotation, $K^1$ is the kinetic energy of rotation of the body;

$\boldsymbol{K^1} = \frac{\boldsymbol{I\omega^2}}{\boldsymbol{2}}$ , Where I is the moment of inertia about the appropriate axis, which is the symmetry axis of the rolling body.

∴ the kinetic energy of a rolling body $\boldsymbol{K = \frac{1}{2}\, I\omega^2 + \frac{1}{2}\, mv\,^2_{cm}}$

Substituting $I = mK^2$ Where K is the corresponding radius of gyration of the body and

$\boldsymbol{v_{cm} = \omega R}$

We get

$$\therefore \boldsymbol{K = \frac{mk^2\, v^2_{cm}}{R^2} + \frac{1}{2}\, mv\,^2_{cm}}$$

$$\therefore \boldsymbol{K = \frac{1}{2}\, mv\,^2_{cm}\, [1 + \frac{k^2}{R^2}]}$$

6. State and explain the principle of conservation of angular momentum in case of (i) swivel chair (ii) an acrobat.

In the absence of the external torque, the total angular momentum of a body rotating about a fixed axis remains constant.

If the external torque is Zero, $I_Z = \boldsymbol{I\omega}$ = constant.

Swivel chair: Sit on a swivel chair with arms folded and feet not resting on. i.e., away from the ground. Now rotate the chair rapidly. While the chair is rotating with considerable angular speed stretch the arms horizontally. The angular speed reduces now. If the arms are folded back, the angular speed increases again. This is because of the law of conservation of angular momentum. If the friction of the rotational mechanism is neglected, there is no external torque about the axis of rotation of the chair and hence $\boldsymbol{I\omega}$ is constant. Stretching the arms increases I about the axis of rotation, resulting decreasing the angular speed $\boldsymbol{\omega}$. Bringing the arms closer to body has the opposite effect.

An acrobat make use of this principle during the course of his performance. Sometimes he stretches out his hands and legs to increase the M.I. of the body and to decrease the angular speed. On the other hand to increase the angular speed he brings hands and legs near his body, the moment of inertia decreases. i.e., principle of angular momentum can be used to perform somer-saults in air by an acrobat.

7. Three equal masses are kept at $\boldsymbol{P}\,(x_1\, y_1)$, $\boldsymbol{Q}\,(x_2\, y_2)$ and $\boldsymbol{R}\,(x_3\, y_3)$ in a co-ordinate system. Show that, their centre of mass coincides with the centroid of the triangle PQR.

    a. Let the masses of the three particles be $m_1$, $m_2$ and $m_3$ respectively, the centre of mass C of the system of the three particles is defined and located by the co-ordinates (x, y) given by

$$X = \frac{m_1\, x_1 + m_2\, x_2 + m_3\, x_3}{m_1 + m_2 + m_3}$$

$$Y = \frac{m_1\, y_1 + m_2\, y_2 + m_3\, y_3}{m_1 + m_2 + m_3}$$

For the particles of equal mass $m_1 = m_2 = m_3 = m$

$$x = \frac{m(x_1 + x_2 + x_3)}{3m} = \frac{x_1 + x_2 + x_3}{3}$$

$$y = \frac{m(y_1 + y_2 + y_3)}{3m} = \frac{y_1 + y_2 + y_3}{3}$$

Thus, for three particle of equal mass, the centre of mass coincides with the centroid of the triangle formed by the particle.

8. Show that torque is equal to rate of change of angular momentum of a particle.

We have $\boldsymbol{l} = \boldsymbol{r} \times p$

Where r – position vector, p – momentum.

Differentiating the above equation

$\frac{\boldsymbol{dl}}{\boldsymbol{dt}} = \frac{\boldsymbol{d}}{\boldsymbol{dt}} (\boldsymbol{r}\times\boldsymbol{p})$

Using product rule, on R.H. S.

$\frac{\boldsymbol{d}}{\boldsymbol{dt}} (\boldsymbol{r}\times\boldsymbol{p}) = \frac{\boldsymbol{dr}}{\boldsymbol{dt}} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\boldsymbol{dp}}{\boldsymbol{dt}}$

Now the velocity of the particle is $\boldsymbol{v} = \frac{\boldsymbol{dr}}{\boldsymbol{dt}}$ and $\boldsymbol{p} = \boldsymbol{mv}$

$\therefore, \frac{\boldsymbol{dr}}{\boldsymbol{dt}} \times \mathrm{p} = \boldsymbol{v} \times \boldsymbol{mv} = \mathbf{0}$

