MISS

MATHEMATICAL INDUCTION SEQUENCES and SERIES

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Contents

This booklet contains eleven lectures on the topics:

Exercises on these topics are on the following pages:

Solutions to the exercises in this booklet are available at the Web-site:

`www-history.mcs.st-andrews.ac.uk/~john/MISS_solns/`

Mathematical induction

This is a method of "pulling oneself up by one's bootstraps" and is regarded with suspicion by non-mathematicians.

Example
Suppose we want to sum an *Arithmetic Progression*:

$$1+2+3+\ldots+n=\tfrac{1}{2}n(n+1).$$

Engineers' induction
Check it for (say) the first few values and then for one larger value — if it works for those it's bound to be OK.
Mathematicians are scornful of an argument like this — though notice that if it fails for some value there is no point in going any further.

Doing it more carefully:
We define a sequence of "propositions" $P(1)$, $P(2)$, ...
where $P(n)$ is "$1+2+3+\ldots+n=\tfrac{1}{2}n(n+1)$"
First we'll prove $P(1)$; this is called "*anchoring the induction*".
Then we will prove that **if $P(k)$ is true** for some value of k, then so is $P(k + 1)$; this is called "*the inductive step*".

Proof of the method
If $P(1)$ is OK, then we can use this to deduce that $P(2)$ is true and then use this to show that $P(3)$ is true and so on. So if n is the first value for which the result is false, then $P(n - 1)$ is true and we would get a contradiction. $\square$

So let's look hard at the above example.
$P(1)$ is certainly OK: $1=\tfrac{1}{2}\times 1\times 2$.
Now suppose that $P(k)$ is true for some value of k.
Then try and prove $P(k + 1)$:
Now $1 + 2 + 3 + \ldots + k + (k + 1) = \tfrac{1}{2}k(k + 1) + (k + 1)$ (using $P(k)$, which we are allowed to assume).
But this simplifies to $(k + 1)(\tfrac{1}{2}k + 1) = \tfrac{1}{2}(k + 1)(k + 2)$ and this is exactly what $P(k + 1)$ says.

Hence the result is true for all values. $\square$

Remarks
Of course, proving something by induction assumes that you know (by guesswork, numerical calculation, ...) what the result you are trying to prove is.

More examples

1. *Summing a Geometric Progression*
Let r be a fixed real number. Then
$1+r+r^2+r^3+\dots+r^n=\dfrac{1-r^{n+1}}{1-r}$. This is $P(n)$.
Proof
Clearly $P(0)$ is true. (Note that we can anchor the induction where we like.)
So we suppose that $P(k)$ is true and we'll try and prove $P(k+1)$.
So look at $\left(1+r+r^2+r^3+\dots+r^k\right)+r^{k+1}$.
By $P(k)$ the term in brackets is $\dfrac{1-r^{k+1}}{1-r}$ and so we can simplify this to
$$\frac{1-r^{k+1}}{1-r}+r^{k+1}=\frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}=\frac{1-r^{k+2}}{1-r}$$ which is what $P(k+1)$ predicts. $\square$

2. *Summing another series*
$1^2+2^2+\dots+n^2=\frac{1}{6}n(n+1)(2n+1)$
Proof
$P(1)$ is true, so we assume that $P(k)$ holds and work on $P(k+1)$.
$1^2+2^2+\dots+k^2+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$ using $P(k)$.
The RHS simplifies to
$(k+1)\left[\frac{1}{6}k(2k+1)+k+1\right]=\frac{1}{6}(k+1)(2k^2+7k+6)=$
$\frac{1}{6}(k+1)(k+2)(2k+3)$ which is what $P(k+1)$ says it should be. $\square$

Remark
For any positive integer s the sum $1^s+2^s+\dots+n^s$ is a polynomial of degree $s+1$ in n.
For example $1^7+2^7+\dots+n^7=\frac{1}{8}n^8+\frac{1}{2}n^7+\frac{7}{12}n^6-\frac{7}{24}n^4+\frac{1}{12}n^2$.

The Swiss mathematician Jacob Bernoulli (1654 – 1705) worked out a formula for all such sums using what are now called *Bernoulli numbers*.

3. *A result in Number Theory*

For any positive integer n we have $n^5 - n$ is divisible by 5.

n	1	2	3	4	...	12
$n^5 - n$	0	30	240	1020		248820

So it looks reasonable!

Proof

$P(1)$ is true.

So suppose $P(k)$ is true and we will try and prove $P(k + 1)$.

$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 =$

$(k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$

The first bracket on the RHS is divisible by 5 by $P(k)$ and the second is clearly a multiple of 5. Hence we have proved $P(k + 1)$. □

Remark

The above table suggests that in fact $n^5 - n$ is always divisible by 30.

Use the fact that $n^5 - n = (n-1)n(n+1)(n^2+1)$ to prove that it is divisible by 2 and 3 as well as 5.

4. *A result from Geometry*

The internal angles of an n-sided polygon total $(n - 2)\pi$ radians (or $180n - 360$ degrees).

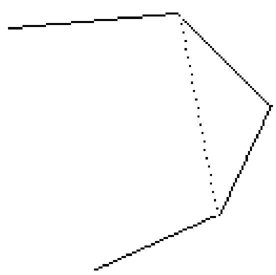
Proof

The result for $n = 3$ (a triangle) is classical.

So $P(3)$ is true.

Now assume $P(k)$ and we'll try and prove $P(k + 1)$.

Consider a $(k + 1)$-gon:

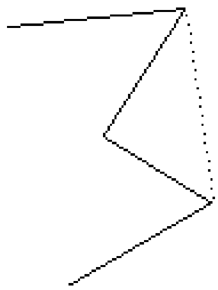


Join two "next but one" vertices as shown and we get a k-gon (with angle sum $(k - 2)\pi$ and an extra triangle – which gives an extra π.

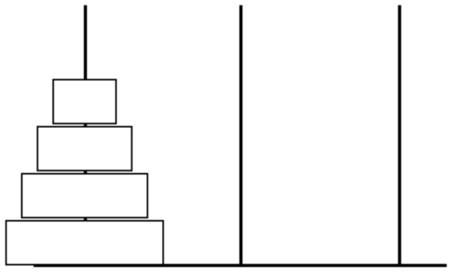
So the total of all the angles is $(k - 2)\pi + \pi =$

$\{(k + 1) - 2\}\pi$ which is what $P(k + 1)$ says it should be.

