11

CIRCLES

Notice the path in which the tip of the hand of a watch moves. (see Fig. 11.1)

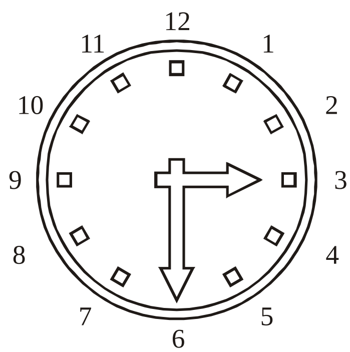

Fig. 11.1

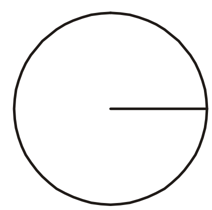

Fig. 11.2

Again, notice the curve traced out when a nail is fixed at a point and a thread of certain length is tied to it in such a way that it can rotate about it, and on the other end of the thread a pencil is tied. Then move the pencil around the fixed nail keeping the thread in a stretched position (See Fig 11.2)

Certainly, the curves traced out in the above examples are of the same shape and this type of curve is known as a ***circle***.

The distance between the tip of the pencil and the point, where the nail is fixed is known as the ***radius*** of the circle.

We shall discuss about the curve traced out in the above examples in more details.

OBJECTIVES

After studying this lesson, you will be able to :

- derive and find the equation of a circle with a given centre and radius;

- state the conditions under which the general equation of second degree in two variables represents a circle;
- derive and find the centre and radius of a circle whose equation is given in general form;
- find the equation of a circle passing through :

  (*i*) three non-collinear points (*ii*) two given points and touching any of the axes;
- derive the equation of a circle in the diameter form;
- find the equation of a circle when the end points of any of its diameter are given; and
- find the parametric representation of a circle with given centre and radius.

EXPECTED BACKGROUND KNOWLEDGE

- Terms and concepts connected with circle.
- Distance between two points with given coordinates.
- Equation of a straight line in different forms.

11.1 DEFINITION OF THE CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point in the same plane remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

11.2 EQUATION OF A CIRCLE

Can we find a mathematical expression for a given circle?

Let us try to find the equation of a circle under various given conditions.

11.2.1 WHEN COORDINATES OF THE CENTRE AND RADIUS ARE GIVEN

Let C be the centre and a be they radius of the circle. Coordinates of the centre are given to be (h, k), say.

Take any point $P(x, y)$ on the circle and draw perpendiculars CM and PN on OX. Again, draw CL perpendicular to PN.

We have

$$CL = MN = ON - OM = x - h$$

and $PL = PN - LN = PN - CM = y - k$

In the right angled triangle CLP, $CL^2 + PL^2 = CP^2$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2 \qquad \text{...(1)}$$

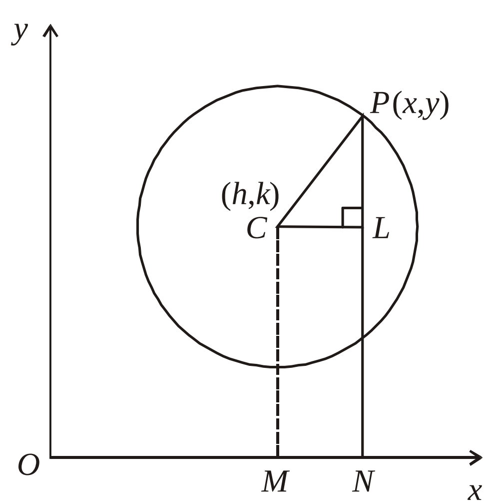

Fig. 11.3

This is the required equation of the circle under given conditions. This form of the circle is known as ***standard form*** of the circle.

Conversely, if (x, y) is any point in the plane satisfying (1), then it is at a distance 'a' from (h, k). So it is on the circle.

What happens when the

(*i*) circle passes through the origin?

(*ii*) circle does not pass through origin and the centre lies on the x-axis?

(*iii*) circle passes through origin and the x-axis is a diameter?

(*iv*) centre of the circle is origin?

(*v*) circle touches the x-axis?

(*vi*) circle touches the y-axis?

(*vii*) circle touches both the axes?

We shall try to find the answer of the above questions one by one.

