SYSTEM OF PARTICLES AND ROTATIONAL MOTION

Important Points:

1. Centre of Mass:

It is the imaginary point at which the total mass of the system is supposed to be concentrated.

2. There need not be any mass at the centre of mass

Ex.: Hollow sphere, ring etc.

3. Internal forces cannot change the position of centre of mass.

4. The algebraic sum of moments of masses of all the particles about the centre of mass is zero.

5. **Centre of Gravity:**

An imaginary point at which the total weight of the system is supposed to be concentrated is called centre of gravity.

6. For small objects Centre of mass and Centre of gravity coincide but for large or extended objects like hills, buildings they do not coincide.

7. If r_1 and r_2 be the distances of the particles of masses m_1 and m_2 from their centre of mass respectively, then

$$m_1 r_1 = m_2 r_2$$

8. Co-Ordinates of Centre of Mass:

Let us consider a system of n particles of masses m_1, m_2, ..., m_n whose co-ordinates are $(x_1, y_1, z_1), (x_2, y_2, z_2)$ ...... (x_n, y_n, z_n), respectively. Then co-ordinates of their centre of mass are

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + ... + m_ny_n}{m_1 + m_2 + ... + m_n}$$

And $z_{cm} = \frac{m_1 z_1 + m_2 z_2 + ... + m_n z_n}{m_1 + m_2 + ... + m_n}$

9. Velocity of Centre of Mass:

$$\vec{V}_{cm} = \frac{m_1 \vec{V}_1 + m_2 \vec{V}_2 + ... + m_n \vec{V}_n}{m_1 + m_2 + ... + m_n}$$

10. Momentum of Centre of Mass:

$$M \vec{V}_{cm} = m_1 \vec{V}_1 + m_2 \vec{V}_2 + ... + m_n \vec{V}_n$$

$$M \vec{V}_{cm} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n$$

11. Acceleration of Centre of Mass:

$$\vec{a}_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + ... + m_n \vec{a}_n}{m_1 + m_2 + ... + m_n}$$

12. Vector or Cross Product:

a) The vector product of two vectors is a vector which is the product of their magnitude and sine of the angle between them.

$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to plane containing $\vec{A} \times \vec{B}$.

b) The direction of cross product of two vectors is always perpendicular to the plane formed by those vectors

c) Vector product does not obey commutative law $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

d) Vector product obeys distributive law

$$\vec{A} \times \left(\vec{B} + \vec{C} \right) = \left(\vec{A} \times \vec{B} \right) + \left(\vec{A} \times \vec{C} \right)$$

e) i x j = j x j = k x k = 0

i x j = k   j x i = -k

j x k = i   k x j = -i

k x i = j   i x k = -j

f) If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$$\text{Then } \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \hat{i}\,(A_2B_3 - A_3B_2) - \hat{j}\,(A_1B_3 - A_3B_1) + \hat{k}\,(A_1B_2 - A_2B_1)$$

13. Moment of Inertia:

a) Moment of inertia of a body about an axis is defined as the sum of the products of the masses and the squares of their distances of different particles from the axis of rotation.

b) $I = m_1r_1^2 + m_2r_2^2 + \ldots\ldots + m_nr_n^2$ or I =

Unit: $kg.m^2$

Dimensional formula: $M L^2 T^0$

c) For a rigid body $I = mk^2$ where K is called radius of gyration.

d) **Radius of Gyration:**

It is the effective distance of all particles of the body from the axis of rotation.

$$K = K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots.. + r_n^2}{n}}$$

e) MI depends on the mass, distribution of mass, the axis of rotation, shape, size and temperature of the body.

f) MI opposes the change in the rotary motion.

14. Moment of Inertia of Different Bodies:

a) **Uniform Rod -** axis passing through its centre and perpendicular to its length

$$I = \frac{M\ell^2}{12}$$

M = mass and l = length

b) **Rectangular plate** axis passing through its center and perpendicular to the plane $I = \left(\frac{Ml^2}{12} + \frac{Mb^2}{12}\right)$

Where M = mass, l = length and b = breadth

c) **Uniform Circular Disc** axis passing through its centre and perpendicular to its plane $I = \frac{MR^2}{2}$

d) **Solid Cylinder -** about its natural axis $I = \frac{MR^2}{2}$

e) **Uniform Circular Ring-** about an axis which is perpendicular to its plane and passing through its centre is $I = MR^2$

15. Theorems of Moment of Inertia

a) Perpendicular Axes Theorem:

Moment of inertia of a plane laminar about an axis perpendicular to its plane passing through a point is equal to the sum of moments of inertia of the lamina about any two mutually perpendicular axes in its plane and passing through same point.