Exercise
The proof is incomplete, because when we join two vertices we might get a picture:
Do the inductive step in this case.



5. *The Towers of Hanoi*
This is a "game with three "towers". On the LH tower is a pile of n discs. One may move the discs one at a time – but may never put a larger disc on top of a smaller one. The object is to transfer all the discs to one of the other towers.

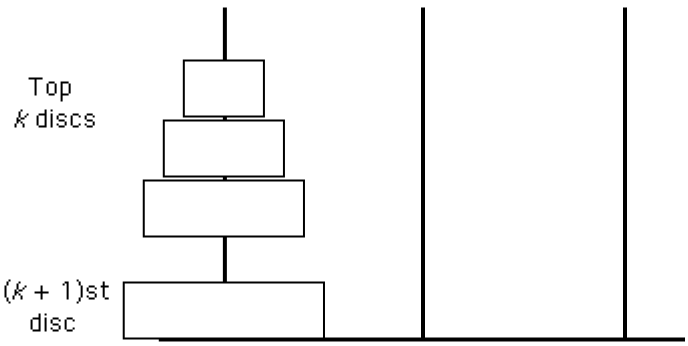


Remark
This game was invented by the French mathematician Edouard Lucas (1842 – 1891) who did his most important work in the theory of prime numbers.

Theorem
There is a solution with $2^n - 1$ moves.
Proof
We use induction on n. $P(1)$ is easy!
So to do the inductive step, we suppose we know how to do it with k discs.
Now let's try it with $k + 1$ discs.

a) Use the k-disc case and $2^k - 1$ moves to move the top k discs to the middle.

b) Use 1 move to move the bottom disc to the RH tower.



c) Use another $2^k - 1$ moves to
shift the k discs on the middle to the RH tower.

That's a total of $2(2^k - 1) + 1 = 2^{k+1} - 1$ moves just as $P(k + 1)$ predicts. □

Remark
By looking more carefully at this proof and noting that one can *only* move the bottom disc when all the others are stacked on one tower, one can prove that this is the *minimum* number of moves. By thinking carefully you should see how to do it in this number of moves.

6. *A result about inequalities*
For all integers $n > 4$ we have $n! > 2^n$
Proof
We anchor the induction at $P(4)$ (since $P(3)$ is false!).
So now assume that $P(k)$ holds: $k! > 2^k$ and look at $P(k+1)$.
Since $k + 1 > 2$ $(k+1)! = k! \times (k+1) > 2^k \times 2 = 2^{k+1}$ which is what we needed to prove.
$\square$

7. *A result from Integral Calculus*
For any integer $n \geq 0$ we have $\int_0^\infty x^n e^{-x} dx = n!$
Proof
Let the above statement be $P(n)$ and we start the induction at $n = 0$: $\int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1 = 0!$
So assume the above holds for some value of k and then look at $\int_0^\infty x^{k+1} e^{-x} dx$.
Integrate this by parts to get
$$\int_0^\infty x^{k+1} e^{-x} dx = \left[x^{k+1} \times (-e^{-x})\right]_0^\infty + \int_0^\infty (k+1) x^k e^{-x} dx =$$
$$0 + (k+1)\int_0^\infty x^k e^{-x} dx = (k+1)k!$$
by the inductive hypothesis and this is $(k+1)$! as $P(k+1)$ predicts. $\square$

8. *The Binomial Theorem*

We define symbols called *binomial coefficients*: $\binom{n}{r}$ (sometimes written ^nC_r or $_nC_r$) by the formula:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n}y^n$$

We shall prove that these satisfy $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$ for $r > 1$ and $\binom{n}{0} = 1$ for any n and so they fit into *Pascal's Triangle*:

n \ r	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Proof

Note that we are not using induction here.

We always have $\binom{n}{0} = 1$ since $(x+y)^n = 1x^n + \ldots$.

We'll work on $(x+y)^{n+1} = (x+y)(x+y)^n$.

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\left[\ldots + \binom{n}{r-1}x^{n-r+1}y^{r-1} + \binom{n}{r}x^{n-r}y^r + \ldots\right]$$

The coefficient of $x^{n-r+1}y^r$ on the LHS is $\binom{n+1}{r}$ while on the RHS (after multiplying out the brackets) it is $1\binom{n}{r-1} + 1\binom{n}{r}$ and so the result follows.

□

Now we will use induction on n to prove:

Theorem

For $0 \le r \le n$ the binomial coefficient

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.\ldots.r} = \frac{n!}{(n-r)!\,r!}$$

and is 0 for other values of r.

Proof

Take $P(n)$ to be the above statement and interpreting 0! as 1 the result holds for $k = 0$.

Then $\binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r} = \frac{k!}{(k-r+1)!(r-1)!} + \frac{k!}{(k-r)!r!}$ and collecting the terms on the RHS over the same denominator we get

$k!\left(\frac{r}{(k-r+1)!r!} + \frac{k-r+1}{(k-r+1)!r!}\right) = k!\left(\frac{k+1}{(k-r+1)!r!}\right) = \frac{(k+1)!}{(k-r+1)!r}$ which is what $P(k+1)$ says it should be. □

Exercises on Mathematical Induction

1. Use induction to show that the following series sums are valid for all $n \geq 1$.

(a) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$

(By coincidence this is the square of $\sum_{i=1}^{n} i$)

(b) $\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(c) $1 + 3 + 5 + \ldots + (2n-1) = n^2$

(d) $1 + 2x + 3x^2 + 4x^3 + \ldots + n\,x^{n-1} = \frac{1-(n+1)x^n + n\,x^{n+1}}{(1-x)^2}$

for real numbers $x \neq 1$.

2. Prove by induction that $13^n - 4^n$ is divisible by 9 for every integer $n \geq 1$.

3. Prove by induction that for every integer $n \geq 1$ and real number $x > -1$

$(1+x)^n \geq 1 + nx$.

4. Prove by induction that for every integer $n \geq 1$

$$\frac{d^n}{dx^n}(x\,e^{2x}) = 2^{n-1}(2x+n)e^{2x}.$$

5. Define a sequence $x_1, x_2, \ldots$ by $x_1 = 5$, $x_2 = 13$ and $x_{n+1} = 5x_n - 6x_{n-1}$ for $n \geq 2$. Prove by induction that

$$x_n = 2^n + 3^n.$$

6. Let A be the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. Prove that $A^n = \begin{pmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{pmatrix}$ for $n \geq 1$.