$since \frac{dp}{dt} = F$

$r \times \frac{\boldsymbol{dp}}{\boldsymbol{dt}} = r \times \mathrm{F} = \tau$

$\therefore \frac{\boldsymbol{dl}}{\boldsymbol{dt}} = \tau$

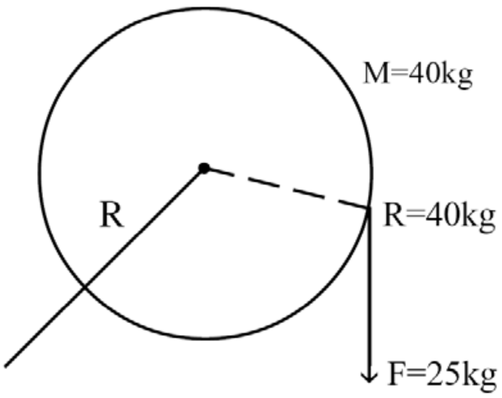
## Five marks questions

1. A cord of negligible mass is wound round the rim of a flywheel of mass 40 kg and radius 40 cm. A steady pull of 25 N is applied on the cord as shown in the figure. The flywheel is mounted on a horizontal axle with frictionless bearings.

   Compute the angular acceleration of the wheel.

   Find work done by the pull, when two metre of cord is unwound.

   Find also the kinetic energy of the wheel at this point assume that the wheel starts from rest.




   a. We have I∝= τ

      the torque τ = FR

      $= \mathbf{25} \times \mathbf{0.40}\boldsymbol{Nm}$

      $= \mathbf{10}\boldsymbol{N}$

I = M.I. of fly wheel about its axis $\frac{MR^2}{2}$

$= \frac{40 \times (0.40)^2}{2} = \mathbf{20 \times 0.16 = 3.2kgm^2}$

$\propto = angular\ acceleration = \frac{\mathbf{10Nm}}{3.2\ \mathbf{kgm^2}} = \mathbf{3.125S^{-2}}$

b) work done by the pull unwinding 2m of the cord

= 25 X 2m = 50j

c) Let $'\boldsymbol{\omega}'$ be the final angular velocity. The kinetic energy gained

$$K = \frac{1}{2} I\boldsymbol{\omega}^2$$

Since the wheel starts from rest, $\boldsymbol{\omega}^2 = \boldsymbol{\omega}_0^2 + 2\propto \theta$

$$\boldsymbol{\omega_0 = 0}$$

The angular displacement $\theta = \frac{length\ of\ un\ wound\ string}{radius\ of\ wheel}$

$$= \frac{\mathbf{2m}}{\mathbf{0.4}} = \mathbf{5\ rad}$$

$$\therefore \boldsymbol{\omega}^2 = \mathbf{2 \times 3.125 \times 5 = 31.25}\ (rad/s)^2$$

$$\therefore \text{K.E gained} = \frac{\mathbf{1}}{\mathbf{2}} \times \mathbf{3.2 \times 31.25} \quad (\boldsymbol{K} = \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{I\omega}^2)$$

$$= \mathbf{1.6 \times 31.25}$$

= 50 J

2. Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They starts from rest. The radii of the bodies are identical which of the bodies reaches the ground with maximum velocity?

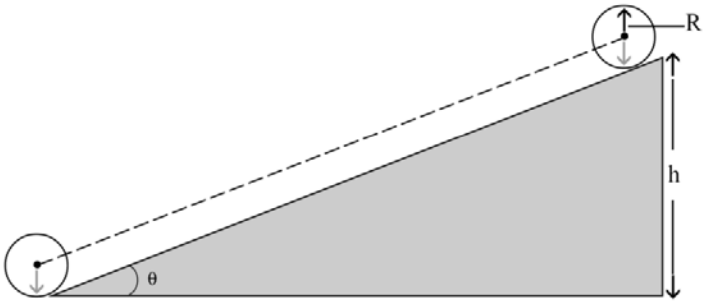
   Since energy of a rolling body is conserved potential energy lost by the body in rolling down the inclined plane (mgh) Must be equal to K.E. gained.

   Since the bodies start from rest the K.E. gained = the final K.I. of the bodies.

   We have K.E. of a rolling body $\boldsymbol{K} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{mv^2}\left[\mathbf{1} + \frac{\boldsymbol{k^2}}{\boldsymbol{R^2}}\right]$ ………….(1)

Where $\boldsymbol{v}$ is the final velocity of the body.

