R.K.MALIK'S
NEWTON CLASSES
JEE (MAIN & ADV.), MEDICAL + BOARD, NDA, IX & X

# CHAPTER 3 : TRIGONOMETRIC FUNCTIONS-II

In the previous lesson, you have learnt trigonometric functions of real numbers, drawn and interpretd the graphs of trigonometric functions. In this lesson we will establish addition and subtraction formulae for $\cos(A \pm B)$, $\sin(A \pm B)$ and $\tan(A \pm B)$. We will also state the formulae for the multiple and sub multiples of angles and solve examples thereof. The general solutions of simple trigonometric functions will also be discussed in the lesson.

## OBJECTIVES

After studying this lesson, you will be able to :

- write trigonometric functions of $-x, \frac{x}{2}, x \pm y, \frac{\pi}{2} \pm x, \pi \pm x$ where x, y are real nunbers;
- establish the addition and subtraction formulae for :

  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$,

  $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- solve problems using the addition and subtraction formulae;
- state the formulae for the multiples and sub-multiples of angles such as cos2A, sin 2A, tan 2A, cos 3A, sin 3A, tan 3A, $\sin\frac{A}{2}, \cos\frac{A}{2}$ and $\tan\frac{A}{2}$; and
- solve simple trigonometric equations of the type :

  $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, $\tan x = \tan\alpha$

## EXPECTED BACKGROUND KNOWLEDGE

- Definition of trigonometric functions.
- Trigonometric functions of complementary and supplementary angles.
- Trigonometric identities.

## 4.1 ADDITION AND MULTIPLICATION OF TRIGONOMETRIC FUNCTIONS

In earlier sections we have learnt about circular measure of angles, trigonometric functions, values of trigonometric functions of specific numbers and of allied numbers.

You may now be interested to know whether with the given values of trigonometric functions of any two numbers $A$ and $B$, it is possible to find trigonometric functions of sums or differences.

You will see how trigonometric functions of sum or difference of numbers are connected with those of individual numbers. This will help you, for instance, to find the value of trigonometric functions of $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ etc.

$\frac{\pi}{12}$ can be expressed as $\frac{\pi}{4}-\frac{\pi}{6}$ and $\frac{5\pi}{12}$ can be expressed as $\frac{\pi}{4}+\frac{\pi}{6}$

How can we express $\frac{7\pi}{12}$ in the form of addition or subtraction?

In this section we propose to study such type of trigonometric functions.

### 4.1.1 Addition Formulae

For any two numbers A and B,

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

In given figure trace out

$\angle SOP = A$

$\angle POQ = B$

$\angle SOR = -B$

where points P, Q, R, S lie on the unit circle.

Coordinates of P, Q, R, S will be $(\cos A, \sin A)$,

$[\cos(A+B), \sin(A+B)]$,

$[\cos(-B), \sin(-B)]$, and $(1, 0)$.

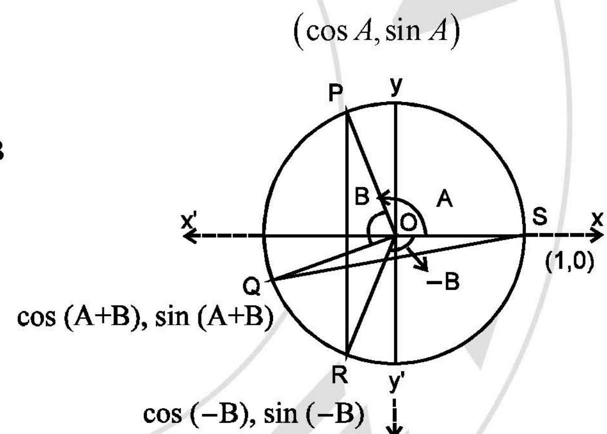


**Fig. 4.1**

From the given figure, we have

side OP = side OQ, $\angle POR = \angle QOS$ (each angle $= \angle B + \angle QOR$), side OR = side OS

