CHAPTER 10 : STRAIGHT LINES

We have read about lines, angles and rectilinear figures in geometry. Recall that a line is the join of two points in a plane continuing endlessly in both directions. We have also seen that graphs of linear equations, which came out to be straight lines

Interestingly, the reverse problem of the above is finding the equations of straight lines, under different conditions, in a plane. The analytical geometry, more commonly called coordinate geomatry, comes to our help in this regard. In this lesson. We shall find equations of a straight line in different forms and try to solve problems based on those.

OBJECTIVES

After studying this lesson, you will be able to :

- derive equations of a line parallel to either of the coordinate axes;
- derive equations of a line in different forms (slope-intercept, point-slope, two point, intercept, and perpendicular)
- find the equation of a line in the above forms under given conditions;
- state that the general equation of first degree represents a line;
- express the general equation of a line into

  (i) slope-intercept form (ii) intercept form and (iii) perpendicular form;
- derive an expression for finding the distance of a given point from a given line;
- calculate the distance of a given point from a given line;
- derive the equation of a line passing through a given point and parallel/perpendicular to a given line;
- find equation of family of lines passing through the point of intersection of two lines.

EXPECTED BACKGROUND KNOWLEDGE

- Congruence and similarity of traingles

14.1 STRAINGHT LINE PARALLEL TO AN AXIS

If you stand in a room with your arms stretched, we can have a line drawn on the floor parallel to one side. Another line perpendicular to this line can be drawn intersecting the first line between your legs.

In this situation the part of the line in front of you and going behind you is the y-axis and the one being parallel to your arms is the x-axis.

The direction part of the y-axis in front of you is positive and behind you is negative.

The direction of the part x-axis to your right is positive and to that to your left is negative.

Now, let the side facing you be at b units away from you, then the equation of this edge will be $y = b$ (parallel to x-axis)

where b is equal in absolute value to the distance from the x-axis to the opposite side.

If $b > 0$, then the line lies in front of you, i.e., above the x-axis.

If $b < 0$, then the line lies behind you, i.e., below the x-axis.

If $b = 0$, then the line passes through you and is the x-axis itself.

Again, let the side of the right of you is at c units apart from you, then the equation of this line will be $x = c$ (parallel to y - axis)

where c is equal in absolute value, to the distance from the y-axis on your right.

If $c > 0$, then the line lies on the right of you, i.e., to the right of y-axis.

If $c < 0$, then the line lies on the left of you, i.e., to the left of y-axis

If $c = 0$, then the line passes through you and is the y-axis.

Example 14.1 Find the equation of the line passing through $(-2, -3)$ and

(i) parallel to x-axis (ii) parallel to y-axis

Solution :

(i) The equation of any line parallel to x-axis is $y = b$

Since it passes through $(-2, -3)$, hence $-3 = b$

$\therefore$ The required equation of the line is $y = -3$

(i) The equation of any line parallel to y-axis is $x = c$

Since it passes through $(-2, -3)$, hence $-2 = c$

$\therefore$ The required cquation of the line is $x = -2$

14.2 DERIVATION OF THE EQUATION OF STRAIGHT LINE IN VARIOUS STANDARD FORMS

So far we have studied about the inclination, slope of a line and the lines parallel to the axes. Now the questions is, can we find a relationship between x and y, where (x, y) is any arbitrary point on the line?

The relationship between x and y which is satisfied by the co-ordinates of arbitrary point on the line is called the equation of a straight line. The equation of the line can be found in various forms under the given conditions, such as

(a) When we are given the slope of the line and its intercept on y-axis.

(b) When we are given the slope of the line and it passes through a given point.

(c) When the line passes through two given points.

(d) When we are given the intercepts on the axes by the line.

(e) When we are given the length of perpendicular from origin on the line and the angle which the perpendicualr makes with the positive direction of x-axis.

We will discuss all the above cases one by one and try to find the equation of line in its standard forms.

(A) SLOPE-INTECEPT FORM

Let AB be a straight line making an angle θ with x-axis and cutting off an intercept $OD = c$ from OY.

As the line makes intercept OD = c on y-axis, it is called y-intercept.

Let AB intersect OX' at T.

Take any point $P(x, y)$ on AB. Draw $PM \perp OX$.

The $OM = x$, $MP = y$.

Draw $DN \perp MP$.

From the right-angled triangle DNP, we have

$$\tan\theta = \frac{NP}{DN} = \frac{MP - MN}{OM}$$

$$= \frac{y - OD}{OM}$$

$$= \frac{y - c}{x}$$

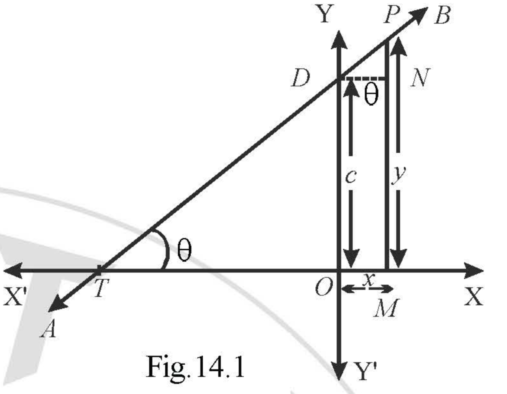

Fig. 14.1

$\therefore \quad y = x\tan\theta + c$

$\tan\theta = m$ (slope)

$\therefore \quad y = mx + c$

Since, this equation is true for every point on AB, and clearly for no other point in the plane, hence it represents the equation of the line AB.