Equating the equation (1) With the potential energy lost weight




$$mgh = \frac{1}{2}mv^2\left[1+\frac{k^2}{R^2}\right]$$

or $v^2 = \frac{2gh}{\left[1+\frac{k^2}{R^2}\right]}$

for a ring $K^2 = R^2$

$v_{\text{ring}} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$

for a solid cylinder $k^2 = \frac{R^2}{2}$

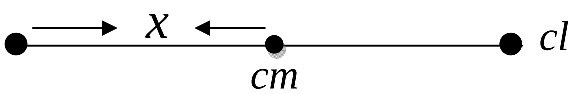
$\therefore v_{\text{disc}} = \sqrt{\frac{2gh}{1+\frac{1}{2}}} = \sqrt{\frac{4gh}{3}}$

For a solid sphere $k^2 = \frac{2R^2}{5}$

$\therefore v_{\text{sphere}} = \sqrt{\frac{2gh}{1+\frac{2}{5}}} = \sqrt{\frac{10gh}{7}}$

From the result obtained it is clear that among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

3. In the $\boldsymbol{Hcl}$ molecule, the separation between the nuclei of the two atoms is about $\mathbf{1.27}\boldsymbol{A}^0$ ($\boldsymbol{IA}^0$=$10^{-10}\boldsymbol{m}$) Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of and atom is concentrated in its nucleus.



Let us consider hydrogen nucleus as the origin for measuring distance. If `m' is the mass of the hydrogen atom, then mass of the chlorine atom = **35.5m**

Distance of the centre of moss of $\boldsymbol{Hcl}$ molecule from the origin is given by

$$x = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

Here $x_1 = 0$, $x_2 = 1.27 \times 10^{-10}\, metre$

$$\therefore x = \frac{m\times o+35.5\text{m} \times 1.27\times10^{-10}}{m+35.5m}$$

$$\therefore x = \frac{35.5 \times 1.27\times10^{-10}}{36.5}$$

$$= 1.23 \times 10^{-10m} = 1.235A^0$$

4. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greatest angular acceleration after a given time?

Moment of inertia of a cylinder about its axis of symmetry $\boldsymbol{I_c = MR^2}$

Moment of inertia of a sphere about its diameter $\boldsymbol{I_s = \frac{2}{5}MR^2}$

Angular acceleration $\propto = \frac{\tau}{I}$

$$\propto_c = \frac{\tau}{I_c} = \frac{\tau}{MR^2}$$

also $\propto_s = \frac{\tau}{I_s} = \frac{\tau}{\frac{2}{5}MR^2} = \mathbf{2.5}\frac{\tau}{MR^2} = \mathbf{2.5} \propto_c$

the sphere will acquire a greatest angular acceleration.

5. A solid cylinder of mass 20 kg rotates about its axis with angular speed $\mathbf{100}\boldsymbol{radS^1}$ The radius of the cylinder is 0.25m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Sol. (i) Kinetic energy of rotation $\boldsymbol{K = \frac{1}{2}I\omega^2}$

Here, $\boldsymbol{I} = \frac{1}{2}\boldsymbol{MR^2}$

$\therefore \boldsymbol{I} = \frac{20}{2} \times \ (\mathbf{0.25})^2$ = 0.625kgm$^2$

and $\quad \boldsymbol{\omega} = \mathbf{100} \boldsymbol{radS^{-1}}$

$\therefore$ Kinetic energy of rotation = $\frac{1}{2} \times \ \mathbf{0.625} \times \mathbf{100^2}$ = 3125 j

Also angular momentum $\boldsymbol{L} = \boldsymbol{I\omega}$

$$= (\mathbf{0.625}) \times \mathbf{100} = \mathbf{62.5}\boldsymbol{kgm^2S^{-1}}$$

6. (a) A child stands at the centre of a turn table with his arm outstretched. The turn table is set rotating with an angular speed of 40 rev/min. how much is the angular speed of the child, if he folds his hand back and thereby reduces his moment of inertia to $\frac{2}{5}$ times the initial value? Assume that the turntable rotates without friction.

(b) Show that the child's new K.E. of rotation is more than the initial K.E. of rotation. How do you account for this increase in Kinetic. Energy?