7. Prove that $\dfrac{1}{4.1^2-1} + \dfrac{1}{4.2^2-1} + \dfrac{1}{4.3^2-1} + \ldots + \dfrac{1}{4n^2-1} = \dfrac{n}{2n+1}$.

Sequences

One of the most important ideas in calculus (and indeed in mathematics) is the notion of an *approximation*. If we take an approximation (say a_1) to some fixed real number α and then a better approximation a_2 and so on we get a *sequence*

$$a_1, a_2, a_3, \ldots$$

of better and better approximations which "converges to α".

Writing this out formally:

Definition
A *real sequence* is an ordered set $(a_1, a_2, a_3, \ldots)$ of real numbers. We write this as (a_n) or (rarely) as $(a_n)_{n \in \mathbf{N}}$.

We say that the sequence (a_n) *converges to a limit* α if all the terms of the sequence eventually get close to α.

Then we write $(a_n) \to \alpha$ or $\lim\limits_{n \to \infty}(a_n) = \alpha$.
If a sequence does not converge to any limit we call it *divergent*.

Examples

1. The sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \to 0$. i.e. $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

2. The sequence defined by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for $n \geq 1$

   i.e. $\left(1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots\right)$ or (approximately) (1, 1.5, 1.41667, 1.4142156. ...) converges to the number $\sqrt{2} \approx 1.4142136\ldots$ though this is not obvious.

3. The sequence (0, 1, 0, 1, ...) *diverges*. It cannot converge to a real number α because then we would have to have α close to both 0 and 1.

4. The sequence (1, 2, 3, 4, ...) diverges also. Although we write $(a_n) \to \infty$ this does not mean that the sequence converges since ∞ is not a real number.

Remarks

For a sequence to converge to a finite limit α we insist that if we are given any "error" ε then *all* the terms far enough down the sequence approximate α with less that this error.

You will see more about this definition in later courses.

Example

If we take (say) $\varepsilon = 10^{-3}$ then all the terms after the 1000th in example 1. approximate 0 by better than this. In example 2. above, all terms after the 3rd approximate √2 by better than this

More sequence examples

5. Given a real number α with an infinite decimal expansion, define a_i to be the number we get by cutting off the expansion after i decimal places. Then the sequence $(a_i) \to \alpha$.

   e.g. (1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, ...) $\to \sqrt{2}$

6. The sequence (1, 1, 2, 3, 5, 8, 13, ...) defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$ is called the *Fibonacci sequence* (after Leonard "Fibonacci" of Pisa (1175 – 1250) who investigated it in connection with a problem involving rabbits).

   It is a divergent sequence.

However, the sequence of ratios of successive terms $\left(\frac{f_{n+1}}{f_n}\right)$ is

$\left(1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \ldots\right)$ which is approximately

(1, 2, 1.5, 1.6667, 1.6, 1.625, 1.6153, 1.6190, 1.6176, 1.6182, 1.6180, ...)

and this *does* converge (in fact to the number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803...$ which the Ancient Greeks called the *Golden Ratio*.)

Here if we take $\varepsilon = 10^{-3}$ then all the terms after the 8th are within ε of the limit ϕ.

Note that some of the terms are > ϕ and some are < ϕ. The sequence *oscillates* about its limit.

7. Define a sequence by $a_1 = 1$ and then $a_{n+1} = \cos(a_n)$.
   a) Use your calculator to find out what happens to this sequence. (Remember to set it to *radians*.)
      In fact this sequence *does* converge. What to ?
   b) Look at the same sequence but with sin instead of cos.
   c) The same sequence but with tan instead of cos.

8. The *Geometric Progression* $\left(1, r, r^2, r^3, \ldots\right)$ converges to 0 if | r | < 1 and diverges if | r | > 1. (It converges at r = 1 and diverges at r = –1.)

9. *The Compound Interest formula*

   If we invest at (say) 100% *per annum* interest and start with £1 then at the end of the year we have 1 + 1 = 2 pounds.

   (*Compound Interest* means that we would earn 2 pounds in the next year, and so on. Under *Simple Interest* we would get just 1 pound more the next year, and so on.)

   Now, if instead of waiting 12 months to calculate the interest, we "compounded" the interest after 6 months then we have $\left(1+\frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$ pounds at the end of the year.

   Compounding three times gives $\left(1+\frac{1}{3}\right)^3 = \frac{64}{27} \approx 2.37$ pounds and so on.

   Clearly, the more often you compound the more you earn. Do you become infinitely rich ?

Answer: No, because $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.718281828$

Remark

If the interest rate is r then compounding infinitely often will produce $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$ pounds at the end of the year

Some sequences are always increasing; some are always decreasing — others "jiggle about". The main result about sequences (which we don't have time to prove) is as follows.

Definition

A sequence (a_n) is *monotonic increasing* if $a_{n+1} \geq a_n$ for all n
(and *monotonic decreasing* if $a_{n+1} \leq a_n$ for all n).

Monotonic sequences theorem

A monotonic sequence which is bounded is convergent.

We will see some applications of this result later. We could use this to prove:

Arithmetic properties of sequences

If sequences $(a_n) \to \alpha$ and $(b_n) \to \beta$ then the following sequences converge.
$(a_n + b_n) \to \alpha + \beta$, $(a_n - b_n) \to \alpha - \beta$, $(a_n \times b_n) \to \alpha \times \beta$
and (provided that each b_i and $\beta \neq 0$) $\left(\frac{a_n}{b_n}\right) \to \frac{\alpha}{\beta}$.

Example

Find $\lim_{n\to\infty}\left(\frac{n+2}{2n+7}\right)$.

We have $\frac{n+2}{2n+7} = \frac{1+2/n}{2+7/n}$ and since $\lim_{n\to\infty}\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{7}{n}\right) = 0$ applying the results above gives the limit as $\frac{1+0}{2+0} = \frac{1}{2}$.