(*i*) In this case, since (0, 0) satisfies (1), we get

$$h^2 + k^2 = a^2$$

Hence the equation (1) reduces to

$$x^2 + y^2 - 2hx - 2ky = 0 \quad ...(2)$$

(*ii*) In this case $k = 0$

Hence the equation (1) reduces to

$$(x - h)^2 + y^2 = a^2 \quad ...(3)$$

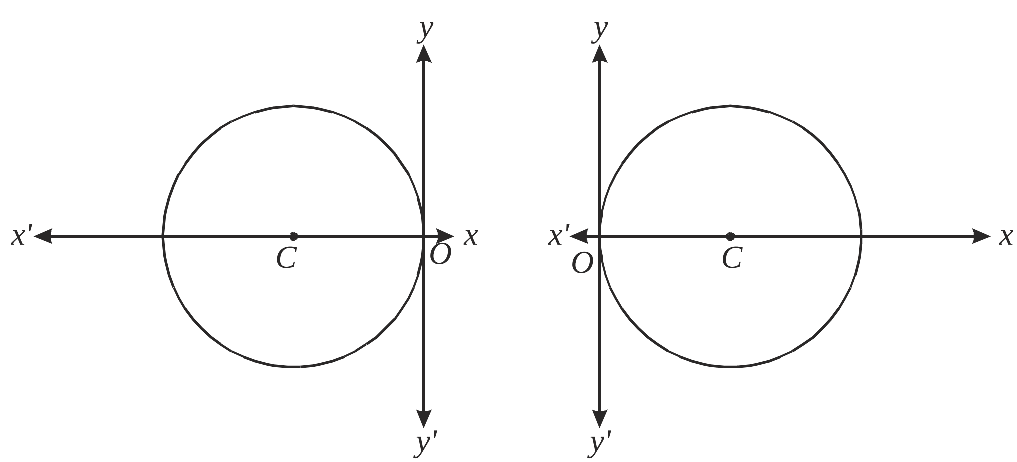

Fig. 11.4

(*iii*) In this case $k = 0$ and $h = \pm a$ (see Fig. 11.4)

Hence the equation (1) reduces to $x^2 + y^2 \pm 2ax = 0$ ...(4)

(*iv*) In this case $h = 0 = k$

Hence the equation (1) reduces to $x^2 + y^2 = a^2$ ...(5)

(*v*) In this case $k = a$ (see Fig. 11.5)

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ay + h^2 = 0$ ...(6)

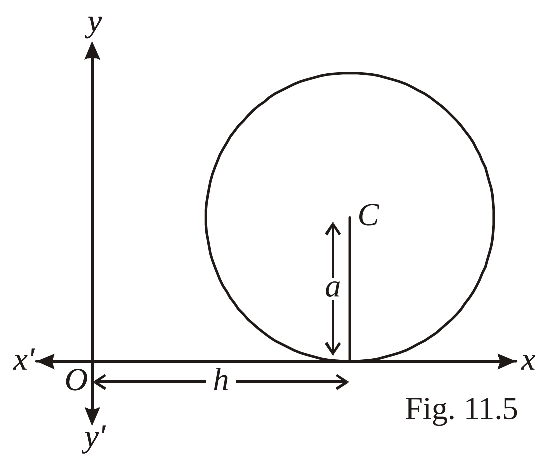

Fig. 11.5

(*vi*) In this case $h = a$

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ky + k^2 = 0$ ...(7)

(*vii*) In this case $h = k = a$. (See Fig. 11.6)

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(8)

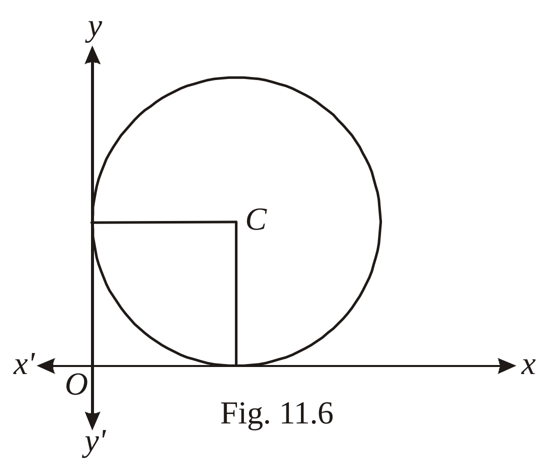

Fig. 11.6

Example 11.1 Find the equation of the circle whose centre is $(3, -4)$ and radius is 6.

Solution : Comparing the terms given in equation (1), we have

$h = 3$, $k = -4$ and $a = 6$.