Given (a) $\boldsymbol{I_{final}} = \frac{2}{5}$ Initial, $\boldsymbol{\omega_i} = \ \mathbf{40}\ \boldsymbol{rev\ min^{-1}}$ using the principle of conservation of angular momentum

We get $\boldsymbol{I_i\omega_i} = \boldsymbol{I_F\omega_F}$

Or $\boldsymbol{\omega_F} = \frac{I_i\omega_i}{I_F} = \frac{I_i \times 40}{\frac{2}{5}I_i} = \mathbf{100}\ \boldsymbol{rev\ min^{-1}}$

(b) $\dfrac{Final\ K.E.of\ rotation}{initia\ K.E.of\ rotation} = \dfrac{\frac{1}{2}\,\mathrm{I_F}\boldsymbol{\omega_F^2}}{\frac{1}{2}\,\mathrm{I_i}\boldsymbol{\omega_i^2}}$

$= \left(\frac{I_F}{\mathrm{I_i}}\right)\left(\frac{\omega_F}{\omega_i}\right)^2 = \frac{2}{5} \times \left(\frac{100}{40}\right)^2 = \frac{5}{2} = 2.5$

$\therefore Final\ kinetic = \ \mathbf{2.5} \times$ initial K.E.

Final K.E. is more than initial K.E. because the child uses his internal energy when the folds his hands.

7. A rope of negligible mass is wound round a hollow cylinder of mass 3kg and radius 40cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30N? What is the linear acceleration of the rope? Assume that there is no slipping.

   Given: $\boldsymbol{M = 3\,kg, R = 40\,cm = 0.4m}$

   MI of the hollow cylinder about its axis $\boldsymbol{I = MR^2 = 3 \times (0.4)^2 = 0.48\,kgm^2}$

   When the force of $\boldsymbol{30N}$ is applied over the rope wound round the cylinder, the torque will act on the cylinder.

   $\boldsymbol{\tau = R \times F = 30 \times 0.4 = 12\,Nm}$

   If $\propto$ be the angular acceleration produced, then $\boldsymbol{\tau = I \propto}$

   OR $\propto = \frac{\tau}{I} = \frac{12}{0.48} = \boldsymbol{25radS^{-2}}$

8. To maintain a rotor at a uniform angular speed of $\boldsymbol{200radS^{-1}}$ ,an engine needs to transmit a torque of $\boldsymbol{180Nm}$. What is the power required by the engine?

   Given :

   $\tau = 180\text{Nm}$, $\omega = \boldsymbol{200radS^{-1}}$

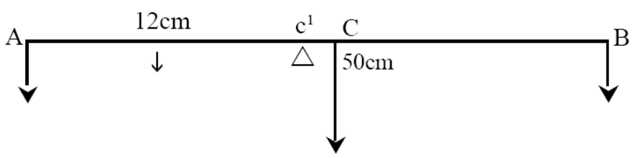
   Power $p = \tau\omega$

   $p = \boldsymbol{180 \times 200}$

   $= \boldsymbol{36000\,Watt}.$

9. A metre stick is balanced on a knife edge at its centre. When two Coins, each of mass 5g are put on the top of the other at 12.0cm mark, the stick is found to be balanced at 45.0cm. What is the mass of the metre stick?

   Let $\boldsymbol{m}$ be the mass of the metre stick. It is concentrated at $\boldsymbol{C}$ the 50 cm mark for equilibrium about $\boldsymbol{C^1}$ , at 45cm mark




$$\boldsymbol{10\,g\,(45 - 12) = mg(50 - 45\;)}$$

$$\boldsymbol{10g \times 33 = mg \times 5}$$

$$m = 10 \times \frac{33}{5} = 2 \times 33 = 66g$$

10. A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. Will it reach the bottom with the same speed in each case? Will it take longer to roll down one plane than the other? If so, which one and why?

The kinetic energy of a rolling body K=$\frac{1}{2}$I$\omega^2$+ ½ $mv^2$

Where $v$ is the velocity of the centre of mass at the bottom of the inclined plane.

According to the principle of conservation of energy

½ $mv^2$+½ $I\omega^2 = mgh$ (P.E. lost by the body)

For a sphere $I = \frac{2}{5}mR^2$ ∴ $\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\omega^2 = mgh$

as $R\omega = v$

$$\frac{1}{2}mv^2 + \frac{1}{5}mv^2 = mgh$$

$$v = \sqrt{\frac{10}{7}gh}$$

Since the inclined planes have the same height, $v$ must be same and also the time.

11. A loop of radius 2m weighs 100 kg. It rolls along a horizontal floor. So that its centre of mass has a speed of $20cm\ /S$. How much work is done to stopit?