$\Delta POR \cong \Delta QOS$ (by SAS) $\therefore$ PR = QS

$$PR=\sqrt{(\cos A-\cos(-B))^2+(\sin A-\sin(-B))^2}$$

$$QS=\sqrt{(\cos(A+B)-1)^2+(\sin(A+B)-0)^2}$$

Since $PR^2 = QS^2 \therefore \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B + 2\sin A \sin B$

$$= \cos^2(A+B) + 1 - 2\cos(A+B) + \sin^2(A+B)$$

$\Rightarrow \quad 1 + 1 - 2(\cos A \cos B - \sin A \sin B) = 1 + 1 - 2\cos(A+B)$

$\Rightarrow \quad \cos A \cos B - \sin A \sin B = \cos(A+B)$ (I)

**Corollary 1**

For any two numbers A and B, cos (A – B) = cos A cos B + sin A sin B

**Proof :** Replace B by $-B$ in (I)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$[\because \cos(-B) = \cos B$ and $\sin(-B) = -\sin B]$

**Corollary 2**

For any two numbers A and B, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

**Proof :** We know that $\cos\left(\frac{\pi}{2} - A\right) = \sin A$ and $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$$\therefore \quad \sin(A+B) = \cos\left[\left(\frac{\pi}{2} - (A+B)\right)\right] = \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

or $\quad \sin(A+B) = \sin A \cos B + \cos A \sin B$ .....(II)

**Corollary 3**

For any two numbers A and B, $\sin(A-B) = \sin A \cos B - \cos A \sin B$

**Proof :** Replacing B by $-B$ in (2), we have

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

or $\quad \sin(A-B) = \sin A \cos B - \cos A \sin B$

**Example 4.1**

(a) Find the value of each of the following :

(i) $\sin\frac{5\pi}{12}$ (ii) $\cos\frac{\pi}{12}$ (iii) $\cos\frac{7\pi}{12}$

(b) If $\sin A = \frac{1}{\sqrt{10}}, \sin B = \frac{1}{\sqrt{5}}$ show that $A + B = \frac{\pi}{4}$

**Solution :**

(a) (i) $\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Observe that $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$

(iii) $\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$$= \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(b) $\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos A = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}}$ and $\cos B = \sqrt{1-\frac{1}{5}} = \frac{2}{\sqrt{5}}$

Substituting all these values in the above formula, we get

$$\sin(A + B) = \frac{1}{\sqrt{10}}\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} + \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \text{ or } A + B = \frac{\pi}{4}$$

**Corollary 4 :** $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

**Proof :** $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}=\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$

Dividing by cos A cos B, we have

$$\tan(A+B)=\dfrac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

or $$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B} \qquad \text{......(III)}$$

**Corollary 5 :** $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

**Proof :** Replacing B by $-B$ in (III), we get the required result.

**Corollary 6 :** $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

**Proof :** $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$

Dividing by sin A sin B, we have ......(IV)

$$\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$$

**Corollary 7 :** $\tan\left(\dfrac{\pi}{4}+A\right)=\dfrac{1+\tan A}{1-\tan A}$

**Proof :** $$\tan\left(\dfrac{\pi}{4}+A\right)=\dfrac{\tan\dfrac{\pi}{4}+\tan A}{1-\tan\dfrac{\pi}{4}\cdot\tan A}=\dfrac{1+\tan A}{1-\tan A} \text{ as } \tan\dfrac{\pi}{4}=1$$

Similarly, it can be proved that $\tan\left(\dfrac{\pi}{4}-A\right)=\dfrac{1-\tan A}{1+\tan A}$

**Example 4.2** Find $\tan\frac{\pi}{12}$

**Solution :** $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \dfrac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}} = \dfrac{1-\frac{1}{\sqrt{3}}}{1+1.\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$$= \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

$\therefore \qquad \tan\frac{\pi}{12} = 2-\sqrt{3}$

**Example 4.3** Prove the following :

(a) $\dfrac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}} = \tan\frac{4\pi}{9}$