$\therefore \quad (x-3)^2 + (y+4)^2 = 6^2$

or $\quad x^2 + y^2 - 6x + 8y - 11 = 0$

Example 11.2 Find the centre and radius of the circle given by $(x+1)^2 + (y-1)^2 = 4$.

Solution: Comparing the given equation with $(x-h)^2 + (y-k)^2 = a^2$ we find that

$-h = 1$, $-k = -1$, $a^2 = 4$

$\therefore \quad h = -1$, $k = 1$, $a = 2$.

So the given circle has its centre $(-1,1)$ and radius 2.

11.3 GENERAL EQUATION OF THE CIRCLE IN SECOND DEGREE IN TWO VARIABLES

The standard equation of a circle with centre (h, k) and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \ldots (1)$$

or

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \qquad \ldots (2)$$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (3)$$

$$\Rightarrow \quad (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$\Rightarrow \quad [x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2 \qquad \ldots (4)$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2$$

where $h = -g, \quad k = -f, \quad r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$

11.3.1 CONDITONS UNDER WHICH THE GENERAL EQUATION OF SECOND DEGREE IN TWO VARIABLES REPRESENTS A CIRCLE

Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) It is a second degree equation in x, y in which coefficients of the terms involving x^2 and y^2 are equal.

(ii) It contains no term involving xy

Note : *In solving problems, we keep the coefficients of x^2 and y^2 unity.*

Example 11.3 Find the centre and radius of the circle

$$45x^2 + 45y^2 - 60x + 36y + 19 = 0$$

Solution : Given equation can be written on dividing by 45 as

$$x^2 + y^2 - \frac{4}{3}x + \frac{4}{5}y + \frac{19}{45} = 0$$

Comparing it with the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ we get}$$

$$g = -\frac{2}{3},\ f = \frac{2}{5} \text{ and } c = \frac{19}{45}$$

Thus, the centre is $\left(\frac{2}{3}, -\frac{2}{5}\right)$ and radius is $\sqrt{g^2 + f^2 - c} = \frac{\sqrt{41}}{15}$

Example 11.4 Find the equation of the circle which passes through the points (1, 0), (0, – 6) and (3, 4).

Solution: Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

Since the circle passes through three given points so they will satisfy the equation (1). Hence

$$1 + 2g + c = 0 \qquad \cdots(2)$$

and

$$36 - 12f + c = 0 \qquad \cdots(3)$$

$$25 + 6g + 8f + c = 0 \qquad \cdots(4)$$

Subtracting (2) from (3) and (3) from (4), we have

$$2g + 12f = 35$$

and $6g + 20f = 11$

Solving these equations for g and f, we get $g = -\frac{71}{4}, \quad f = \frac{47}{8}$

Substituting g in (2), we get $c = \frac{69}{2}$

and substituting g, f and c in (1), the required equation of the circle is

$$4x^2 + 4y^2 - 142x + 47y + 138 = 0$$

Exmaple 11. 5 Find the equation of the circles which touches the axis of x and passes through the points (1,–2) and (3, –4).

Solution : Since the circle touches the x-axis, put $k = a$ in the standard form (See result 6) of the equation of the circle, we have

$$x^2 + y^2 - 2hx - 2ay + h^2 = 0 \qquad \text{... (1)}$$

This circle passes through the point (1, –2)

$$\therefore \qquad h^2 - 2h + 4a + 5 = 0 \qquad \text{... (2)}$$

Also, the circle passes through the point (3, – 4)

$$\therefore \qquad h^2 - 6h + 8a + 25 = 0 \qquad \text{...(3)}$$

Eliminatiog 'a' from (2) and (3), we get

$$\Rightarrow \qquad h^2 + 2h - 15 = 0$$

$$h = 3 \ or \ h = -5.$$

From (3) the corresponding values of a are – 2 and –10 respectively. On substituting the values of h and a in (1) we get

$$x^2 + y^2 - 6x + 4y + 9 = 0 \qquad \text{... (4)}$$

and $$x^2 + y^2 + 10x + 20y + 25 = 0 \qquad \text{... (5)}$$

(4) and (5) represent the required equations.

11.4 EQUATION OF A CIRCLE WHEN END POINTS OF ONE OF ITS DIAMETERS ARE GIVEN.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given end points of the diameter AB (See Fig. 11.7)

Let $P(x, y)$ be any point on the circle drawn on AB as diameter. Join AP and BP. Since the angle in a semi-circle is a right angle.