$I_Z = I_x + I_y$.

b)Parallel axes Theorem:

Moment of inertia of a rigid body about any axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of mass and the product of the mass of the body and square of the perpendicular distance between the two axes.

$I_Z = I_{Cm} + Mr^2$

16. Torque:

The turning effect of a force about the axis of rotation is called moment of force or torque.

Torque = Force x Perpendicular distance of line of action of force from axis of rotation.

$\bar{\tau} = \bar{r} \times \bar{F}$ and $|\bar{\tau}| = rF \sin\theta$

17. Relation Between τ and I:

$\tau = r\,F = r\,m\,a = r\,m(r\,\alpha) = mr^2\,\alpha = I\alpha$

Where $\alpha =$ angular acceleration

18. Angular Momentum (L):

Moment of linear momentum of a particle about axis of rotation is known as Angular momentum

Angular momentum is an axial vector

$\vec{L} = \vec{r} \times \vec{P}$

$L = rp\sin\theta$

$\vec{r}$ = Position vector.

19. Law of Conservation of Angular Momentum:

Angular momentum of a rotating body remains constant when no external torque acting on it.

$$\vec{\tau} = \frac{d\left(\vec{L}\right)}{dt}$$

If $\vec{\tau} = 0$

$\frac{d\left(\vec{L}\right)}{dt} = 0 \Rightarrow L$ is constant.

Where L = Iω, $\quad \therefore$ Iω is constant

$I_1\,\omega_1 = I_2\,\omega_2$

20. Rotational KE $= \frac{1}{2} I\omega^2$

21. K.E of a Rolling Body:

$$K.E_{total} = K.E_{translatory} + K.E_{rotational}$$

$$= \frac{1}{2} mV_c^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} mV_c^2 \left(1 + \frac{K^2}{R^2}\right)$$

Where V_c = Velocity of C.M

K = radius of gyration

R = radius

Very Short Answer Questions

1. Is it necessary that a mass should be present at the centre of mass of any system?

A. No, it is not necessary that a mass should be present at the centre of mass of any system.

Ex: For a uniform circular ring the centre of mass lies at the centre of the ring where there is no mass.

2. What is the difference in the positions of a girl carrying a bag in one of her hands and another girl carrying a bag in each of her two hands?

A. A girl with a bag in one of her hands, slightly bends towards the side in which bag is located due to more mass on that side.

A girl with a bag in each of her two hands, position of the body will not change, due to uniform distribution of mass on each side.

3. Two rigid bodies have same moment of inertia about their axes of symmetry. Of the two, which body will have greater kinetic energy?

A: Rotational $K.E = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$

Since the moment of inertia is same, $K.E \propto L^2$

A body of greater angular momentum will have greater kinetic energy.

4. Why are spokes provided in a bicycle wheel?

A. The spokes to the cycle wheel increases the moment of inertia due to the increase in the distribution of mass. This opposes the change in the rotary motion of the wheel. Thus spokes fitted to the cycle wheel gives a steady motion.

5. We cannot open or close the door by applying force at the hinges. Why?

A. Torque $(\bar{\tau}) = \bar{r} \times \bar{F} = rF\sin\theta$

$r\sin\theta \propto \frac{1}{F}\ (\because \tau = \text{constant})$

If the force is applied at the hinge, then r = 0

Hence force required to rotate the door becomes infinity.

6. Why do we prefer a spanner of longer arm as compared to the spanner of shorter arm?

A: Torque $(\tau) = \vec{r} \times \vec{F} = rF\sin\theta$

If same force is applied both the spanners $\tau \propto r$, for a spanner of longer arm, r is more.

$\therefore$ Torque produced by spanner of longer arm is more than that of spanner shorter arm.

7. By spinning eggs on a table top, how will you distinguish a hardboiled egg from a raw egg?

A. When the raw egg is rotated on a table top, the liquid in it move away from the axis of rotation due to centrifugal force. Hence moment of inertia increases and angular velocity decreases It comes to rest quickly. But the boiled egg rotates more time.

8. Why should a helicopter necessarily have two propellers?

A: If the helicopter had only one propeller, then due to conservation of angular momentum, the helicopter itself would turn in the opposite direction.