(b) $\tan 7A - \tan 4A - \tan 3A = \tan 7A\ \tan 4A \cdot \tan 3A$

**Solution :** (a) Dividing numerator and denominator by $\cos\frac{7\pi}{36}$, we get

$$\text{L.H.S.} = \frac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}} = \frac{1+\tan\frac{7\pi}{36}}{1-\tan\frac{7\pi}{36}} = \frac{\tan\frac{\pi}{4}+\tan\frac{7\pi}{36}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{7\pi}{36}}$$

$$= \tan\left(\frac{\pi}{4}+\frac{7\pi}{36}\right) = \tan\frac{16\pi}{36} = \tan\frac{4\pi}{9} = \text{R.H.S.}$$

(b) $\tan 7A = \tan(4A+3A) = \dfrac{\tan 4A + \tan 3A}{1-\tan 4A\ \tan 3A}$

or $\quad \tan 7A - \tan 7A\ \tan 4A\ \tan 3A = \tan 4A + \tan 3A$

or $\quad \tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \tan 3A$

## 4.2 TRANSFORMATION OF PRODUCTS INTO SUMS AND VICE VERSA

### 4.2.1 Transformation of Products into Sums or Differences

We know that $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

By adding and subtracting the first two formulae, we get respectively

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B) \quad .....(1)$$

and $$2 \cos A \sin B = \sin(A+B) - \sin(A-B) \quad .....(2)$$

Similarly, by adding and subtracting the other two formulae, we get

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B) \quad ....(3)$$

and $$2 \sin A \sin B = \cos(A-B) - \cos(A+B) \quad ....(4)$$

We can also quote these as

$$2 \sin A \cos B = \sin(\text{sum}) + \sin(\text{difference})$$

$$2 \cos A \sin B = \sin(\text{sum}) - \sin(\text{difference})$$

$$2 \cos A \cos B = \cos(\text{sum}) + \cos(\text{difference})$$

$$2 \sin A \sin B = \cos(\text{difference}) - \cos(\text{sum})$$

### 4.2.2 Transformation of Sums or Differences into Products

In the above results put

$$A+B = C \text{ and } A-B = D$$

Then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$ and (1), (2), (3) and (4) become

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

### 4.2.3 Further Applications of Addition and Subtraction Formulae

We shall prove that (i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$ or $\cos^2 B - \sin^2 A$

**Proof :** (i) $\sin(A+B)\sin(A-B)$

$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B = \cos^2 A - \sin^2 B$

$= (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$

**Example 4.4** Express the following products as a sum or difference

(i) $2\sin 3\theta\cos 2\theta$ (ii) $\cos 6\theta\cos\theta$ (iii) $\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

**Solution :**

(i) $2\sin 3\theta\cos 2\theta = \sin(3\theta+2\theta) + \sin(3\theta-2\theta) = \sin 5\theta + \sin\theta$

(ii) $\cos 6\theta\cos\theta = \frac{1}{2}(2\cos 6\theta\cos\theta) = \frac{1}{2}[\cos(6\theta+\theta) + \cos(6\theta-\theta)]$

$= \frac{1}{2}(\cos 7\theta + \cos 5\theta)$

(iii) $$\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \frac{1}{2}\left[2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{5\pi-\pi}{12}\right) - \cos\left(\frac{5\pi+\pi}{12}\right)\right] = \frac{1}{2}\left[\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right]$$

**Example 4.5** Express the following sums as products.

(i) $\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9}$ (ii) $\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36}$

**Solution :**

(i) $$\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 2\cos\frac{5\pi+7\pi}{9\times 2}\cos\frac{5\pi-7\pi}{9\times 2}$$

$$= 2\cos\frac{2\pi}{3}\cos\frac{\pi}{9} \quad \left[\because \cos\left(-\frac{\pi}{9}\right) = \cos\frac{\pi}{9}\right]$$

$$= 2\cos\left(\pi - \frac{\pi}{3}\right)\cos\frac{\pi}{9} = -2\cos\frac{\pi}{3}\cos\frac{\pi}{9}$$

$$= -\cos\frac{\pi}{9} \quad \left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

(ii) $$\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36} = \sin\left(\frac{\pi}{2} - \frac{13\pi}{36}\right) + \cos\frac{7\pi}{36}$$

$$= \cos\frac{13\pi}{36} + \cos\frac{7\pi}{36}$$

$$= 2\cos\frac{13\pi+7\pi}{36\times 2}\cos\frac{13\pi-7\pi}{36\times 2} = 2\cos\frac{5\pi}{18}\cos\frac{\pi}{12}$$