Given: $R = 2m, M = 100kg, V = 20cm/s \ = 0.2m/s$

Total energy of the loop $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(MR^2)\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mv^2 \ (R\omega = v)$$

$$= mv^2$$

∴ Work required to stop the loop = total energy of the loop.

$$= 100\ (0.2)^2$$

= 4 J

12. The oxygen molecule has a mass of $\boldsymbol{5.30\times10^{-26}kg}$ and a M.I. of $\boldsymbol{I=1.94\times10^{-46}kgm^2}$, about and axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is $\boldsymbol{500m/s}$ and that its K.E. of rotation is two thirds of its K.E. of translation. Find the average angular velocity of the molecule.

Given : $\boldsymbol{m=5.30\times10^{-26}kg}$

$$\boldsymbol{I=1.94\times10^{-46}kgm^2,\ v=500m/s}$$

An oxygen molecule contains two atoms. If m is the mass of the oxygen molecule then mass of each atom of oxygen $=\frac{m}{2}$

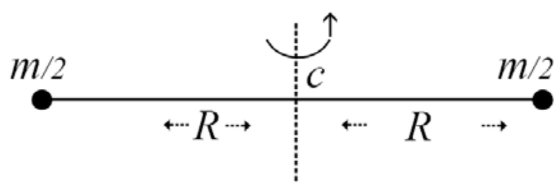
If 2R is distance between the two atoms.

Then moment of inertia of oxygen molecule $\boldsymbol{I=\frac{mR^2}{2}+\frac{m}{2}R^2=mR^2}$

$$\boldsymbol{mR^2=1.94\times10^{-46}}$$

$$\boldsymbol{R^2=\frac{1.94\times10^{-46}}{m}=\frac{1.94\times10^{-46}}{5.30\times10^{-23}}}$$

$$\boldsymbol{R=\sqrt{\frac{1.94\times10^{-24}}{5.3}}=0.61\times10^{-10}m}$$




But K.E of rotation $=\frac{2}{3}$ K.E. of translation

$$\boldsymbol{\frac{1}{2}I\omega^2=\frac{2}{3}\times\frac{1}{2}mv^2}$$

$$\boldsymbol{I=mR^2}$$

$$\boldsymbol{\therefore\frac{1}{2}mR^2\omega^2=\frac{1}{3}mv^2}$$

OR $\boldsymbol{\omega=\sqrt{\frac{2}{3}}\times\frac{v}{R}=\sqrt{\frac{2}{3}}\times\frac{500}{0.61\times10^{-10}}}$

$$\boldsymbol{=6.7\times10^{12}\ rad/S}$$

13. A solid cylinder rolls up an inclined plane at the bottom of the inclined plane, the centre of mass of the cylinder has a speed of $\boldsymbol{5m/s.}$ How far will the cylinder go up the plane?

Given : $\boldsymbol{v=5m/s}$

If `$\boldsymbol{h}$` is the height attained by the cylinder.

Then $\frac{1}{2}\boldsymbol{mv^2} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{I\omega^2} = \boldsymbol{mgh}$

$\frac{1}{2}\boldsymbol{mv^2} + \frac{\mathbf{1}}{\mathbf{2}}(\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{mR^2})\boldsymbol{\omega^2} = \boldsymbol{mgh}$

$\boldsymbol{i.e}$; $\frac{\mathbf{3}}{\mathbf{4}}\boldsymbol{mv^2} = \boldsymbol{mgh}$

$\boldsymbol{h} = \frac{\mathbf{3}\boldsymbol{v^2}}{\mathbf{4}\boldsymbol{g}} = \frac{\mathbf{3\times5^2}}{\mathbf{4\times9.8}} = \mathbf{1.913}\boldsymbol{m}$

14. A man stands on a rotating plat form, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight form the axis changing from 90cm to 20cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to $\mathbf{76}\boldsymbol{kgm^2}$.

   (a). What is his new angular speed? (neglect friction)

   (b) Is kinetic energy conserved in the process? If not, from where does the change come from?

   given $\omega_i = 30\ rpm.$

Ans: (a) Initial = inertia of the man together with the platform + moment inertia of the out stretched weight.

$= 7.6 + 2(MR^2)$

$= 7.6 \times 2 \times 5 \times (0.9)^2$

$= 7.6 \times 10 \times 0.81$

$= 15.7\ kgm^2$

$I_{\text{Final}} = 7.6 + 2 \times (MR^2)$

$= 7.6 \times 2 \times 5 \times (0.2)^2 = 8.0kgm^2$

Using the principle of conservation of angular momentum

$I_1\omega_1 = I_2\omega_2$

$\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{15.7 \times 30}{8} = 58.88rpm$

(b) Kinetic energy is not conserved. As the moment of inertia decreases, the K.E. of rotation increases. This change comes from the work done by the man in bringing his arms close to his body.