Note : *(1) When $c = 0$ and $m \neq 0 \Rightarrow$ the line passes through the origin and its equation is $y = mx$*

(2) When $c = 0$ and $m = 0 \Rightarrow$ the line coincides with x – axis and its equation is of the form $y = $ o

(3) When $c \neq 0$ and $m = 0 \Rightarrow$ the line is parallel to x-axis and its equation is of the form $y = $ c

Example 14.2 Find the equation of a line with slope 4 and y-intercept 0.

Solution : Putting $m = 4$ and $c = 0$ in the slope intercept form of the equation, we get $y = 4x$

This is the desired equation of the line.

Example 14.3 Determine the slope and the y-intercept of the line whose equation is

$8x + 3y = 5$.

Solution : The given equation of the line is

$$8x + 3y = 5 \quad \text{or,} \quad y = -\frac{8}{3}x + \frac{5}{3}$$

Comparing this equation with the equation $y = mx + c$ (Slope intercept form) we get

$$m = -\frac{8}{3} \textit{ and } c = \frac{5}{3}$$

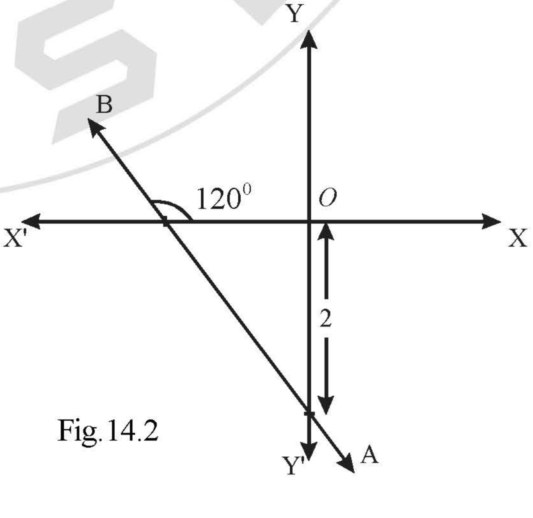

Fig. 14.2

Therefore, slope of the line is $-\frac{8}{3}$ and its y-intercept is $\frac{5}{3}$.

Example 14.4 Find the equation of the line cutting off an intercept of length 2 from the negative direction of the axis of y and making an angle of 120^0 with the positive direction x-axis

Solution : From the slope intercept form of the line $\therefore \quad y = x \tan 120° + (-2)$

$$= -\sqrt{3}\,x - 2 \text{ or, } y + \sqrt{3}\,x + 2 = 0$$

Here $m = \tan 120°$, and $c = -2$, because the intercept is cut on the negative side of y -axis.

(b) POINT-SLOPE FORM

Here we will find the equation of a line passing through a given point $A(x_1, y_1)$ and having the slope m.

Let $P(x, y)$ be any point other than A on given the line. Slope $(\tan\theta)$ of the line joining $A(x_1, y_1)$ and $P(x, y)$ is given by

$$m = \tan\theta = \frac{y - y_1}{x - x_1}$$

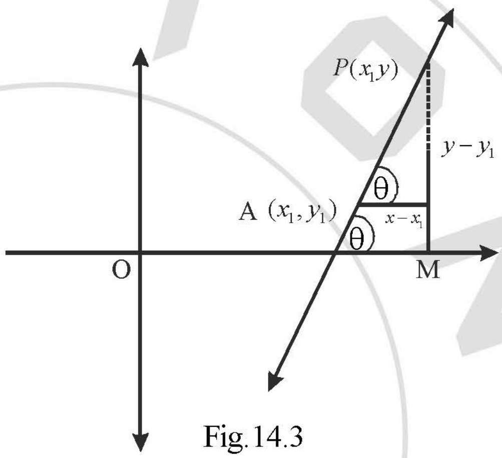

Fig.14.3

The slope of the line AP is given to be m.

$$\therefore \quad m = \frac{y - y_1}{x - x_1}$$

$\therefore$ The equation of the required line is, $y - y_1 = m(x - x_1)$

***Note :** Since, the slope m is undefined for lines parallel to y-axis, the point-slope form of the equation will not give the equation of a line though A (x_1, y_1) parallel to y-axis. However, this presents no difficulty, since for any such line the abscissa of any point on the line is x_1. Therefore, the equation of such a line is $x = x_1$.*

Example 14.5 Determine the equation of the line passing through the point $(2, -1)$ and having slope $\frac{2}{3}$.

Solution : Putting $x_1 = 2, y_1 = -1$ and $m = \frac{2}{3}$ in the equation of the point-slope form of the line we get, $y - (-1) = \frac{2}{3}(x - 2)$

$$\Rightarrow \quad y+1=\frac{2}{3}(x-2) \Rightarrow \quad y=\frac{2}{3}x-\frac{7}{3}$$

which is the required equation of the line.

(c) TWO POINT FORM

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given distinct points.

Slope of the line passing through these points is given by

$$m=\frac{y_2-y_1}{x_2-x_1} \qquad (x_2 \neq x_1)$$

From the equation of line in point slope form, we get

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

which is the required equation of the line in two-point form.

Example 14.6 Find the equation of the line passing through (3, – 7) and (– 2,– 5).

Solution : The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \cdots \text{(i)}$$

Since $x_1 = 3$, $y_1 = -7$ and $x_2 = -2$, and $y_2 = -5$, equation (i) becomes,

$$y+7=\frac{-5+7}{-2-3}(x-3)$$

or, $\quad y+7=\frac{2}{-5}(x-3)$ or, $\quad 2x+5y+29=0$

(d) INTERCEPT FORM

We want to find the equation of a line which cuts off given intercepts on both the co-ordinate axes.

Let PQ be a line meeting x-axis in A and y-axis in B. Let $OA = a$, $\quad OB = b$.

Then the co-ordinates of A and B are $(a,0)$ and (0, b,) respectively.