**Example 4.6** Prove that $\dfrac{\cos 7A - \cos 9A}{\sin 9A - \sin 7A} = \tan 8A$

**Solution :**

$$\text{L.H.S.} = \frac{2\sin\frac{7A+9A}{2}\sin\frac{9A-7A}{2}}{2\cos\frac{9A+7A}{2}\sin\frac{9A-7A}{2}}$$

$$= \frac{\sin 8A\ \sin A}{\cos 8A\ \sin A} = \frac{\sin 8A}{\cos 8A} = \tan 8A = \text{R.H.S.}$$

**Example 4.7** Prove the following :

(i) $\cos^2\left(\frac{\pi}{4}-A\right)-\sin^2\left(\frac{\pi}{4}-B\right)=\sin(A+B)\cos(A-B)$

(ii) $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A$

**Solution :**

(i) Applying the formula

$\cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)$, we have

$$\text{L.H.S.}=\cos\left[\frac{\pi}{4}-A+\frac{\pi}{4}-B\right]\cos\left[\frac{\pi}{4}-A-\frac{\pi}{4}+B\right]$$

$$=\cos\left[\frac{\pi}{2}-(A+B)\right]\cos\left[-(A-B)\right]=\sin(A+B)\cos(A-B)=\text{R.H.S.}$$

(ii) Applying the formula

$\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$, we have

$$\text{L.H.S.}=\sin\left(\frac{\pi}{8}+\frac{A}{2}+\frac{\pi}{8}-\frac{A}{2}\right)\sin\left(\frac{\pi}{8}+\frac{A}{2}-\frac{\pi}{8}+\frac{A}{2}\right)$$

$$=\sin\frac{\pi}{4}\sin A=\frac{1}{\sqrt{2}}\sin A=\text{R.H.S.}$$

**Example 4.8** Prove that

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{3}\cos\frac{4\pi}{9}=\frac{1}{16}$$

**Solution :** L.H.S. $\cos\frac{\pi}{3}\left[\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$

$$=\frac{1}{2}\cdot\frac{1}{2}\left[2\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}\qquad\left[\because\cos\frac{\pi}{3}=\frac{1}{2}\right]$$

$$=\frac{1}{4}\left[\cos\frac{\pi}{3}+\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[2\cos\frac{4\pi}{9}\cos\frac{\pi}{9}\right]$$

$$=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[\cos\frac{5\pi}{9}+\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{8}\cos\frac{4\pi}{9} + \frac{1}{8}\cos\frac{5\pi}{9} + \frac{1}{16} \qquad .....(1)$$

Now $$\cos\frac{5\pi}{9} = \cos\left[\pi - \frac{4\pi}{9}\right] = -\cos\frac{4\pi}{9} \qquad .....(2)$$

From (1) and (2), we get L.H.S. $= \frac{1}{16} =$ R.H.S.

## 4.3 TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

(a) To express sin 2A in terms of sin A, cos A and tan A.

We know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$

By putting $B = A$, we get $\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$

$\therefore$ sin 2A can also be written as

$$\sin 2A = \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A} \quad (\because 1 = \cos^2 A + \sin^2 A)$$

Dividing numerator and denomunator by $\cos^2 A$, we get

$$\sin 2A = \frac{2\left(\frac{\sin A\cos A}{\cos^2 A}\right)}{\frac{\cos^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1+\tan^2 A}$$

(b) To express cos 2A in terms of sin A, cos A and tan A.

We know that $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Putting $B = A$, we have $\cos 2A = \cos A\cos A - \sin A\sin A$

or $\quad \cos 2A = \cos^2 A - \sin^2 A$

Also $\cos 2A = \cos^2 A - (1-\cos^2 A) = \cos^2 A - 1 + \cos^2 A$

i.e, $\quad \cos 2A = 2\cos^2 A - 1 \quad \Rightarrow \quad \cos^2 A = \frac{1+\cos 2A}{2}$

Also $\cos 2A = \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A$

i.e., $\quad \cos 2A = 1 - 2\sin^2 A \quad \Rightarrow \quad \sin^2 A = \frac{1-\cos 2A}{2}$

$$\therefore \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

Dividing the numerator and denominator of R.H.S. by $\cos^2 A$, we have

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(c) To express tan 2A in terms of tan A.