9. If the polar ice caps of the earth were to melt, what would the effect of the length of the day be?

A. When the polar ice caps melt, water flows towards the equator. Then moment of inertia of the earth increases and angular velocity decreases.

$I\omega =$ constant

$\therefore$ As I increases, ω decreases

But, $\omega = \dfrac{2\pi}{T}$

$\therefore$ The time period increases. Hence the length of the day increases.

10. Why is it easier to balance a bicycle in motion?

A: Due to law of conservation of angular momentum, bicycle is balanced in motion.

Short Answer Questions

1. Distinguish between centre of mass and centre of gravity?

A.

Centre of Mass	Centre of Gravity
1. It is a point where the entire mass of the system is concentrated.	1. It is a point where the weight of the system is concentrated.
2. It refers to the mass of the body.	2. It refers to the weight acting on all particles of the body.
3. For small and regular bodies centre of mass and centre of gravity will coincide.	3. For huge bodies centre of gravity and centre of mass do not coincide.
4. It does not depend on acceleration due to gravity.	4. It depends on acceleration due to gravity.

2. Show that a system of particles moves under the influence of an external force as if the force is applied at its centre of mass?

A. Consider a system of the particles of mass m_1, m_2,---- , m_n moving with velocities V_1, V_2--- ,V_n.Then $M = m_1 + m_2 +$ ----- $+ m_n$

Velocity $\overrightarrow{V_{cm}} = \frac{1}{M}\left(m_1\vec{v_1} + m_2\vec{v_2} + ---- + m_n\vec{v_n}\right)$

Acceleration of center of mass $\overrightarrow{a_{cm}} = \frac{d\overrightarrow{V_{cm}}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{v_1}}{dt} + m_2\frac{d\vec{v_2}}{dt} + ---- + m_n\frac{d\vec{v_n}}{dt}\right)$

Or

$$\overrightarrow{a_{cm}} = \frac{1}{M}\left(m_1\vec{a_1} + m_2\vec{a_2} + ---- + m_n\vec{a_n}\right)$$

From Newton's second law, $\overrightarrow{a_{cm}} = \frac{1}{M}\left[\vec{F}_1 + \vec{F}_2 + ------ + \vec{F}_n\right]$

$$\therefore M\overrightarrow{a_{cm}} = \sum \overrightarrow{F}_n = \overrightarrow{F}_{ext}$$

Hence the system of particles moves under the influence of an external force as if the force is applied at its centre of mass.

3. Explain about the Centre of mass of Earth - moon system and its rotation around the sun?

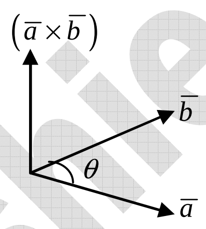
A. The earth -moon system rotates about the common centre of mass. The mass of the earth is about 81 times that of the moon. Hence the centre of mass of the earth-moon system is relatively close to the earth. The gravitational attraction of the sun is an external force that acts on the earth-moon system. The centre of mass of the earth-moon system moves in an elliptical path around the sun.

4. Define vector product .Explain the properties of vector product with two examples?

A. **Vector Product:**

It is a vector which is the product of the magnitudes of the two vectors and the sine of the angle between them. The direction is perpendicular to the plane containing the two vectors.

$\overline{a}\times\overline{b} = ab\sin\theta.\ \hat{n}$ Where $\hat{n}$ is a u nit vector along $\overline{a}\times\overline{b}$.



Properties:

i. Cross product of vectors do not obey commutative law.

$\overline{A}\times\overline{B} \neq \overline{B}\times\overline{A}$ And $\overline{A}\times\overline{B} = -\overline{B}\times\overline{A}$

ii. Cross product obeys distributive law.

$\overline{A}\times(\overline{B}+\overline{C}) = \overline{A}\times\overline{B}+\overline{A}+\overline{C}$

iii. The cross product of two parallel vectors is a null vector. (i.e.) If $\theta = 0^0$, then $\overline{A} \times \overline{B} = 0$

iv. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}; \hat{k} \times \hat{j} = -\hat{i}\ ; \hat{i} \times \hat{k} = -\hat{j}$

Examples:

1. Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

2. Linear velocity $\vec{V} = \vec{\omega} \times \vec{r}$

5. Define angular velocity (ω). Derive $v = r\omega$.

A. Angular velocity (ω):

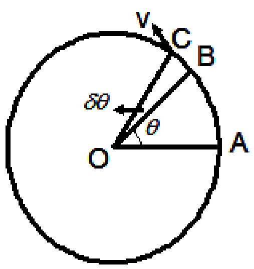
The rate of change of angular displacement of a particle is called angular velocity.

$$\omega = \frac{d\theta}{dt} \text{ rad/sec}$$

To derive $v = r\omega$:

Consider a particle be moving along a circle of radius r. At any time t, let the angular displacement of the particle be θ. Let the particle is displaced through an angle in a time interval δt.