Exercises on sequences

1. Consider the sequences with nth term given below. Find the limit of each sequence as $n \to \infty$ if it exists.

(a) $\dfrac{7-4n^2}{3+2n^2}$ (b) $\dfrac{4}{8-7n}$ (c) $\dfrac{3^n+4^n}{4^n+5^n}$ (d) $\dfrac{4n+5}{8n+6}$

(e) -5 (f) $\dfrac{(-1)^{n+1}3}{n^2+4n+5}$ (g) $1+(-1)^{n+1}$ (h) $8-\left(\dfrac{7}{8}\right)^n$

(i) $\dfrac{n^2}{2n-1}-\dfrac{n^2}{2n+1}$ (j) $\sqrt{n+1}-\sqrt{n}$.

2. Find the limit of the sequence $\left(\dfrac{1}{n}\sin(n)\right)$.

Find the limit of the sequence $\left(n\sin\left(\dfrac{1}{n}\right)\right)$ (remember l'Hôpital).

Series

Series give us one of the most common ways of getting sequences.

Definition

A *real series* is something of the form:

$a_1 + a_2 + a_3 + \ldots$ with $a_i \in \mathbf{R}$.

It is often written $\sum_{i=1}^{\infty} a_i$ using the notation invented by the Swiss mathematician Leonhard Euler (1707 – 1783) where the $\mathbf{\Sigma}$ sign stands for "summation".

The a_i are the *terms* of the series.

From our point of view the important thing about a series is its *sequence of partial sums*:

$(a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots)$

which we can write as (σ_n) with $\sigma_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i$.

Definition

We say that a *series* $a_1 + a_2 + a_3 + \ldots$ *is convergent* if its sequence of partial sums is a convergent sequence and we write $\sum_{i=1}^{\infty} a_i$ for the limit of this sequence.

So $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty}(\sigma_n) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i\right)$ provided that this limit is a real number.

If a series is not convergent we call it *divergent*.

Note that the terms of series are separated by + signs; the terms of a sequence are separated by commas.

Some historical remarks.

Mathematicians have used series to perform calculations from the days of Archimedes of Syracuse (287 BC – 212 BC) onwards, but until quite late in the 19th Century they had little idea about how to handle them rigorously. The Norwegian mathematician Niels Abel (1802 – 1829) expressed their worries as follows.

> *If you disregard the very simplest cases, there is in all of mathematics not a single infinite series whose sum has been rigorously determined. In other words, the most important parts of mathematics stand without a foundation.*
>
> *The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes ... .*

These days we have a better understanding of how to deal with them!

Examples

1. The series 1 + 1 + 1 + ... diverges.

2. The *Geometric Series* $1 + r + r^2 + r^3 + \ldots$ converges if $|r| < 1$.

Proof

The partial sums $\sigma_n = 1 + r + r^2 + \ldots + r^{n-1} = \dfrac{1-r^n}{1-r}$ and as $n \to \infty$ we have $\left(\dfrac{1-r^n}{1-r}\right) \to \dfrac{1-0}{1-r} = \dfrac{1}{1-r}$ provided $|r| < 1$. □

3. The *Harmonic Series* $\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots\right)$ is *divergent*.

Proof

Group the terms as shown with two of them in the third group, then 4 in the next one and then 8 etc.

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \ldots >$$

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \ldots + \frac{1}{16}\right) + \ldots =$$

$1 + \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \ldots$ and this increases without limit. □

Remarks

(a) This is called the *Harmonic Series* because of the musical connection that Pythagoras (569 BC – 475 BC) found between reciprocals of integers and harmony.

(b) Jacob Bernoulli was the first to realise that this series diverges.

(c) In fact although this series diverges to ∞ it does so rather slowly. For example $\displaystyle\sum_{i=1}^{1\,000\,000} \frac{1}{i} \approx 14.5$.

To get the sum up to 20 you would need to take about 300 000 000 terms.

(d) If the *terms* of the series $\sum a_i$ do not form a sequence (a_i) which converges to 0 then the series does not converge. The Harmonic Series shows that the converse of this result is not true.

4. The series $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^s}$ converges for any real $s > 1$ and diverges if $s \le 1$

Remarks

For example, Leonard Euler showed in 1780 that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$.

The function $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$ can also be defined for complex values of s. This is the *Riemann ζ-function* which is very important in several areas of mathematics.

The main tool for deciding whether or not a series is convergent is the following result due to Jacob Bernoulli.

The Comparison Test for positive series

If $0 \leq a_i \leq b_i$ for all i and the series $\sum b_i$ converges then so does the series $\sum a_i$.

Proof

The sequence of partial sums of $\sum a_i$ is monotonic increasing since $a_i \geq 0$ and is bounded above by the limit of $\sum b_i$ and so is convergent by the theorem on monotonic sequences quoted above. $\square$

Corollary

If $0 \leq b_i \leq a_i$ for all i and the series $\sum b_i$ diverges then so does the series $\sum a_i$.

Remarks

We can summarise this by saying that for positive series anything less than a convergent series converges, while anything greater than a divergent series diverges.

To apply the comparison test you need a fund of "standard" series to compare them to.

Examples

1. Since $\sum \frac{1}{i}$ diverges, so does $\sum \frac{1}{i^s}$ for $s < 1$.

2. The series $\sum \frac{1}{i^2}$ converges.

Proof

Look at the series $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots$

The partial sum of this is

$$\sigma_n = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$$

(Compare this with Exercise 1(b) above.)

Hence the sequence of partial sums $(\sigma_n)\to 1$ and the sequence is convergent.

But the series $\frac{1}{2\times 2}+\frac{1}{3\times 3}+\frac{1}{4\times 4}\ldots \quad < \quad \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots$

and so is convergent by the comparison test. $\square$

Remarks

This is a special case of Example 4. above.

It follows that if $s \geq 2$ then $\sum_{i=1}^{\infty}\frac{1}{i^s}$ converges. Proving that this series converges if $1 < s < 2$ is harder.

3. The series $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots \quad < \quad 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

since $n! \geq 2^{n-1}$ for $n \geq 1$.

Since the series on the RHS is a convergent Geometric Series, the series on the LHS converges also (in fact to $e-1$)

In this last example we compared a series to a Geometric Series. If we do this with a general series we can deduce:

The Ratio Test for Positive Series

The positive series $\sum a_i$ *converges* if the ratio $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ exists and is < 1.

It *diverges* if the limit exists and is > 1 (or ∞).

If the limit $=1$ or fails to exist then the test gives no information.

Proof

Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l < r < 1$ for some r.

Then after some point k in this sequence all the terms are within $(r - l)$ of l and so are $< r$.