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$$

Thus we have derived the following formulae :

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}, \quad \cos^2 A = \frac{1 + \cos 2A}{2}, \quad \sin^2 A = \frac{1 - \cos 2A}{2}$$

**Example 4.9** Prove that $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$

**Solution :** $\dfrac{\sin 2A}{1 + \cos 2A} = \dfrac{2 \sin A \cos A}{2 \cos^2 A} = \dfrac{\sin A}{\cos A} = \tan A$

**Example 4.10** Prove that $\cot A - \tan A = 2 \cot 2A$.

**Solution :** $\cot A - \tan A = \dfrac{1}{\tan A} - \tan A = \dfrac{1 - \tan^2 A}{\tan A}$

$$= \frac{2(1 - \tan^2 A)}{2 \tan A}$$

$$= \frac{2}{\left(\dfrac{2 \tan A}{1 - \tan^2 A}\right)}$$

$$= \frac{2}{\tan 2A} = 2 \cot 2A.$$

**Example 4.11** Evaluate $\cos^2 \dfrac{\pi}{8} + \cos^2 \dfrac{3\pi}{8}$.

**Solution :** $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}=\frac{1+\cos\frac{\pi}{4}}{2}+\frac{1+\cos\frac{3\pi}{4}}{2}$

$$=\frac{1+\frac{1}{\sqrt{2}}}{2}+\frac{1-\frac{1}{\sqrt{2}}}{2}=\frac{(\sqrt{2}+1)+(\sqrt{2}-1)}{2\sqrt{2}}=1$$

**Example 4.12** Prove that $\frac{\cos A}{1-\sin A}=\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$.

**Solution :** R.H.S. $=\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}\tan\frac{A}{2}}$

$$=\frac{1+\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{1-\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}=\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$$

$$=\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2}$$

[Multiplying Numerator and Denominator by $\left(\frac{\cos A}{2}-\frac{\sin A}{2}\right)$]

$$=\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\cos\frac{A}{2}\sin\frac{A}{2}}=\frac{\cos A}{1-\sin A}=\text{L.H.S}.$$

### 4.3.1 Trigonometric Functions of 3A in Terms of A

**(a) sin 3A in terms of sin A**

Substituting 2A for B in the formula

$\sin(A+B) = \sin A \cos B + \cos A \sin B$, we get

$\sin(A+2A) = \sin A \cos 2A + \cos A \sin 2A$

$= \sin A\left(1 - 2\sin^2 A\right) + \left(\cos A \times 2\sin A \cos A\right)$

$= \sin A - 2\sin^3 A + 2\sin A\left(1 - \sin^2 A\right)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$\therefore \quad \sin 3A = 3\sin A - 4\sin^3 A \quad \ldots(1)$

**(b) cos 3A in terms of cos A**

Substituting 2A for B in the formula

$\cos(A+B) = \cos A \cos B - \sin A \sin B$, we get

$\cos(A+2A) = \cos A \cos 2A - \sin A \sin 2A$

$= \cos A\left(2\cos^2 A - 1\right) - \left(\sin A\right) \times 2\sin A \cos A$

$= 2\cos^3 A - \cos A - 2\cos A\left(1 - \cos^2 A\right)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$\therefore \quad \cos 3A = 4\cos^3 A - 3\cos A \quad \ldots(2)$

**(c) tan 3A in terms of tan A**

Putting B = 2A in the formula $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, we get

$$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A} = \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\dfrac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\dfrac{1 - \tan^2 A - 2\tan^2 A}{1-\tan^2 A}} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \quad \ldots(3)$$

(d) **Formulae for** $\sin^3 A$ **and** $\cos^3 A$

$\because \quad \sin 3A = 3\sin A - 4\sin^3 A$

$\therefore \quad 4\sin^3 A = 3\sin A - \sin 3A$ or $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