The equation of the line joining A and B is

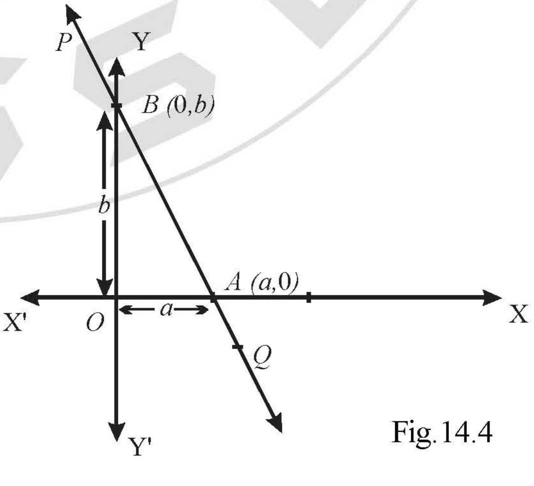

Fig.14.4

$$y - 0 = \frac{b-0}{0-a}(x-a) \text{ or, } \quad y = -\frac{b}{a}(x-a)$$

$$\text{or, } \quad \frac{y}{b} = -\frac{x}{a} + 1 \text{ or, } \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the required equation of the line having intercepts a and b on the axes.

Example 14.7 Find the equation of a line which cuts off intercepts 5 and –3 on x and y axes respectively.

Solution : The intercepts are 5 and –3 on x and y axes respectively. i.e., $a = 5$, $b = -3$

The required equation of the line is

$$\frac{x}{5} + \frac{y}{-3} = 1,\ 3x - 5y - 15 = 0$$

Example 14.8 Find the equation of a line which passes through the point (3, 4) and makes intercepts on the axes equl in magnitude but opposite in sign.

Solution : Let the x-intercept and y-intercept be a and –a respectively

$\therefore$ The equation of the line is, $\frac{x}{a} + \frac{y}{-a} = 1,\ x - y = a$ $\cdots$ (i)

Since (i) passes through (3, 4)

$\therefore$ $3 - 4 = a$ or $a = -1$

Thus, the required equation of the line is

$x - y = -1$ or $x - y + 1 = 0$

Example 14.9 Determine the equation of the line through the point (– 1,1) and parallel to x - axis.

Solution : Since the line is parallel to x-axis, so its slope ia zero. Therefore from the point slope form of the equation, we get, $y - 1 = 0\,[x - (-1)]$, $y - 1 = 0$

which is the required equation of the given line

Example 14.10 Find the intercepts made by the line

$3x - 2y + 12 = 0$ on the coordinate axes

Solution : Equation of the given line is, $3x - 2y = -12$.

Dividing by – 12, we get, $\frac{x}{-4} + \frac{y}{6} = 1$

Comparing it with the standard equation of the line in intercept form, we find $a = -4$ and $b =$

6. Hence the intercepts on the x-axis and y-axis repectively are –4. and 6.

Example 14.11 The segment of a line, intercepted between the coordinate axes is bisected at the point (x_1, y_1). Find the equation of the line

Solution : Let $P(x_1, y_1)$ be the middle point of the segment CD of the line AB intercepted between the axes. Draw $PM \perp OX$

$\therefore \quad OM = x_1$ and $MP = y_1$

$\therefore \quad OC = 2x_1$ and $OD = 2y_1$

Now, from the intercept form of the line

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1 \text{ or, } \quad \frac{x}{x_1} + \frac{y}{y_1} = 2$$

which is the required equation of the line.

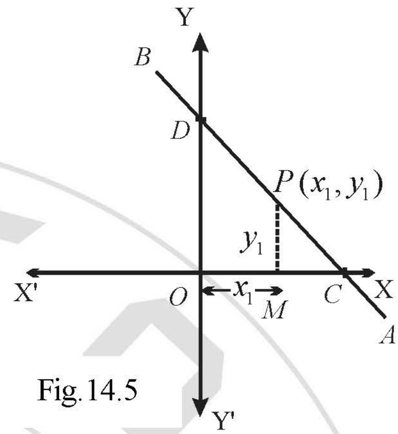

Fig. 14.5

(e) PERPENDICULAR FORM (NORMAL FORM)

We now derive the equation of a line when p be the length of perpendicular from the origin on the line and α, the angle which this perpendicular makes with the positive direction of x-a i x is are given.

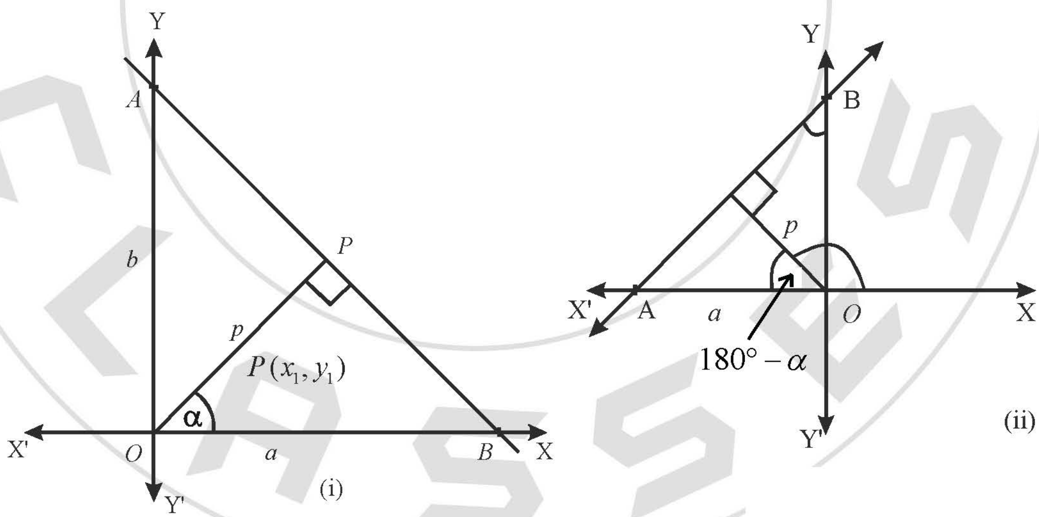

Fig. 14.6

(i) Let AB be the given line cutting off intercepts a and b on x–*axis* and y–*axis* respectively. Let OP be perpendicular from origin O on AB and $\angle POB = \alpha$ (See Fig. 14.6 (i))