Then $a_{k+1} < r\, a_k$; $a_{k+2} < r\, a_{k+1} < r^2\, a_k$; ...

So comparing $a_k + a_{k+1} + a_{k+2} + \ldots$ with $a_k + r\, a_k + r^2\, a_k + \ldots$ gives convergence since the RH series is a convergent Geometric Series.

We may argue similarly to deduce divergence if $l > 1$. □

Examples

1. The series $\sum \frac{x^n}{n!}$ is convergent for any $x \geq 0$.

   (We have to take $x \geq 0$ since the Ratio Test is only for *positive* series.)

   Proof

   $a_n = \frac{x^n}{n!}$; $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$ $\Rightarrow$ $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0$ as $n \to \infty$. Hence the limit exists and is < 1 □

2. $\sum \frac{n^n}{n!}$ is a divergent series.

   Proof

   $a_n = \frac{n^n}{n!}$; $a_{n+1} = \frac{n^{n+1}}{(n+1)!}$ $\Rightarrow$ $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \left(1 + \frac{1}{n}\right)^n \to e$

   as $n \to \infty$. Hence the limit exists and is > 1.

3. The series $\sum \frac{1}{i^s}$ for various values of s sometimes converges and sometimes does not (see above)

   But for all these series the Ratio Test is inconclusive since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

   So the Ratio test is a "less powerful" test than the Comparison test.

We now look at series whose terms are not necessarily all positive. The main idea is:

Definition

A series $\sum a_n$ of (not necessarily positive) terms is called *absolutely convergent* if the series $\sum |a_n|$ is convergent.

Example

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum (-1)^{n+1} \frac{1}{n}$ is convergent but not absolutely convergent

Proof

We may write $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots =$

$\frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \ldots + \frac{1}{(2n-1) \times (2n)} + \ldots < \sum \frac{1}{(2n-1)^2} < \sum \frac{1}{n^2}$ and so this is convergent by the comparison test.

In fact this series converges to $\log_e 2 \approx 0.693$

But the series $\sum |a_n|$ is the Harmonic series which diverges.

Theorem

A series which is absolutely convergent is convergent.

Proof

We have $0 \le |a_i| + a_i \le 2|a_i|$ and so if the series $\sum |a_n|$ converges so does $\sum (|a_i| + a_i)$ by the comparison test.

Then the partial sum $\sigma_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (|a_i| + a_i) - \sum_{i=1}^{n} |a_i|$ and the sequences of partial sums on the RHS both converge. $\square$

Remark

Given a series $\sum a_n$ we may use the comparison test or ratio test on $\sum |a_n|$ to test for *absolute convergence*.

Example

The series $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ is absolutely convergent (by comparison with $\sum\frac{1}{n^2}$) and so is convergent.

For a series where the terms are alternately positive and negative we have the following result proved by the German mathematician Gottfried Leibniz (1646 – 1716).

Leibniz's test for alternating series

The series $\sum(-1)^n a_n$ with $a_n \geq 0$ is convergent if the *sequence* (a_n) is monotonic decreasing with limit 0.

Proof

$\sigma_{2n} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-1} - a_{2n}) =$
$\sigma_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n}$.
Since each $a_i - a_{i-1} \geq 0$ we have that (σ_{2n}) is a monotonic increasing sequence which (by the second line) is bounded above by a_1 . Hence it is convergent by the theorem on monotonic sequences,
Since $(a_n) \to 0$ we have (σ_{2n-1}) converges to the same limit and hence the sequence (σ_n) of partial sums converges. □

Examples

1. $\sum\frac{(-1)^n}{\sqrt{n}}$ is a convergent (but not absolutely convergent) series.
2. The sequence of terms of the series $\frac{1}{2}-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{3^2}+\frac{1}{4}-\frac{1}{4^2}+\ldots$ alternate in sign and converge to 0 but *not* monotonically, so we cannot apply Leibniz's test.
   And indeed: $\left(\frac{1}{2}-\frac{1}{2^2}\right)+\left(\frac{1}{3}-\frac{1}{3^2}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n^2}\right)+\ldots=\frac{1}{4}+\frac{2}{9}+\ldots+\frac{n-1}{n^2}+\ldots$
   which *diverges* by comparison with the series $\frac{1}{2}\sum\frac{1}{n}$.

Exercises on series

1. Determine whether or not the following Geometric series converge and find the sum if it exists.

(a) $3+\frac{3}{4}+\frac{3}{4^2}+\frac{3}{4^3}+\ldots$ (b) $\sum_{n=1}^{\infty} 2^{-n}3^{n-1}$

(c) $3+(x-1)+\frac{(x-1)^2}{3}+\frac{(x-1)^3}{3^2}+\ldots$

2. By looking at the nth term show that the following series are divergent.

(a) $\sum_{n=1}^{\infty}\frac{3n}{5n-1}$ (b) $\sum_{n=1}^{\infty}\log_e\left(\frac{2n}{7n-5}\right)$.

3. Use the Comparison Test to determine which of the following series are convergent and which are divergent.

(a) $\sum_{n=1}^{\infty}\frac{1}{n^4+n^2+1}$ (b) $\sum_{n=1}^{\infty}\frac{2+\cos n}{n^2}$ (c) $\sum_{n=1}^{\infty}\frac{1}{n3^n}$

(d) $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$ (e) $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n^2}\right)$.

4. Find $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ for each of the following series and use the Ratio Test to determine whether or not the series converges.

(a) $\sum_{n=1}^{\infty}\frac{3n+1}{2^n}$ (b) $\sum_{n=1}^{\infty}\frac{3^n}{n^2+4}$ (c) $\sum_{n=1}^{\infty}\frac{100^n}{n!}$ (d) $\sum_{n=1}^{\infty}\frac{n!}{e^n}$

5. Determine whether the following series are convergent using any appropriate test.

(a) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$ (b) $\sum_{n=1}^{\infty}\frac{2}{n^3+e^n}$ (c) $\sum_{n=1}^{\infty}3^{1/n}$

(d) $\frac{1}{2}+\frac{1\times 4}{2\times 4}+\frac{1\times 4\times 7}{2\times 4\times 6}+\frac{1\times 4\times 7\times 10}{2\times 4\times 6\times 8}+\frac{1\times 4\times 7\times 10\times 13}{2\times 4\times 6\times 8\times 10}+\ldots$

6. Determine whether or not the following Alternating series are convergent and absolutely convergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+7}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2/3}}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}5}{n^3+1}$.