Similarly, $\cos 3A = 4\cos^3 A - 3\cos A$

$\therefore \quad 3\cos A + \cos 3A = 4\cos^3 A$ or $\cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

**Example 4.13** Prove that

$$\sin\alpha \sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right) = \frac{1}{4}\sin 3\alpha$$

**Solution :** $\sin\alpha \sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right)$

$$= \frac{1}{2}\sin\alpha\left[\cos 2\alpha - \cos\frac{2\pi}{3}\right] = \frac{1}{2}\sin\alpha\left[1-2\sin^2\alpha - \left(1-2\sin^2\frac{\pi}{3}\right)\right]$$

$$= 2\frac{1}{2}\sin\alpha\left[\sin^2\frac{\pi}{3} - \sin^2\alpha\right]$$

$$= \sin\alpha\left[\frac{3}{4} - \sin^2\alpha\right] = \frac{3\sin\alpha - 4\sin^3\alpha}{4} = \frac{1}{4}\sin 3\alpha$$

**Example 4.14** Prove that $\cos^3 A \sin 3A + \sin^3 A \cos 3A = \dfrac{3}{4}\sin 4A$

**Solution :** $\cos^3 A \sin 3A + \sin^3 A \cos 3A$

$$= \cos^3 A\left(3\sin A - 4\sin^3 A\right) + \sin^3 A\left(4\cos^3 A - 3\cos A\right)$$

$$= 3\sin A\cos^3 A - 4\sin^3 A\cos^3 A + 4\sin^3 A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos A\left(\cos^2 A - \sin^2 A\right) = \left(3\sin A\cos A\right)\cos 2A$$

$$= \frac{3\sin 2A}{2}\times\cos 2A = \frac{3}{2}\frac{\sin 4A}{2} = \frac{3}{4}\sin 4A.$$

**Example 4.15** Prove that $\cos^3\frac{\pi}{9}+\sin^3\frac{\pi}{18}=\frac{3}{4}\left(\cos\frac{\pi}{9}+\sin\frac{\pi}{18}\right)$

**Solution :** L.H.S. $=\frac{1}{4}\left[3\cos\frac{\pi}{9}+\cos\frac{\pi}{3}\right]+\frac{1}{4}\left(3\sin\frac{\pi}{18}-\sin\frac{\pi}{6}\right)$

$$=\frac{3}{4}\left[\cos\frac{\pi}{9}+\sin\frac{\pi}{18}\right]+\frac{1}{4}\left(\frac{1}{2}-\frac{1}{2}\right)=\frac{3}{4}\left[\cos\frac{\pi}{9}+\sin\frac{\pi}{18}\right]=\text{R.H.S.}$$

## 4.4 TRIGONOMETRIC FUNCTIONS OF SUBMULTIPLES OF ANGLES

$\frac{A}{2},\frac{A}{3},\frac{A}{4}$ are called submultiples of A.

It has been proved that

$$\sin^2 A=\frac{1-\cos 2A}{2},\ \cos^2 A=\frac{1+\cos 2A}{2},\ \tan^2 A=\frac{1-\cos 2A}{1+\cos 2A}$$

Replacing A by $\frac{A}{2}$, we easily get the following formulae for the sub-multiple $\frac{A}{2}$ :

$$\sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}},\ \cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}}\text{ and }\tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

We will choose either the positive or the negative sign depending on whether corresponding value of the function is positive or negative for the value of $\frac{A}{2}$. This will be clear from the following examples

**Example 4.16** Find the values of $\sin\left(-\frac{\pi}{8}\right)$ and $\cos\left(-\frac{\pi}{8}\right)$.

**Solution :** We use the formula $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

and take the lower sign, i.e., negative sign, because $\sin\left(-\frac{\pi}{8}\right)$ is negative.

$$\sin\left(-\frac{\pi}{8}\right) = -\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = -\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

$$= -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

Similarly, $$\cos\left(-\frac{\pi}{8}\right) = +\sqrt{\frac{1+\cos\left(-\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