Instantaneous angular velocity $\omega = \underset{\delta t \to 0}{Lim}\left(\frac{\delta\theta}{\delta t}\right) = \frac{d\theta}{dt}$



Linear velocity of the particle is given by

$$v = \lim_{\delta t \to 0} \frac{BC}{\delta t} = \lim_{\delta t \to 0} \frac{r\delta\theta}{\delta t} \quad = r\frac{d\theta}{dt} \quad \left(\because \delta\theta = \frac{BC}{r}\right)$$

$$\therefore\ V = r\omega$$

6. Define angular acceleration and torque. Establish the relation between angular acceleration and torque.

A. Angular acceleration:

Rate of change of angular velocity is called Angular Acceleration (α). Its unit is rad s^{-2}.

$$\alpha = \frac{change\ in\ angular\ velocity(d\omega)}{time(dt)}$$

Torque:

The turning effect of a force about the axis of rotation is called moment of force or torque.

Torque = Force x Perpendicular distance of line of action of force from axis of rotation.

$\bar{\tau} = \bar{r} \times \bar{F}$ and $|\bar{\tau}| = rF\sin\theta$

Relation between angular acceleration and torque:

By definition, torque $\bar{\tau} = \bar{r} \times \bar{F}$

$$\tau = r \times ma = r\,m(r\alpha) = mr^2\alpha = I\alpha$$

$$\therefore \tau = I\alpha$$

7. Write the equations of motion for a particle rotating about a fixed axis?

A: The equations of rotator motion of a rigid body are similar to the equations of linear motion. They are given below

1) $\omega = \omega_0 + \alpha t$

2) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

3) $\omega^2 - \omega_0 t = 2\alpha\theta$

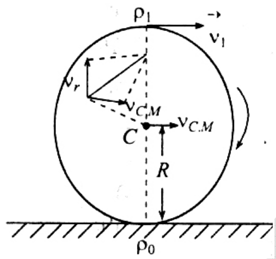
Where ω_0 is initial angular velocity, ω is final angular velocity, α is angular acceleration and θ is angular displacement

8. Derive expressions for the final velocity and total energy of a body rolling without slipping.

A: Final Velocity of Rolling Body:

Let us consider the rolling motion (with out slipping) of a circular disc on a level surface. Let $V_{C.M}$ is the velocity of centre of mass. Let V_r is the rotational velocity.

$V_r = r\omega$, where ω is angular velocity.



The disc on rolling motion without slipping. The essential condition is $v_{C.M} = R\omega$

Velocity of point P_1 at the top of the disc,

$$v_t = v_{C.M} + R\omega = 2v_{C.M}$$

K.E of the Rolling Body:

The total K.E of the rolling body = K.E of transitional motion of centre of mass + K.E of rotational motion.

$$K = K_T + K_R$$

$$K = \frac{1}{2}mv_{C.M}^2 + \frac{1}{2}I\omega^2$$

From $v_{C.M} = K\omega \Rightarrow \omega = \frac{v_{C.M}}{R}$ and $I = mK^2$

Where I is moment of inertia and K is radius of gyration

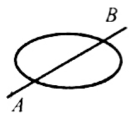
$$K = \frac{1}{2}mv_{C.M}^2 + \frac{1}{2}mK^2\left(\frac{v_{C.M}}{R}\right)^2$$

$$K = \frac{1}{2}mv_{C.M}^2\left(1 + \frac{K^3}{R^2}\right)$$

Long Answer Questions

1. a) State And Prove Parallel Axes Theorem?

b) For a thin flat circular disk, the radius of gyration about a diameter as axis is k. If the disk is cut along a diameter AB as shown in to equal pieces, then find the radius of gyration of each piece about AB?