Power Series

We now look at an important application of series.

Definition

A series of the form $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ with $c_n \in \mathbf{R}$ is called a *power series*.

We are interested when this represents a function on the real line for different values of x.

Theorem

If a power series is convergent at a point $x = r > 0$ then it is convergent at every point of the interval $(-r, r)$.

The maximum value of r for which the power series is convergent in the interval $(-r, r)$ is called the *radius of convergence* of the power series. and is written R.

Proof

Suppose the series is convergent at $x = r > 0$. Choose an x with $|x| < r$. Then we'll show that the series is absolutely convergent at x.

Since $\sum c_n r^n$ is convergent we have $\left|c_n r^n\right|$ is bounded (since if this became arbitrarily large then the series could not settle down to a finite value). Suppose $\left|c_n r^n\right| < M$ for all n. Then $\left|c_n\right| < \dfrac{M}{r^n}$ and so $\sum\left|c_n x^n\right| < \sum \dfrac{M}{r^n}|x|^n = M\sum\left|\dfrac{x}{r}\right|^n$ and the series on the right is a convergent Geometric series. Hence the power series is absolutely convergent by the comparison test.

$\square$

Remarks

So the power series defines a real-valued function at any point x in its *interval of convergence*. The power series may or may not be convergent at the ends $\pm R$ of this interval.

In fact a similar result holds for *complex* power series. The area in which the convergence takes place is then a circle of radius R. This explains the name.

We can usually use the Ratio test to find the radius of convergence of a power series.

Examples

1. $1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$.

   Using the ratio test for absolute convergence:
   $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| \to |x|$ as $n \to \infty$. So provided $|x| < 1$ the series is convergent and if $|x| > 1$ it diverges. Thus $R = 1$.

   Note that at $x = +R$ and $x = -R$ the series diverges.

2. $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots = \sum_{n=1}^{\infty} \frac{1}{n} x^n$

   Using the ratio test again $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}\left|\frac{x^{n+1}}{x^n}\right| = \frac{n}{n+1}|x| \to |x|$ as $n \to \infty$.

   Again the series converges provided $|x| < 1$ and we have $R = 1$.

   This time the series converges at $x = -R$, but not at $x = +R$.

3. $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ (with 0! interpreted as 1).

   $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!}\left|\frac{x^{n+1}}{x^n}\right| = \frac{1}{n+1}|x| \to 0$ as $n \to \infty$ for any x. Hence the series is convergent for *all* x and we say it has radius of convergent $R = \infty$.

Exercises on power series

1. Determine for which x the following power series converge. (Remember to check for convergence at the ends of the interval of convergence.)

   (a) $\sum_{n=0}^{\infty} \frac{x^n}{n^2+4}$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{4^n \sqrt{n}}$ (c) $\sum_{n=0}^{\infty} \frac{10^n}{n!} x^n$.

Taylor Series

In the above we were able to use a power series to define a function. Conversely, given a (nice-enough) function we may be able to find a power series to represent it.

Definition

If a function $f(x)$ on an interval of the real line is the limit of a power series $\sum_{n=0}^{\infty} c_n x^n$ then this is called the *Taylor Series* or *Maclaurin Series* of *f*.

Theorem

If *f* has a Taylor series then we may write it as

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \ldots = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

Proof

If $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

then it is reasonable to suppose:

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$$
$$f''(x) = 2c_2 + 3.2c_3 x + 4.3c_4 x^2 + 5.4c_5 x^3 + \ldots$$
$$f'''(x) = 3.2c_3 + 4.3.2c_4 x + 5.4.3c_5 x^2 + \ldots$$
$$\ldots$$
$$f^{(n)}(x) = n!c_n + n!c_{n-1} x + \ldots$$

So putting $x = 0$ in the above we get $f(0) = c_0, f'(0) = c_1, \ldots, f^{(n)}(0) = n!c_n$ and so $c_n = \frac{1}{n!} f^{(n)}(0)$ as required. □

Remarks

a) The English mathematician Brook Taylor (1685 – 1731) published the following more general result in 1715.

$$f(a+x) = f(a) + f'(a)x + \frac{1}{2!}f''(a)x^2 + \frac{1}{3!}f'''(a)x^3 + \ldots = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{x^n}{n!}$$

or $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \ldots$

The previous result (which is the case of a Taylor Series about $a = 0$) was proved independently by the Scottish mathematician Colin Maclaurin (1698 - 1746).

b) To prove this result "properly" one needs to find a way of estimating "the error" in the approximation. Otherwise one cannot be sure that differentiating the above series "term by term" is justified.

c) Note that only really "nice" functions have Taylor Series. For example, the function $f(x) = |\, x \,|$ is not differentiable at the origin and so does not have a Taylor Series about 0.

Examples

1. Let $f(x) = \exp(x) = e^x$. Then $f'(x) = e^x, \quad f''(x) = e^x, \quad \ldots$

   and so the Taylor Series about 0 of exp(x) is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

   This series has radius of convergence ∞ and converges to e^x at any value of x.