$\therefore \quad AP \perp BP$

$\therefore \quad$ (slope of AP) $\times$ (slope of BP) $= -1$.

Now, the slope of $AP = \dfrac{y - y_1}{x - x_1}$

and the slope of $BP = \dfrac{y - y_2}{x - x_2}$

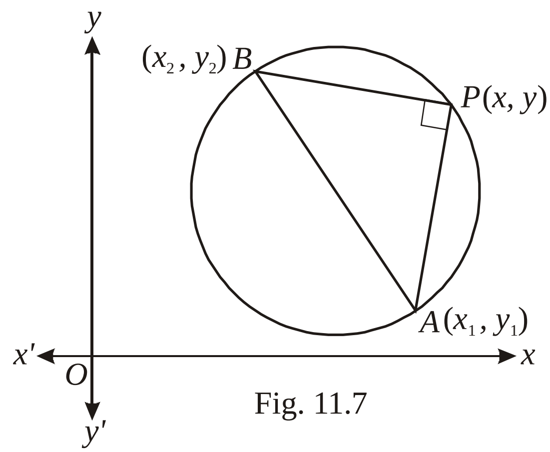

Fig. 11.7

$$\therefore \quad \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

or $\quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \qquad$ ....(1)

Since (1) is true for every point on the circle, and for no other point in the plane:

$\therefore \quad$ (1) represents the equation of the circle in ***diameter from.***

Example 11.6 Find the equation of the circle described on the line joining the origin and the point (2,– 4) as diameter.

Solutoon: Here $x_1 = 0,\ y_1 = 0;\ \ x_2 = 2, y_2 = -4$

Using the Equation (1) required the equation of the circle is,

$$(x - 0)(x - 2) + (y - 0)[y - (-4)] = 0$$

or $\quad x^2 - 2x + y^2 + 4y = 0$

or $\quad x^2 + y^2 - 2x + 4y = 0$

Example 11.7 The equation of a chord of the circle $x^2 + y^2 - 2ax = 0$ is $y = mx$. Find the equation of the circle described on this chord as diameter.

Solution: The coordinates of the points of intersection of the circle and the given chord are,

$$(0, 0) \text{ and } \left(\frac{2a}{1+m^2}, \frac{2ma}{1+m^2}\right).$$

Now, for the required equation of the circle these points are the end points of the diameter, so by equation (1)

$$(x - 0)\left(x - \frac{2a}{1+m^2}\right) + (y - 0)\left(y - \frac{2ma}{1+m^2}\right) = 0$$

or $\quad x^2 + y^2 - \dfrac{2a}{1+m^2}x - \dfrac{2ma}{1+m^2}y = 0$

or $\quad (1+m^2)x^2 + (1+m^2)y^2 - 2ax - 2may = 0$

This is the required equation of the circle.

11.5 EQUATION OF A CIRCLE WHEN RADIUS AND ITS INCLINATION ARE GIVEN (PARAMATRIC FORM)

In order to find the equation of the circle whose centre is the origin and whose radius is r. Let $P(x, \ y)$ be any point on the circle. Draw $PM \perp OX$

$\therefore \quad OM = x, \ MP = y$. Join OP.

Let $\quad \angle XOP = \theta$ and OP = r

Now, x = OM = $r \cos \theta$

and $\quad y = MP = r \sin \theta$

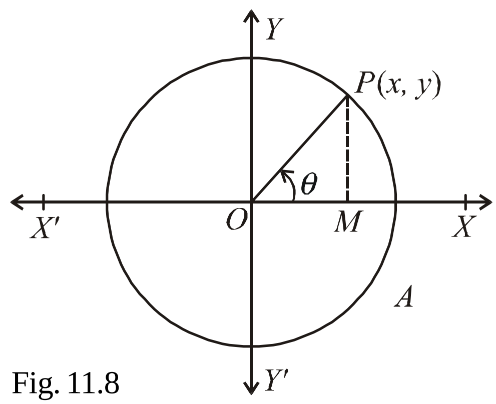

Fig. 11.8

Hence the two equations $x = r \cos \theta$ and $y = r \sin \theta$ taken together represent a circle. These are known as ***parametric form of the equations*** of the circle, where θ is a parameter.

Note : *If the centre of the circle is at h and k, then the parametric form of the equation of the circle is* $x = h + r \cos \theta$ and $y = k + r \sin \theta$ where θ is a parametre which lies in the interval $[0, 2\pi]$.