A. a) Parallel Axes Theorem:

Statement:

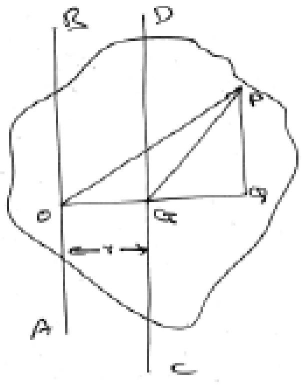
The moment of inertia (I) of a rigid body about any axis is equal to the sum of moment of inertia about a parallel axis passing through its centre of mass (I_G) and the product of the mass of the body (M) and the square of the perpendicular distance (r) between the two parallel axes.

$I = I_0 + Mr^2$

Proof:

Consider a rigid body of mass M. Let I and I_G be the moments of inertia of the body about two parallel axes AB and CD. The axis CD passes through the centre of mass of the body. Let 'r' be the perpendicular distance between the AB and CD.

Consider a particle P of mass m. Extend OG and draw a perpendicular PQ on to OG produced.



Moment of inertia of the body about AB

$$I = \Sigma m(OP)^2$$

Moment of inertia of the body about CD

$$I_G = \Sigma m(GP)^2$$

In the triangle OPQ,

$$OP^2 = OQ^2 + PQ^2 = (OG + GQ)^2 + PQ^2$$

$$= OG^2 + GQ^2 + 2.OG.GQ + PQ^2$$

$$= OG^2 + GP^2 + 2.OG.GQ \ (\because GQ^2 + PQ^2 = GP^2)$$

$$\therefore I = \Sigma m(OP)^2$$

$$= \Sigma m[OG^2 + GP^2 + 2.OG.GQ]$$

$$= \Sigma mr^2 + \Sigma mGP^2 + \Sigma m.2OG.GQ$$

$$\therefore I = I_G + Mr^2 \ (\because \Sigma m = M)$$

($\Sigma m.GQ = 0$ .This is the sum of the moments of all masses about CM.)

b) Moment of inertial of circular disk

$I = \dfrac{MR^2}{4}$ (About diameter)

$$K = \sqrt{\frac{I}{M}} = \sqrt{\frac{MR^2}{4M}} = \frac{R}{2}$$

If the disc is cut along the diameter AB into two equal pieces, then radius of gyration of each piece about AB = K

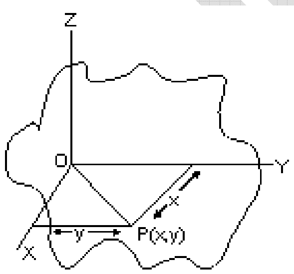
2. a) State And Prove Perpendicular Axes Theorem?

b) If a thin circular ring and a thin flat circular disk of same mass have same moment of inertia about their respective diameters as axes. Then find the ratio of their radii

A. a) Perpendicular Axes Theorem:

Statement:

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other in its plane intersecting each other at the point through where the perpendicular axis passes. $\therefore\ I_z = I_x + I_y$.



Proof:

Consider a particle of mass 'm' at 'P' in the plane lamina. Let X and Y axis lie in the plane of the lamina and Z axis is perpendicular to the lamina. .Let the particle is at a

distance "r' from the Z- axis. Let the moments of inertia of the plane lamina about X, Y and Z axes are I_x, I_y and I_z respectively.

$$\therefore I_x = \Sigma my^2 \quad \text{and} \quad I_y = \Sigma mx^2$$

$$I_z = \sum mr^2 = \sum m\left(x^2 + y^2\right) \qquad \left(\because r^2 = x^2 + y^2\right)$$

$\therefore I_z = I_x + I_y$

Hence the theorem is proved.

b) Moment of inertia of a circular ring about diameter $I_1 = \frac{M_1 R_1^2}{2} \rightarrow (1)$

Moment of inertia of a circular disc about diameter $I_2 = \frac{M_2 R_2^2}{4} \rightarrow (2)$

Given $I_1 = I_2, M_1 = M_2$

$$\frac{R_1^2}{2} = \frac{R_2^2}{4} \Rightarrow \left(\frac{R_1^2}{2}\right)^2 = \frac{1}{2}$$

$$\therefore \frac{R_1}{R_2} = \frac{1}{\sqrt{2}}$$

3. State and prove the principle of conservation of angular momentum. Explain the principle of conservation of angular momentum with examples?

A. Statement:

In the absence of resultant external torque on a rotating system, the angular momentum (L) of the system remains constant both in magnitude and direction.