$\therefore \quad \frac{p}{a} = \cos\alpha \Rightarrow a = p \sec\alpha$ and, $\frac{p}{b} = \sin\alpha \Rightarrow b = p \operatorname{cosec}\alpha$

$\therefore \quad$ The equation of line AB is

$$\frac{x}{p \sec\alpha} + \frac{y}{p \operatorname{cosec}\alpha} = 1$$

or, $x \cos \alpha + y \sin\alpha = p$

(ii) $\frac{p}{a} = \cos(180° - \alpha) = -\cos\alpha$ [From Fig. 14.6 (ii)]

$\Rightarrow$ $a = -p \sec \alpha$

similary, b $= p \operatorname{cosec} \alpha$

$\therefore$ The equation of the line AB is $\frac{x}{-a} + \frac{y}{b} = 1$ *or* $x \cos \alpha + y \sin \alpha = p$

> ***Note :*** *1. p is the length of perpendicular from the origin on the line and is always taken to be positive.*
>
> *2. α is the angle between positive direction of x-axis and the line perpendicular from the origin to the given line.*

Example 14.12 Determine the equation of the line with $\alpha = 135°$ and perpendicular distance $p = \sqrt{2}$ from the origin.

Solution : From the standard equation of the line in normal form have

$$x \cos 135° + y \sin 135° = \sqrt{2}$$

or, $-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$ or, $-x + y - 2 = 0$

or, $x - y + 2 = 0$, which is the required equation of the straight line.

Example 14.13 Find the equation of the line whose perpendicular distance from the origin is 6 units and the perpendicular from the origin to line makes an angle of $30°$ with the positive direction of x-axis.

Solution : Here $\alpha = 30°, p = 6$ $\therefore$ The equation of the line is, $x \cos 30° + y \sin 30° = 6$

or, $x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) = 6$ or, $\sqrt{3}\,x + y = 12$

14.3 GENERAL EQUATION OF FIRST DEGREE

You know that *a* linear equation in two variables x and y is given by $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ ...(1)

In order to understand its graphical representation, we need to take the following thres cases.

Case-1: (When both A and B are equal to zero)

In this case C is automaticaly zero and the equation does not exist.

Case-2: (When A = 0 and B $\neq$ 0)

In this case the equation (1) becomes By + C = 0.

or $y = -\frac{C}{B}$ and is satisfied by all points lying on a line which is parallel to x-axis and the y-coordinate of every point on the line is $-\frac{C}{B}$. Hence this is the equation of a straight line. The case where B = 0 and A $\neq$ 0 can be treated similarly.

Case-3: (When A $\neq$ 0 and B $\neq$ 0)

We can solve the equation (1) for y and obtain., $y = -\frac{A}{B}x - \frac{C}{B}$

Clearly, this represents a straight line with slope $-\frac{A}{B}$ and y – intercept equal to $-\frac{C}{B}$.

14.3.1 CONVERSION OF GENERAL EQUATION OF A LINE INTO VARIOUS FORMS

If we are given the general equation of a line, in the form Ax + By + C = 0, we will see how this can be converted into various forms studied before.

14.3.2 CONVERSION INTO SLOPE-INTERCEPT FORM

We are given a first degree equation in x and y as $Ax + By + C = O$

Are you able to find slope and y-intercept ?

Yes, indeed, if we are able to put the general equation in slope-intercept form. For this purpose, let us re-arrange the given equation as.

$Ax + By + C = 0$ as, $By = -Ax - C$

or $\quad y = -\frac{A}{B}x - \frac{C}{B}$ (Provided $B \neq 0$)

which is the required form. Hence, the slope $= -\frac{A}{B}$, y – intercept $= -\frac{C}{B}$.

Example 14.14 Reduce the equation $x + 7y - 4 = 0$ to the slope – intercept form.

Here find its slope and y intercept.

Solution : The given equation is, $x + 7y - 4 = 0$

or $\quad 7y = -x + 4$, or $y = -\frac{1}{7}x + \frac{4}{7}$

Here slope $= -1/7$ and y interecept $= 4/7$

14.3.3 CONVERSION INTO INTERCEPT FORM

Suppose the given first degree equation in x and y is $\quad Ax + By + C = 0. \quad$ ...(i)

In order to convert (i) in intercept form, we re arrange it as $Ax + By = -C$ or $\frac{Ax}{-C} + \frac{By}{-C} = 1$

or $\quad \frac{x}{(-\frac{C}{A})} + \frac{y}{(-\frac{C}{B})} = 1$ (Provided $A \neq 0$ and $B \neq 0$)

which is the requied converted form. It may be noted that intercept on x – axis $= \frac{-C}{A}$ and

intercept on y – axis $= \frac{-C}{B}$

Example 14.15 Reduce $3x + 5y = 7$ into the intercept form and find its intercepts on the axes.