2. Let $f(x) = \sin(x)$. Then

   $f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x), \quad \ldots$

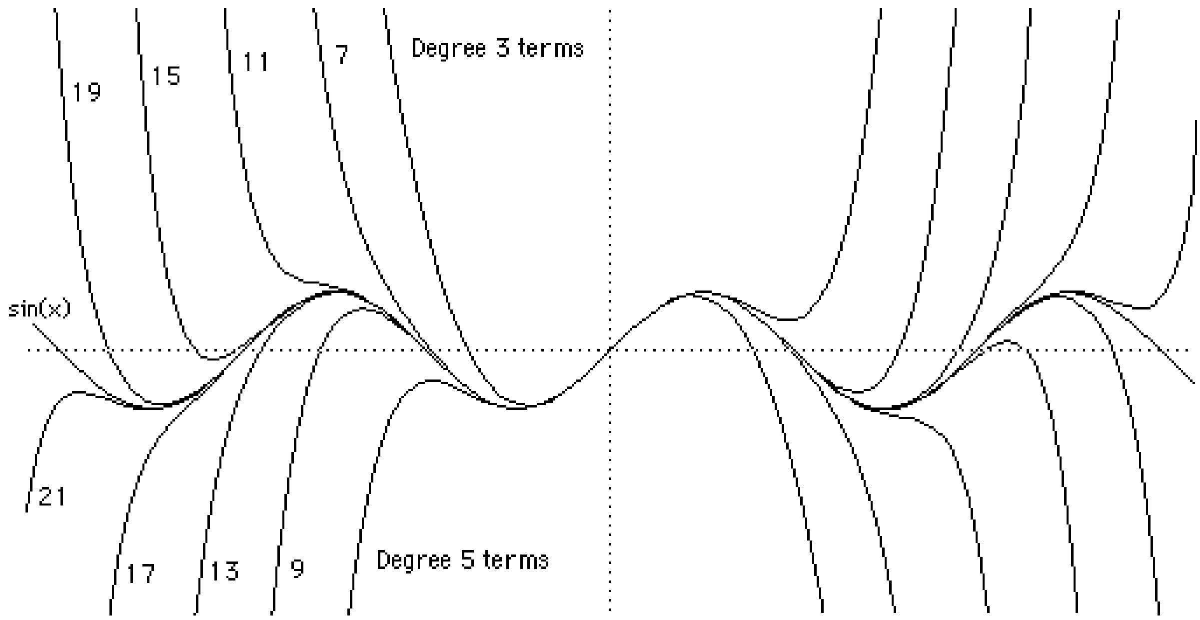
   So the Taylor Series about 0 of sin(x) is

   $$\sin(x) = 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times (-1) + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

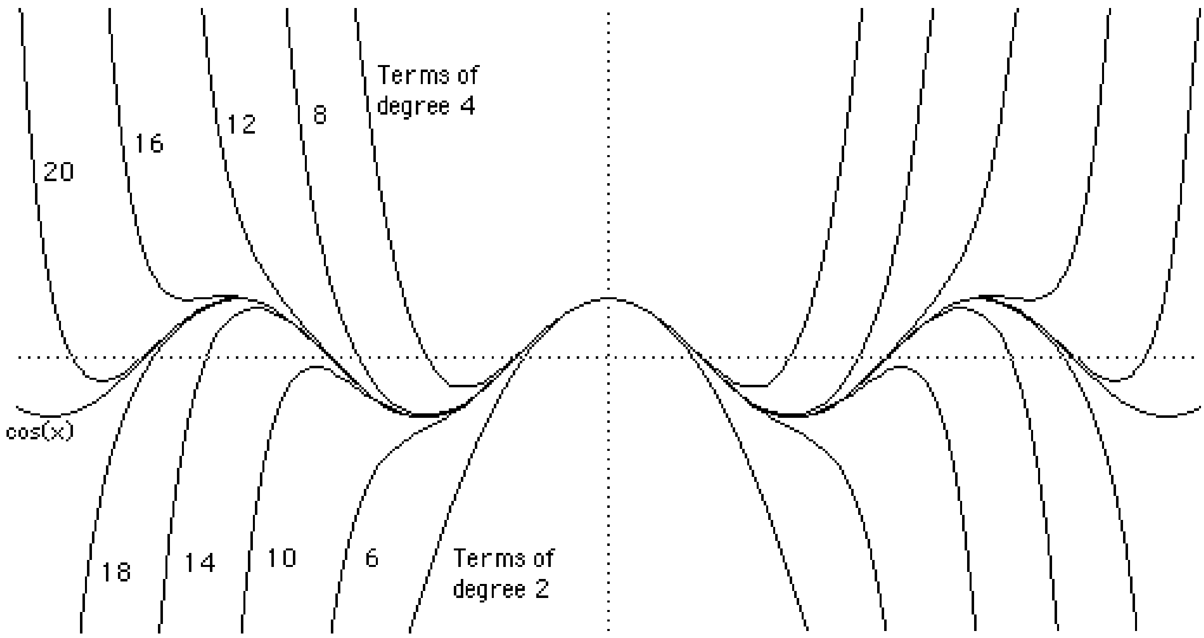
   This series has radius of convergence ∞ and converges to sin(x) at *any* value of x.

However notice that the value of sin(x) always lies between ±1 while the partial sums of the Taylor Series are polynomials and are unbounded on the Real line.

Here is a picture of the graph of sin(x) and of some of the polynomials obtained by looking at partial sums of its Taylor Series.



3. Similarly $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$



4. Let $f(x) = \tan(x)$. Then
$f'(x) = \cos^{-2}(x)$, $f''(x) = 2\cos^{-3}(x)\sin(x)$, $f'''(x) = 6\cos^{-4}(x)\sin^2(x) + 2\cos^{-2}(x)$, ... (and it gets worse!) So the Taylor Series about 0 starts:

$$\tan(x) = 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2 + ... = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + ...$$

This shows the limitations of the method! The coefficients of the odd powers of x depend on the Bernoulli numbers we met earlier.

5. Let $f(x) = \dfrac{1}{1-x}$. Then
$f'(x) = \dfrac{1}{(1-x)^2}$, $f''(x) = \dfrac{2}{(1-x)^3}$, $f'''(x) = \dfrac{2 \times 3}{(1-x)^4}$, ... and so
$f(x) = \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ and we knew that already!
Notice that this Taylor Series has radius of convergence $R = 1$ and converges only on the interval (–1, 1).

6. Let $f(x) = -\log_e(1-x)$. Then
$f'(x) = \dfrac{1}{1-x}$, $f''(x) = \dfrac{1}{(1-x)^2}$, $f'''(x) = \dfrac{2}{(1-x)^3}$, ... and so

$$-\log_e(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + ...$$

Remarks

(a) This is sometimes known as the *Mercator Series* after the Danish-born mathematician Nicolas Mercator (1620 – 1687) who published it in 1668. It was discovered independently by the more famous English mathematician Isaac Newton (1643 –1727) among others.

(b) You can get it by integrating the previous example term by term – if you know that that is allowed!

(c) If you put $x = -1$ in this series you get the (correct) result:
$\log_e 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + ...$ though that is rather hard to prove rigorously!

7. Replace x by $-x^2$ in example 4. to get $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ and integrate this to get $\tan^{-1}(x) = \arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$

Then put $x = 1$ in this to get $\arctan(1) = \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

Remarks

(a) You will need quite a bit more mathematics before you can justify all that!