$I\omega = constant \quad Or \quad I_1\omega_1 = I_2\omega_2$

Proof:

The resultant external torque τ acting on a rotating system is related to its angular momentum L as $\tau = \frac{dL}{dt}$

If the resultant external torque, τ is equal to zero. $\frac{dL}{dt} = 0$ Or L = constant.

$$\therefore I\omega = constant \quad Or \quad I_1\omega_1 = I_2\omega_2$$

Examples:

1. A ballet dancer decreases or increases his angular speed of rotation by stretching the hands or bringing the hands closer to the body.

2. A diver makes rotations in air by bringing the hands and legs closer to the body and increasing the angular velocity. When the diver reaches water, legs and hands are stretched so that moment of inertia increases and decreases.

Problems

1. Show that $\vec{a}.\left(\vec{b}\times\vec{c}\right)$ is equal in magnitude to the volume of the parallelepiped formed on the three vectors $\vec{a},\vec{b}$ and $\vec{c}$.

A: Let a parallelepiped be formed on the three vectors

$\overrightarrow{OA}=\vec{a},\overrightarrow{OB}=\vec{b},\overrightarrow{OC}=\vec{c}$

Thus the moment of inertia of a disc about any of its diameter is $MR^2/4$.

2. A rope of negligible mass is wound round a uniform hollow cylinder of mass 3 kg and radius 40 cm. What is angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is linear acceleration of the rope? Assume that there is no slipping?

A. M = 3 Kg, r = 0.4 m, F = 30 N

i) I = $mr^2=3\times(0.4)^2=0.48 kg-m^2$

$$\tau=rF=I\alpha$$

$$\alpha=\frac{rF}{I}=\frac{0.4\times30}{0.48}=25\ rad\ s^{-2}$$

ii) F = ma

$$a=\frac{F}{m}=\frac{30}{3}=10\ m\ s^{-2}$$

3. A coin is kept at a distance of 10 cm from the centre of a circular turn table. If the coefficient of static friction between the table and the coin is 0.8 find the frequency of rotation of the disc at which the coin will just begin to slip?

A. Condition for just slipping, the required condition is mr $\omega^2 = \mu$ mg

Or $\quad r \times 4\pi^2 n^2 = \mu\ g$

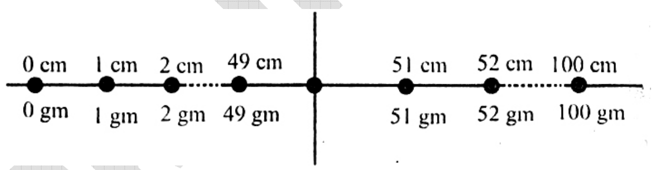
Or $\quad n^2 = \dfrac{\mu g}{4\pi^2 r} = \dfrac{0.8 \times 9.8}{4\pi^2 \times 0.1} = 2$

Or $\ n = \sqrt{2}$ rev/s

4. Particles of masses 1g, 2g, 3g……100g are kept at the marks 1cm, 2cm, 3cm…100cm respectively on a meter scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the meter scale.

A: $m_1 = 1gm = 10^{-3}kg$, $m_2 = 2\times10^{-3}kg$

$m_3 = 3\times10^{-3}kg$,…… $m_{100} = 100\times10^{-3}kg$



$r_1 = 1cm = 10^{-2}m = 1m$, $r_2 = 2\times10^{-2}m,\ \ r_3 = 3\times10^{-2}m,$

$r_{100} = 100\times10^{-2}m = 1m$

The axis of rotation is passing through 50cm. at 50cm a mass of 50gm is placed. Total no.of pairs of particles about an axis of rotation x = 50

Total no. of pairs of particles M = 100gm $= 10^{-1}kg$

The distance of 1st pair of particles (49 & 51) from the axis of rotation

$R_1 = 1cm = 10^{-2}m$

The distance of 2nd pair of particles 48 and 52 from the axis of rotation

$R_2 = 2cm = 10^{-2}m$

$\therefore$ The total moment of inertia of all pair of particles about an axis of rotation

$I = Mr_1^2 + Mr_3^2 + ... + Mr_{50}^2$

$= 10^{-1} \times (10^{-2} + 10^{-1} \times (2 \times 10^{-2}) + ...10^{-1} \times (50 \times 10^{-2})^2$

$I = [1^2 + 2^2 + ... + 50^2]10^{-5}$

$I = \dfrac{50 \times 51 \times 101 \times 10^{-5}}{6} \qquad \left(\because 1^2 + 2^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6} \right)$

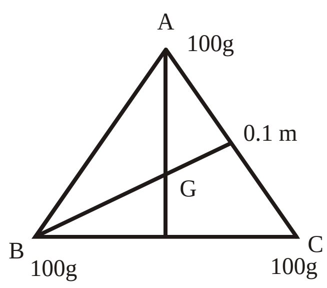
$I = 0.42925 = 0.43 kg - m^2$.