**Example 4.17** If $\cos A = \frac{7}{25}$ and $\frac{3\pi}{2} < A < 2\pi$, find the values of

(i) $\sin\frac{A}{2}$ (ii) $\cos\frac{A}{2}$ (iii) $\tan\frac{A}{2}$

**Solution :** $\because$ A lies in the 4th-quardrant, $\frac{3\pi}{2} < A < 2\pi$

$$\Rightarrow \quad 3\frac{\pi}{4} < \frac{A}{2} < \pi$$

$$\therefore \quad \sin\frac{A}{2} > 0,\ \cos\frac{A}{2} < 0,\ \tan\frac{A}{2} < 0.$$

$$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+\frac{7}{25}}{2}} = -\sqrt{\frac{32}{50}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

and $$\tan\frac{A}{2} = -\sqrt{\frac{1-\cos A}{1+\cos A}} = -\sqrt{\frac{1-\frac{7}{25}}{1+\frac{7}{25}}} = -\sqrt{\frac{18}{32}} = -\sqrt{\frac{9}{16}} = -\frac{3}{4}$$

## 4.5 TRIGONOMETRIC EQUATIONS

You are familiar with the equations like simple linear equations, quadratic equations in algebra. You have also learnt how to solve the same.

Thus, (i) $x - 3 = 0$ gives one value of *x* as a solution.

(ii) $x^2 - 9 = 0$ gives two values of *x*.

You must have noticed, the number of values depends upon the degree of the equation.

Now we need to consider as to what will happen in case *x*'s and *y*'s are replaced by trigonometric functions.

Thus solution of the equation $\sin\theta - 1 = 0$, will give

$$\sin\theta = 1 \text{ and } \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$

Clearly, the solution of simple equations with only finite number of values does not necessarily hold good in case of trigonometric equations.

So, we will try to find the ways of finding solutions of such equations.

### 4.5.1 To find the general solution of the equation sin θ = sin α

It is given that $\sin\theta = \sin\alpha$, $\Rightarrow$ $\sin\theta - \sin\alpha = 0$

$$\text{or} \quad 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\therefore \quad \text{Either } \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2p+1)\frac{\pi}{2} \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p,\ q \in Z$$

$$\Rightarrow \quad \theta = (2p+1)\pi - \alpha \quad \text{or} \quad \theta = 2q\pi + \alpha \quad \ldots(1)$$

From (1), we get

$\theta = n\pi + (-1)^n\alpha,\ n \in Z$ as the geeneral solution of the equation $\sin\theta = \sin\alpha$

### 4.5.2 To find the general solution of the equation cos θ = cos α

It is given that, $\cos\theta = \cos\alpha$, $\Rightarrow$ $\cos\theta - \cos\alpha = 0$

$$\Rightarrow \quad -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\therefore \quad \text{Either, } \sin\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = p\pi \text{ or } \frac{\theta-\alpha}{2} = q\pi,\ p, q \in Z$$

$$\Rightarrow \quad \theta = 2p\pi - \alpha \text{ or } \theta = 2z\pi + \alpha \quad \ldots(1)$$

From (1), we have

$\theta = 2n\pi \pm \alpha, n \in Z$ as the general solution of the equation $\cos\theta = \cos\alpha$

**4.5.3 To find the general solution of the equation $\tan\theta = \tan\alpha$**

It is given that, $\tan\theta = \tan\alpha$, $\Rightarrow \dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin\alpha}{\cos\alpha} = 0$

$\Rightarrow \quad \sin\theta\cos\alpha - \sin\alpha\cos\theta = 0, \Rightarrow \quad \sin(\theta - \alpha) = 0$

$\Rightarrow \quad \theta - \alpha = n\pi, n \in Z, \Rightarrow \quad \theta = n\pi + \alpha, n \in Z$

Similarly, for $\quad \operatorname{cosec}\theta = \operatorname{cosec}\alpha$, the general solution is $\theta = n\pi + (-1)^n \alpha$

and, for $\quad \sec\theta = \sec\alpha$, the general solution is $\theta = 2n\pi \pm \alpha$

and for $\quad \cot\theta = \cot\alpha$, $\theta = n\pi + \alpha$ is its general solution

**Example 4.18** Find the general solution of the following equations :

(a) (i) $\sin\theta = \dfrac{1}{2}$ (ii) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ (b) (i) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\dfrac{1}{2}$