Solution : The given equation is, $3x + 5y = 7$

or, $\quad \frac{3}{7}x + \frac{5}{7}y = 1$ or, $\quad \frac{x}{\frac{7}{3}} + \frac{y}{\frac{7}{5}} = 1$

$\therefore \quad$ The x– intercept $= \frac{7}{3}$ and, y – intercept $= \frac{7}{5}$

14.3.4 CONVERSION INTO PERPENDICULAR FORM

Let the general first degree equation in x and y be, $Ax + By + C = 0 \quad$ ... (i)

We will convert this general equation in perpendicular form. For this purpose let us re-write the given equation (i) as $Ax + By = -C$

Multiplying both sides of the above equation by λ, we have

$$\lambda Ax + \lambda By = -\lambda C \qquad \text{... (ii)}$$

Let us choose λ such that $(\lambda A)^2 + (\lambda B)^2 = 1$

or $\lambda = \frac{1}{\sqrt{(A^2 + B^2)}}$ (Taking positive sign)

Substituting this value of λ in (ii), we have

$$\frac{Ax}{\sqrt{(A^2 + B^2)}} + \frac{By}{\sqrt{(A^2 + B^2)}} = -\frac{C}{\sqrt{(A^2 + B^2)}} \quad \text{... (iii)}$$

This is required conversion of (i) in perpendicular form. Two cases arise according as C is negative or positive.

(i) If $C < 0$, the equation (ii) is the required form.

(ii) If $C > 0$, the R. H. S. of the equation of (iii) is negative.

$\therefore$ We shall multiply both sides of the equation of (iii) by -1.

$\therefore$ The required form will be $-\frac{Ax}{\sqrt{(A^2 + B^2)}} - \frac{By}{\sqrt{(A^2 + B^2)}} = \frac{C}{\sqrt{(A^2 + B^2)}}$

Thus, length of perpendicular from the origin $= \frac{|C|}{\sqrt{(A^2 + B^2)}}$

Inclination of the perpendicular with the positwe direction of x-ax is

is given by $\cos\theta = \mp \frac{A}{\sqrt{A^2 + B^2}}$

or $\sin\theta = \left(\mp \frac{B}{\sqrt{\left(A^2 B^2\right)}}\right)$

where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$. If $C = 0$, the line passes through the origin and there is no perpendicular from the origin on the line.

With the help of the above three cases, we are able to say that

"The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously."

Is the converse of the above statement true? ***The converse of the above statement is that every straight line can be expressed as a general equation of first degree in x and y.***

In this lesson we have studied about the various forms of equation of straight line. For example, let us take some of them as $y = mx + c$, $\frac{x}{a} + \frac{y}{b} = 1$ and $x \cos \alpha + y \sin \alpha = p$. Obviously, all are linear equations in x and y. We can re-arrange them as $y - mx - c = 0$, $bx + ay - ab = 0$ and $x \cos \alpha + y \sin \alpha - p = 0$ respectively. Clearly, these equations are nothing but a different arrangement of general equation of first degree in x and y. Thus, we have established that

"Every straight line can be expressed as a general equation of first degree in x and y".

Example 14.16 Reduce the equation $x + \sqrt{3}\,y + 7 = 0$ into perpendicular form.

Solution : The equation of given line is $x + \sqrt{3}y + 7 = 0$ ... (i)

Comparing (i) with general equation of straight line, we have, $A = 1$ and $B = \sqrt{3}$

$$\therefore \quad \sqrt{A^2 + B^2} = 2$$

Dividing equation (i) by 2, we have, $\frac{x}{2} + \frac{\sqrt{3}}{2}y + \frac{7}{2} = 0$

$$\text{or} \quad \left(-\frac{1}{2}\right)x + \left(-\frac{\sqrt{3}}{2}\right)y - \frac{7}{2} = 0 \quad \text{or} \quad x \cos\frac{4\pi}{3} + y \sin\frac{4\pi}{3} = \frac{7}{2}$$

($\cos\theta$ and $\sin\theta$ being both negative in the third quadrant, value of θ will lie in the third quadrant).

This is the representation of the given line in perpendicular form.

Example 14.17 Find the perpendicular distance from the origin on the line $\sqrt{3}\,x - y + 2 = 0$. Also, find the inclination of the perpendicular from the origin.

Solution : The given equation is $\sqrt{3}\,x - y + 2 = 0$

Dividing both sides by $\sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, we have

$$\frac{\sqrt{3}}{2}x - \frac{1}{2}y + 1 = 0 \text{ or, } \frac{\sqrt{3}}{2}x - \frac{1}{2}y = -1$$

Multiplying both sides by -1, we have, $-\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 1$

or, $x \cos\frac{5\pi}{6} + y \sin\frac{5\pi}{6} = 1$ ($\cos\theta$ is –ve in second quadrant and $\sin\theta$ is +ve in second quadrant, so value of θ lies in the second quadrant).

Thus, inclination of the perpendicular from the origin is 150° and its length is equal to 1.

Example 14.18 Find the equation of a line which passess through the point (3,1) and bisects the portion of the line $3x + 4y = 12$ intercepted between coordinate axes.

Solution : First we find the intercepts on coordinate axes cut off by the line whose equation is

$$3x + 4y = 12 \text{ or } \frac{3x}{12} + \frac{4y}{12} = 1 \text{ or } \quad \frac{x}{4} + \frac{y}{3} = 1$$

Hence, intercepts on x-axis and y-axis are 4 and 3 respectively.

Thus, the coordinates of the points where the line meets the coordinate axes are A (4, 0) and B (0, 3).