Example 11.8 Find the parametric form of each of the following circles:

(i) $(x - 1)^2 + (y + 2)^2 = 9$ (ii) $(x + 2)(x - 4) + (y - 3)(y + 1) = 0$

Solution : (i) The equation of the circle is

$(x - 1)^2 + (y - 2)^2 = 3^2$

Comparing it with the 'standard equation' of the circle, we get centre (1, –2) and radius = 3 for the given circle.

$\therefore$ Parametric form of the equation of the circle is

$x = 1 + 3\cos\theta; \ y = -2 + 3\sin\theta$

(ii) Equation of the circle can be written as

$x^2 + y^2 - 2x - 2y - 11 = 0$

Comparing it with the general equation of the circle and finding centre and radius, we get

Centre (1, 1) and radius = $\sqrt{13}$

$\therefore$ Parametric form of the equation of the circle is

$$x = 1 + \sqrt{13}\cos\theta,\ y = 1 + \sqrt{13}\sin\theta$$

CHECK YOUR PROGRESS 11.1

1. Find the equation of the circle whose

   (a) centre is (0, 0) and radius is 3. (b) centre is (– 2,3) and radius is 4.

2. Find the centre and radius of the circle

   (a) $x^2 + y^2 + 3x - y = 6$ (b) $4x^2 + 4y^2 - 2x + 3y - 6 = 0$

3. Find the equation of the circle which passes through the points (0, 2) (2, 0) and (0, 0).

4. Find the equation of the circle which touches the *y*-axis and passes through the points (–1, 2) and (–2,1)

5. Find the equation of the circle described on the line joining the points (2, 3) and (–2, 6) as diameter.

6. Find the parametric form of the equation of each of the following circles:

   (a) $(x+1)^2 + (y+1)^2 = 4$ (b) $4x^2 + 4y^2 + 2x + 2y - 3 = 0$

   (c) $(x-1)(x+1) + (y-1)(y+1) = 0$

LET US SUM UP

- **Standard form of the circle**

  $(x-h)^2 + (y-k)^2 = a^2$

  Centre is (h, k) and radius is a

- The general form of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

  Its centre: $(-g, -f)$ and radius = $\sqrt{g^2 + f^2 - c}$

- **Diameter form of the circle**

  If the end points of a diameter are (x_1, y_1) and (x_2, y_2), then the equation of the circle is

  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

- **Parametric form of the circle**

  $x = a\cos\theta$, $y = a\sin\theta$ represents the parametric equation of a circle whose centre is at (0, 0) and radius = a

  If the centre of the circle is at (h, k) then the parametric equation of the circle is

  $x = h + a\cos\theta$ and $y = k + a\sin\theta$.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the equation of a circle with centre (4, –6) and radius 7.
2. Find the centre and radius of the circle $x^2 + y^2 + 4x - 6y = 0$
3. Find the equation of the circle passes through the point (1,0), (–1,0) and (0,1)
4. Find the parametric form of the equation of circles given below :

   (a) $x^2 + y^2 = 3$ (b) $x^2 + y^2 - 4x + 6y = 12$

ANSWERS

CHECK YOUR PROGRESS 11.1

1. (a) $x^2+y^2=9$ (b) $x^2+y^2+4x-6y-3=0$

2. (a) $\left(-\frac{3}{2}, 1\right); \frac{\sqrt{37}}{2}$ (b) $\left(\frac{1}{4}, -\frac{3}{8}\right); \frac{\sqrt{109}}{8}$

3. $x^2+y^2-2x-2y=0$ 4. $x^2+y^2+2x-2y+1=0$

5. $x^2+y^2-9y+14=0$

6. (a) $x=-1+2\cos\theta$ and $y=-1+2\sin\theta$.

   (b) $x=-\frac{1}{4}+\sqrt{\frac{7}{8}}\cos\theta \quad y=-\frac{1}{4}+\sqrt{\frac{7}{8}}\sin\theta$

   (c) $x=\sqrt{2}\cos\theta$ and $y=\sqrt{2}\sin\theta$.

TERMINAL EXERCISE

1. $x^2+y^2-8x+12y+3=0$ 2. Centre $(-2, 3)$; Radius = $\sqrt{13}$

3. $x^2+y^2=1$.

4. (a) $x=\sqrt{3}\cos\theta, \quad y=\sqrt{3}\sin\theta$.

   (b) $x=2+5\cos\theta, \quad y=-2+5\sin\theta$.