5. Three particles each of mass 100 g are placed at the vertices of an equilateral triangle of side length 10 cm. Find the moment of inertia of the system about an axis passing through the centroid of the triangle and perpendicular to its plane?

A. Let G be the position of the centroid of the triangle.

Then AG = BG = CG = $\dfrac{0.1}{\sqrt{3}}m$

M.I of the system = 3 mr^2

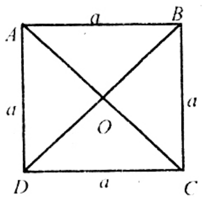


$= 3 \text{ x } 0.1 \text{ x } \left(\frac{0.1}{\sqrt{3}}\right)^2 = 3 \times \frac{1}{10} \times \frac{1}{100} \times \frac{1}{3} = 10^{-3} Kg - m^2$

6. Four particles each of mass 100g are placed at the corners of a square of side 10cm. find the moment of inertia of the system about an axis passing through the centre of the square and perpendicular to its plane. Find also the radius of gyration of the system?

Sol: m = 100gm $= 10^{-1} kg$; a = 10cm $= 10^{-1} m$

From fig $OA = OC = OB = OD = \frac{AV}{2} = \frac{\sqrt{2a}}{2}$



$OA = OC = OB = OD = \frac{a}{\sqrt{2}} = \frac{10^{-1}}{\sqrt{2}} \rightarrow (1)$

i) Total moment of inertia

$I = m(OA)^2 + m(OB)^2 + m(OC)^2 + m(OD)^2$

$I = 4m(OA)^2 = 4 \times 10^{-1} \times \left(\frac{10^{-1}}{\sqrt{2}}\right)^2$

$I = 2 \times 10^{-3} kg - m^2$

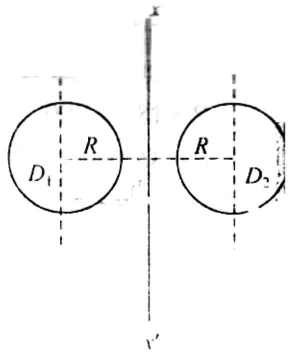
ii) Radius of gyration $K = OA = \frac{10^{-1}}{\sqrt{2}}$

$K = 0.7071 \times 10^{-1} = 0.07071 m$.

7. **Two uniform circular discs, each of mass 1kg and radius 20cm, are kept in contact about the tangent passing through the point of contact. Find the moment of inertia of the system about the tangent passing through the point of contact**

Sol: XX' is the common tangent of two discs D_1 and D_2 at the point of contact M.I of D_1 about a diameter parallel to

$$XX' = I_1^1 = \frac{MR^2}{4}$$



M.I of D_1 about axis $XX' = I_1 = \frac{MR^2}{4} + MR^2 + MR^2 = \frac{5}{4}MR^2$

Similarly M.I of D_2 about axis $XX' = I_2 = \frac{5}{4}MR^2$

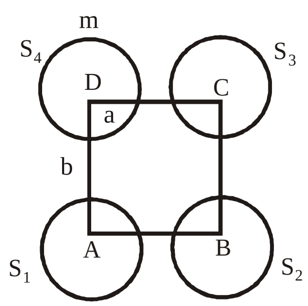
Total M.I of the system $I = I_1 + I_2 = \frac{5}{2}MR^2$

Given M = 1kg, R = 0.2m

$$I = \frac{5}{2}MR^2 = \frac{5}{2}\times 1\times(0.2)^2 = 0.1kg - m^2$$

8. Four spheres each diameter 2a and mass m are placed with their centers on the four corners of a square of the side b. Calculate the moment of inertia of the system about any side of the square?