$\therefore$ Mid – point of AB is . $\left(2, \frac{3}{2}\right)$ is

Hence the equation of the line through (3,1) and

is and $\left(2, \frac{3}{2}\right)$ *is,* $y - 1 = \dfrac{\frac{3}{2} - 1}{2 - 3}(x - 3)$

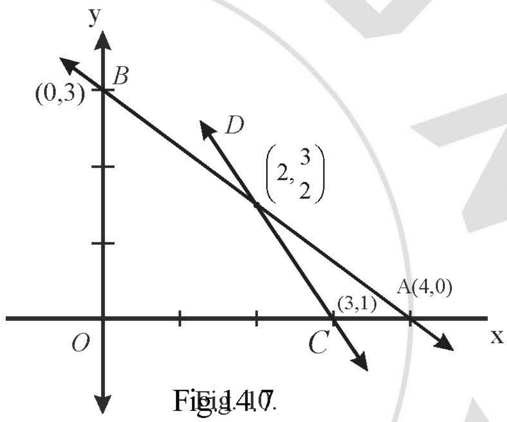

Fig. 14.7

or $\quad y - 1 = -\frac{1}{2}(x - 3)$

or $\quad 2(y - 1) + (x - 3) = 0$

or $\quad 2y - 2 + x - 3 = 0$, or $\quad x + 2y - 5 = 0$

Example 14.19 Prove that the line through (8, 7) and (6, 9) cuts off equal intercepts on coordinate axes.

Solution : The equation of the line passing through (8, 7) and (6, 9) is, $y - 7 = \dfrac{9 - 7}{6 - 8}(x - 8)$

or $\quad y - 7 = -(x - 8)$, or $\quad x + y = 15$

or $\quad \dfrac{x}{15} + \dfrac{y}{15} = 1$

Hence, intercepts on both axes are 15 each.

Example 14.20 Find the ratio in which the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8).

Solution : The equation of the line joining C (–5, 1) and D (1, –3) is

$$y-1=\frac{-3-1}{1+5}(x+5), \text{ or}$$

$$y-1=-\frac{4}{6}(x+5)$$

or $\quad 3y-3=-2x-10$, or $\quad 2x+3y+7=0$ ... (i)

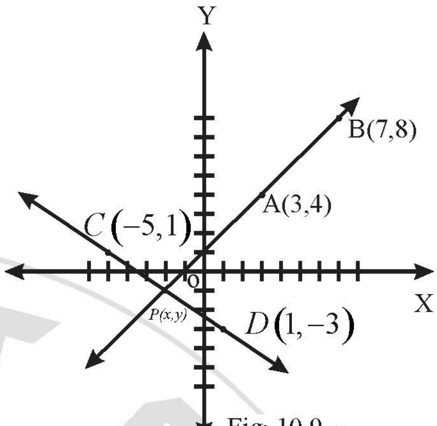

Fig.14.8

Let line (i) divide the join of A (3, 4) and B (7, 8) at the point P.

If the required ratio is $\lambda : 1$ in which line (i) divides the join of A (3, 4) and B (7, 8), then the coordinates of P are

$$\left(\frac{7\lambda+3}{\lambda+1}, \frac{8\lambda+4}{\lambda+1}\right)$$

Since P lies on the line (i), we have

$$2\left(\frac{7\lambda+3}{\lambda+1}\right)+3\left(\frac{8\lambda+4}{\lambda+1}\right)+7=0$$

$$\Rightarrow \quad 14\lambda+6+24\lambda+12+7\lambda+7=0, \Rightarrow \quad 45\lambda+25=0 \Rightarrow \lambda=-\frac{5}{9}$$

Hence, the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8) externally in the ratio 5 : 9.

14.4 DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

In this section, we shall discuss the concept of finding the distance of a given point from a given line or lines.

Let $P(x_1, y_1)$ be the given point and l be the line $Ax+By+C=0$.

Let the line l intersect x axis and y axis R and Q respectively.

Draw $PM \perp l$ and let $PM = d$.

Let the coordinates of M be (x_2, y_2)

$$d=\sqrt{\left\{(x_1-x_2)^2+(y_1-y_2)^2\right\}} \qquad ...(i)$$

$\because$ M lies on l, $\therefore$ $Ax_2 + By_2 + C = 0$ or $C = -(Ax_2 + By_2)$ ...(ii)

The coordinates of R and Q are $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$ respectively.

The slope of $QR = \dfrac{0 + \frac{C}{B}}{-\frac{C}{A} - 0} = -\dfrac{A}{B}$ and,

the slope of $PM = \dfrac{y_2 - y_1}{x_2 - x_1}$

As $PM \perp QR \Rightarrow \dfrac{y_2 - y_1}{x_2 - x_1} \times \left(-\dfrac{A}{B}\right) = -1.$ or $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{B}{A}$ ...(iii)

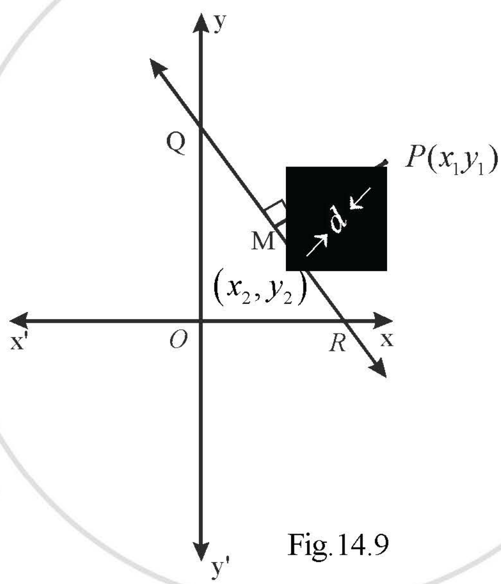

Fig.14.9

From (iii) $\dfrac{x_1 - x_2}{A} = \dfrac{y_1 - y_2}{B} = \dfrac{\sqrt{\left\{(x_1 - x_2) + (y_1 - y_2)^2\right\}}}{\sqrt{(A^2 + B^2)}}$ ...(iv)