(b) This is called *Gregory's Series* after the Scottish mathematician James Gregory (1638 – 1675) who discovered it in 1671. He was the first Regius professor of Mathematics at St Andrews and also invented the first reflecting telescope and built an astronomical observatory in St Andrews.

(c) Gregory's series converges too slowly to be a useful way of calculating π directly. A mathematician called Abraham Sharp (1653 – 1742) used the above series for arctan(x) with $x = \dfrac{1}{\sqrt{3}}$ to calculate about 72 decimal places of π in 1699.

Exercises on Taylor series

1. Find the Taylor series (about 0) of the following functions.

   (a) $f(x) = e^{3x}$ (b) $f(x) = \sin 2x$ (c) $f(x) = \dfrac{1}{1-2x}$.

2. Find the following Taylor series about the point a.

   (a) $f(x) = \sin x$ at $a = \dfrac{\pi}{4}$ (b) $f(x) = \dfrac{1}{x}$ at $a = 1$.

3. Find the following Taylor Series as far as the term in x^2.

   (a) $f(x) = \tan x$ at $a = \dfrac{\pi}{4}$ (b) $f(x) = \sin^{-1} x = \arcsin(x)$ at $a = \dfrac{1}{2}$.

Summary

Mathematical Induction

To prove that a proposition $P(n)$ holds for all $n \in \mathbf{N}$ first prove that $P(1)$ is true. Then prove that **if** $P(k)$ is true **then** $P(k + 1)$ is also true.

Convergence of sequences

The real sequence $(a_1, a_2, a_3, \ldots)$ *converges to a limit* α if all the terms of the sequence eventually get close to α.

Convergence of series

A series $\sum_{i=1}^{\infty} a_i$ *is convergent* if its sequence of partial sums forms a convergent sequence.

The Geometric series $\sum_{n=0}^{\infty} \frac{1}{r^n}$ is convergent if $|r| < 1$ and divergent if $|r| > 1$.

The series $\sum_{n=0}^{\infty} \frac{1}{n^k}$ is convergent if $k > 1$ and is divergent if $k \leq 1$.

Tests for *positive* series

Comparison: If $0 \leq a_i \leq b_i$ for all i and $\sum a_i$ converges then so does $\sum b_i$.

Ratio: $\sum a_i$ *converges* if the ratio $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ exists and is < 1. It *diverges* if the limit exists and is > 1 (or ∞).

Leibniz's test for alternating series

If the terms of $\sum a_i$ alternate in sign and $\lim_{n \to \infty} a_n = 0$ *monotonically* then the series converges.

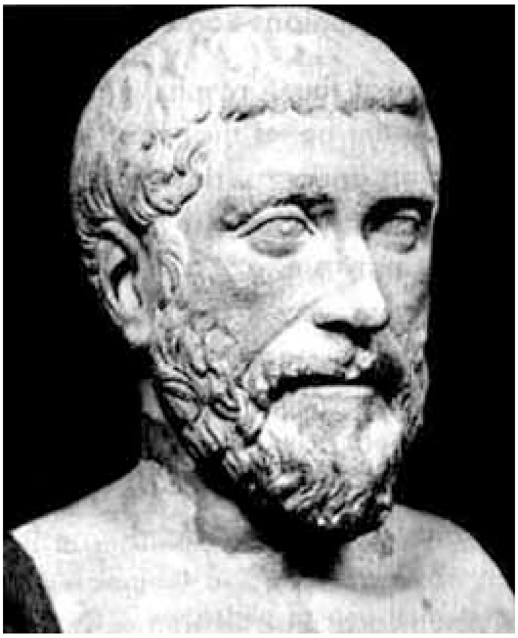
Power series

The maximum value of r for which the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent in the interval $(-r, r)$ is called the *radius of convergence* of the power series and is written R. It may (often) be calculated using the Ratio test.

Taylor Series

If a function $f(x)$ can be represented on an interval by a power series $\sum_{n=0}^{\infty} c_n x^n$ then this is its Taylor series (about 0) and for many functions $c_n = \frac{1}{n!} \frac{d^n f}{dx^n}(0)$.

Some of the mathematicians mentioned in these lectures

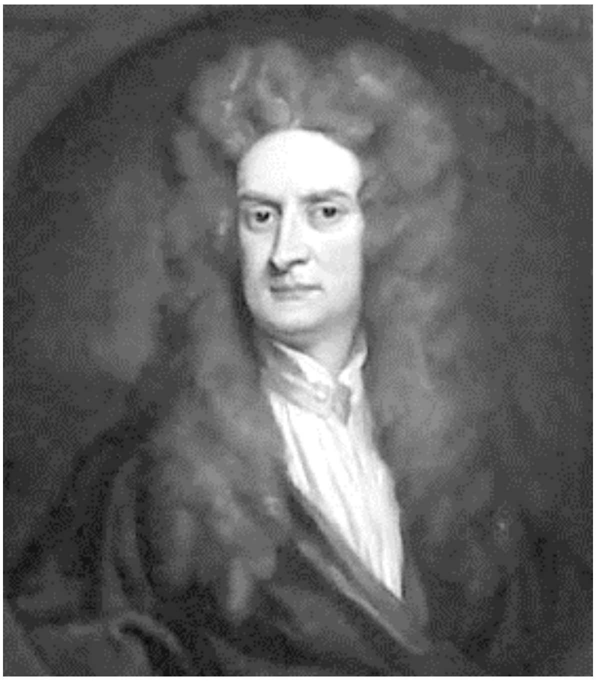


Pythagoras
569BC – 475BC
Born: Samos, Greece

Archimedes
287BC – 212BC
Born: Syracuse, Sicily, Italy

Fibonacci of Pisa
1175 – 1250
Born: Pisa, Italy

James Gregory
1638 – 1675
Born: Aberdeen, Scotland



Isaac Newton
1643 – 1727
Born: Grantham, England

Gottfried von Leibniz
1646 – 1716
Born: Leipzig, Germany

Jacob Bernoulli
1654 – 1705
Born: Basel, Switzerland

Brooke Taylor
1685 – 1731
Born: Middlesex, England

Colin Maclaurin
1698 – 1746
Born: Kilmodan, Scotland

Leonard Euler
1707 – 1783
Born: Basel, Switzerland

Niels Abel
1802 – 1829
Born: Stavanger, Norway

Edouard Lucas
1842 – 1891
Born: Amiens, France