A. Let S_1, S_2, S_3 and S_4 be the given 4 spheres.



For each sphere r = a

Side of the square = b

Let I_1, I_2, I_3 and I_4 be the moments of inertia of spheres

S_1, S_2, S_3 and S_4 respectively about AB

$$\therefore I_1 = I_2 = \frac{2ma^2}{5}$$

$$I_3 = I_4 = \frac{2ma^2}{5} + mb^2$$

$$\therefore \text{M.I of the whole system} = I_1 + I_2 + I_3 + I_4 = \left(2 \times \frac{2ma^2}{5}\right) + 2\left(\frac{2ma^2}{5} + mb^2\right)$$

$$= \frac{8ma^2}{5} + 2mb^2$$

9. To maintain a rotor at a uniform angular speed of $200 rad\, s^{-1}$, an engine needs to transmit a torque of 180Nm. What is the power required by the engine? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque) Assume that the engine is 100% efficient

Sol: $\omega = 200\, rad / s$, $\tau = 180 Nm$

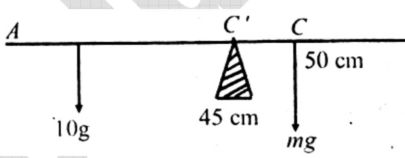
Power (P) $= \tau\omega$

P = 180 x 200

P = 36000W = 36kW

10. A meter stick is balanced on a knife edge at its centre. When two coins, each of mass 5g are put one on top of the other at the 12.0cm mark, the stick is found to be balanced at 45.0cm. What is the mass of the meter stick?

Sol: Let m be the mass of the stick connected at C, the 50cm mark for equilibrium about C, the 45cm mark



10g (45-12) = mg(50-45)

10g x 33 = mg x 5

$$m = \frac{10 \times 33}{5} = 66g$$

11. Determine the kinetic energy of a circular disc rotating with a speed of 60rpm about an axis passing through a point on its circumference and perpendicular to its plane. The circular disc has a mass of 5kg and radius 1m

Sol: $n = 60 rpm = \frac{60}{60} = 1 rps$

$\omega = 2\pi r n = 2\pi \times 1 = 2\pi$, M = 5kg, r = 1m

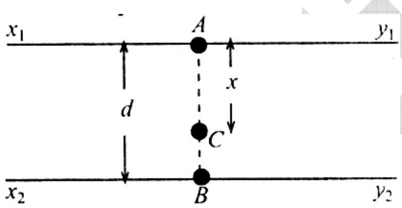
$$I = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

$$I = \frac{3}{2} \times 5 \times (1)^2 = \frac{15}{2}$$

Rotational K.E $= \frac{1}{2}I\omega^2 = \frac{1}{2} \times \frac{15}{2} \times (2\pi)^2 = 15\pi^2 = 148.16J$

12. Two particles, each of mass m and speed u, travel in opposite direction along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken

Sol: Vector angular moment of the two particle system about any point A on x_1y_1



$$\overrightarrow{L_A} = \overrightarrow{mv} \times 0 + \overrightarrow{mv} \times d = \overrightarrow{mv}d$$

Similarly vector angular momentum of two particle system about any point B on X_2Y_2

$$\overrightarrow{L_B} = m\vec{v}d + m\vec{v} \times 0 = m\vec{v}d$$

$\therefore$ vector angular moment of the two particle system about C is $(\because AC = x)$

$$\overrightarrow{L_C} = m\vec{v}(x) + m\vec{v}(d - x) = m\vec{v}d$$

$$\therefore \overrightarrow{L_A} = \overrightarrow{L_B} = \overrightarrow{L_C}$$

13. The moment of inertia of a fly wheel making 300 revolutions per minute is 0.3 kgm². Find torque required to bring it to rest in 20 s?

A. I = 0.3 Kg-m², n = 300/60 = 5 rev/s

$\omega_0 = 2\pi n = 2\pi \times 5 = 10\pi$ rad/s

$\omega = \omega_0 + \alpha t$ Or $0 = 10\pi + 20$

Or $\alpha = -10\pi/20 = -\pi/2$ rad s^{-2}

$\tau = I\alpha = 0.3 \text{ x } \pi/2 = 0.3 \text{ x} \frac{22}{7} \times \frac{1}{2} = 0.471$ Nm

14. When 100 J of work is done on a fly wheel, its angular velocity is increased from 60 rpm to180 rpm. What is the moment of inertia of the wheel?

A. W = 100 J, $\omega_1 = 60$ rpm = 2π rad s^{-1}

$\omega_2 = 180$ rpm = 6π rad s^{-1}

Work done = change in rotational K.E

$$100 = \frac{1}{2} I(\omega_2^2 - \omega_1^2) = \frac{1}{2} I(36\pi^2 - 4\pi^2)$$

Or $I = \frac{200}{32\pi^2} = \frac{5}{8} = 0.63 Kg - m^2$