(Using properties of Ratio and Proportion)

Also $\dfrac{x_1 - x_2}{A} = \dfrac{y_1 - y_2}{B} = \dfrac{A(x_1 - x_2) + B(y_1 - y_2)}{A^2 + B^2}$ ...(v)

From (iv) and (v), we get

$$\frac{\sqrt{\left\{(x_1-x_2)^2(y_1-y_2)^2\right\}}}{\sqrt{(A^2+B^2)}}=\frac{A(x_1-x_2)+B(y_1-y_2)}{A^2+B^2}$$

or $$\frac{d}{\sqrt{A^2+B^2}}+\frac{Ax_1+By_1-(Ax_2+By_2}{A^2+B^2}$$ [Using (i)]

or $$\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}$$ [Using (ii)]

Since the distance is always positive, we can write

$$d=\left|\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}\right|$$

***Note :** The perpendicular distance of the origin (0, 0) from Ax + By + C = 0 is*

$$\frac{A(0)+B(0)+C}{\sqrt{(A^2+b^2)}}=\frac{C}{\sqrt{(A^2+B^2)}}$$

Example 14.21 Find the points on the x-axis whose perpendicular distance from the straight line $\frac{x}{a}+\frac{y}{b}=1$ is a.

Solution : Let $(x_1, 0)$ be any point on x-axis.

Equation of the given line is $bx+ay-ab=0$. The perpendicular distance of the point $(x_1, 0)$ from the given line is, $a=\pm\frac{bx_1+a.0-ab}{\sqrt{(a^2+b^2)}}$ $\therefore$ $x_1=\frac{a}{b}\left\{b\pm\sqrt{(a^2+b^2)}\right\}$

Thus, the point on x-axis is $x_1=\left(\frac{a}{b}b\pm\sqrt{(a^2+b^2)},0\right)$

14.6 EQUATION OF PARALLEL (OR PERPENDICULAR) LINES

Till now, we have developed methods to find out whether the given lines are prallel or perpendicular. In this section, we shall try to find, the equation of a line which is parallel or perpendicular to a given line.

14.6.1 EQUATION OF A STRAIGHT LINE PARALLEL TO THE GIVEN LINE

$$\boldsymbol{Ax + By + c = 0}$$

Let $A_1x + B_1y + C_1 = 0$ ...(i)

be any line parallel to the given line, $Ax + By + C = 0$

The condition for parallelism of (i) and (ii) is ...(ii)

$$\frac{A_1}{A} = \frac{B_1}{B} = K_1 \quad \text{(say)} \Rightarrow \qquad A_1 = AK_1,\ B_1 = \text{B}K_1$$

with these values of A_1 and B_1, (i) gives

$$AK_1x + BK_1\,y + C_1 = 0 \quad \text{or} \qquad Ax + By + \frac{C_1}{K_1} = 0$$

or $\quad Ax + By + K = 0, \qquad$ where $K = \dfrac{C_1}{K_1}$ ... (iii)

This is a line parallel to the given line. From equations (ii) and (iii) we observe that

(i) coefficients of x and y are same

(ii) constants are different, and are to evaluated from given conditions.

Example 14.22 Find equation of the straight line, which passes through the point (1, 2) and which is parallel to the straight line $2x + 3y + 6 = 0$.

Solution : Equation of any straight line parallel to the given equation can be written if we put

(i) the coefficients of x and y as same as in the given equation.

(ii) constant to be different from the given equation, which is to be evaluated under given condition.

Thus, the required equation of the line will be, $2x + 3y + K = 0$ for some constant K

Since it passes through the point (1, 2) hence, $2 \times 1 + 3 \times 2 + K = 0$

or $\quad K = -8$

$\therefore$ Required equation of the line is $2x + 3y = 8$.

14.7 STRAIGHT LINE PERPENDICULAR TO THE GIVEN LINE

Ax + By + C = 0

Let $\quad A_1 x + B_1 y + C_1 = 0 \quad$ ... (i), be any line perpendicular to the given line

$$Ax + By + C = 0$$

Condition for perpendicularity of lines (i) and (ii) is ... (ii)

$$AA_1 + BB_1 = 0 \Rightarrow \frac{A_1}{B} = -\frac{B_1}{A} = K_1 \quad \text{(say)}$$

$\Rightarrow \quad A_1 = BK_1$ and $B_1 = -AK_1$

With these values of A_1 and B_1, (i) gives, $Bx - Ay + \frac{C_1}{K_1} = 0 = 0$

or $\quad Bx - Ay + K = 0$ where $K = \frac{C_1}{K_1}$ ... (iii)

Hence, the line (iii) is perpendicular to the given line (ii)

We observe that in order to get a line perpendicular to the given line we have to follow the following procedure : (i) Interchange the coefficients of x and y

(ii) Change the sign of one of them.

(iii) Change the Constant term to a new constant K (say), and evaluate it from given condition.

Example 14.23 Find the equation of the line which passes through the point (1, 2) and is perpendicular to the line $2x + 3y + 6 = 0$.

Solution : Following the procedure given above, we get the equation of line perpendicular to the given equation as $3x - 2y + K = 0$ ...(i)

(i) passes through the point (1, 2), hence

$3 \times 1 - 2 \times 2 + K = 0$ or $K = 1$

$\therefore$ Required equation of the straight line is $3x - 2y + 1 = 0$.

Example 14.24 Find the equation of the line which passes through the point (x_2, y_2) and is perpendicular to the straight line $y y_1 = 2a(x + x_1)$.

Solution : The given straight line is $yy_1 - 2ax - 2ax_1 = 0$ ...(i)

Any straight line perpendicular to (i) is $2ay + xy_1 + C = 0$

This passes through the point (x_2, y_2) $\therefore$ $2ay_2 + x_2 y_1 + C = 0$

$\Rightarrow$ $C = -2ay_2 - x_2y_1$

$\therefore$ Required equation of the spraight line is, $2a(y - y_2) + y_1(x - x_2) = 0$

14.8 EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES :

Let $l_1 : a_1x + b_1y + c_1 = 0$ ...(i)

and $l_2 : a_2x + b_2y + c_2 = 0$, be two intersecting lines.

Let P(h, k) be the point of intersection of l_1 and l_2, then

$$a_1h + b_1k + c_1 = 0 \quad ...(iii)$$

and $$a_2h + b_2k + c_2 = 0 \quad ...(iv)$$

Now consider the equation

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0 \quad ...(v)$$

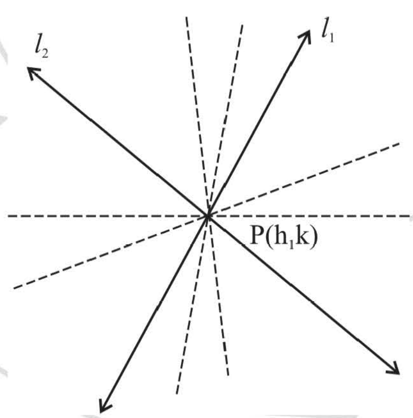

Fig.14.10

It is a first degree equation in x and y. So it will represent different lines for different values of λ. If we replace x by h and y by k we get

$$(a_1h + b_1k + c_1) + \lambda(a_2h + b_2k + c_2) = 0 \quad ...(vi)$$

using (iii) and (iv) in (vi) we get

$0 + \lambda 0 = 0$ i.e. $0 = 0$ which is true.

So equation (v) represents a family of lines passing through the point (h, k) i.e. the point of intersection of the given lines l_1 and l_2.

- A particular member of the family is obtained by giving a particular value to λ. This value of λ can be obtained from other given conditions.

Example 14.25 Find the equation of the line passing through the point of intersection of the lines $x + y + 1 = 0$ and $2x - y + 7 = 0$ and containing the point (1, 2).

Solution : Equation of family of lines passing through the intersection of given lines is $(x + y + 1) + \lambda\,(2x - y + 7) = 0$

This line will contain the point (1, 2) if

$(1 + 2 + 1) + \lambda\,(2 \times 1 - 1 \times 2 + 7) = 0$

i.e. $\quad 4 + 7\lambda = 0 \Rightarrow \lambda = -\dfrac{4}{7}.$

Therefore the equation of required line is, $(x + y + 1) - \dfrac{4}{7}(2x - y + 7) = 0$

i.e. $\quad 7(x + y + 1) - 4(2x - y + 7) = 0$ i.e. $-x + 11y - 21 = 0$

or $\quad x - 11y + 21 = 0$

Example 14.26 Find the equation of the line passing through the intersection of lines $3x + y - 9 = 0$ and $4x + 3y - 7 = 0$ and parallel to y-axis.

Solution : Equation of family of lines passing through the intersection of given lines is

$$(3x + y - 9) + \lambda(4x + 3y - 7) = 0, \text{ i.e. } (3 + 4\lambda)x + (1 + 3\lambda)y - (9 + 7\lambda) = 0 \quad \text{...(i)}$$

We know that if a line is parallel to y-axis then co-efficient of y in its equation must be zero.

$\therefore\ 1 + 3\lambda = 0 \Rightarrow \lambda = -1/3.$

Hence, equation of the required line is, $\left\{3 + 4\left(-\dfrac{1}{3}\right)\right\}x + 0y - \left\{9 + 7\left(\dfrac{-1}{3}\right)\right\} = 0$

i.e. $\quad x = 4$

LET US SUM UP

- The equation of a line parallel to y-axis is $x = a$ and parallel to x-axis is $y = b$.
- The equation of the line which cuts off intercept c on y-axis and having slope m is $y = mx + c$
- The equation of the line passing through $A(x_1, y_1)$ and having the slope m is $y - y_1 = m(x - x_1)$
- The equation of the line passing through two points $A(x, y_1)$ and $B(x_2, y_2)$ is
  $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
- The equation of the line which cuts off intercepts a and b on x – axis and y – axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$
- The equation of the line in normal or perpendicular form is $x \cos\alpha + y \sin\alpha = p$

  where p is the length of perpendicular from the origin to the line and α is the angle which this perpendicular makes with the positive direction of the x-axis.

- The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously.
- From general equation $Ax + By + C = 0$ we can evaluate the following :

  (i) Slope of the line $= -\frac{A}{B}$ (ii) x-intercept $= -\frac{C}{A}$ (iii) y-intercept $= -\frac{C}{B}$

  (iv) Length of perpendicular from the origin to the line $= \frac{|C|}{\sqrt{(A^2 + B^2)}}$

  (v) Inclination of the perpendicular from the origin is given by

  $$\cos\alpha = \frac{\mp A}{\sqrt{(A^2 + B^2)}} ; \sin\alpha = \frac{\mp B}{\sqrt{(A^2 + B^2)}}$$

  where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$; but if $C = 0$ then either only the upper sign or only the lower sign are taken.

- Distance of a given point (x_1, y_1) from a given line $Ax + By + C = 0$ is $d = \left|\frac{Ax_1 + By_1 + C}{\sqrt{(A^2 + B^2)}}\right|$
- Equation of a line parallel to the line $Ax + By + C = 0$ is $Ax + By + k = 0$
- Equation of a line perpendicular to the line $Ax + By + C = 0$ is $Bx - Ay + k = 0$
- Equation of a line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$