(c) $\cot\theta = -\sqrt{3}$ (d) $4\sin^2\theta = 1$

**Solution :** (a) (i) $\sin\theta = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$

$\therefore \quad \theta = n\pi + (-1)^n \dfrac{\pi}{6}, \quad n \in Z$

(ii) $\sin\theta = \dfrac{-\sqrt{3}}{2} = -\sin\dfrac{\pi}{3} = \sin\left(\pi + \dfrac{\pi}{3}\right) = \sin\dfrac{4\pi}{3}$

$\therefore \quad \theta = n\pi + (-1)^n \dfrac{4\pi}{3}, \quad n \in Z$

(b) (i) $\cos\theta = \dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6}, \therefore \quad \theta = 2n\pi \pm \dfrac{\pi}{6}, \quad n \in Z$

(ii) $\cos\theta = -\dfrac{1}{2} = -\cos\dfrac{\pi}{3} = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\dfrac{2\pi}{3}$

$\therefore \quad \theta = 2n\pi \pm \dfrac{2\pi}{3}, \quad n \in Z$

(c) $\quad \cot\theta = -\sqrt{3}$, $\tan\theta = -\dfrac{1}{\sqrt{3}} = -\tan\dfrac{\pi}{6} = \tan\left(\pi - \dfrac{\pi}{6}\right) = \tan\dfrac{5\pi}{6}$

$\therefore \quad \theta = n\pi + \dfrac{5\pi}{6}, \quad n \in Z$

(d) $\quad 4\sin^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2 = \sin^2\dfrac{\pi}{6}$

$\Rightarrow \quad \sin\theta = \sin\left(\pm\dfrac{\pi}{6}\right) \quad \therefore \theta = n\pi \pm \dfrac{\pi}{6}, \quad n \in Z$

**Example 4.19** Solve the following to find general solution :

(a) $2\cos^2\theta + 3\sin\theta = 0$ (b) $\cos 4x = \cos 2x$

(c) $\cos 3x = \sin 2x$ (d) $\sin 2x + \sin 4x + \sin 6x = 0$

**Solution :**

(a) $2\cos^2\theta + 3\sin\theta = 0$, $\Rightarrow 2\left(1-\sin^2\theta\right) + 3\sin\theta = 0$

$\Rightarrow 2\sin^2\theta - 3\sin\theta - 2 = 0$, $\Rightarrow \left(2\sin\theta + 1\right)\left(\sin\theta - 2\right) = 0$

$\Rightarrow \sin\theta = -\dfrac{1}{2}$ or $\sin\theta = 2$, Since $\sin\theta = 2$ is not possible.

$\therefore \sin\theta = -\sin\dfrac{\pi}{6} = \sin\left(\pi + \dfrac{\pi}{6}\right) = \sin\dfrac{7\pi}{6}$

$\therefore \theta = n\pi + (-1)^n \cdot \dfrac{7\pi}{6}$, $n \in Z$

(b) $\cos 4x = \cos 2x$ i.e., $\cos 4x - \cos 2x = 0$

$\Rightarrow -2\sin 3x \sin x = 0$

$\Rightarrow \sin 3x = 0$ or $\sin x = 0$

$\Rightarrow 3x = n\pi$ or $x = n\pi$

$\Rightarrow x = \dfrac{n\pi}{3}$ or $x = n\pi$ $n \in Z$

(c) $\cos 3x = \sin 2x \Rightarrow \cos 3x = \cos\left(\dfrac{\pi}{2} - 2x\right)$

$\Rightarrow 3x = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x\right)$ $n \in Z$

Taking positive sign only, we have $3x = 2n\pi + \dfrac{\pi}{2} - 2x$

$\Rightarrow 5x = 2n\pi + \dfrac{\pi}{2}$ $\Rightarrow x = \dfrac{2n\pi}{5} + \dfrac{\pi}{10}$

Now taking negative sign, we have

$$3x = 2n\pi - \frac{\pi}{2} + 2x \quad \Rightarrow \quad x = 2n\pi - \frac{\pi}{2} \quad n \in Z$$

(d) $\sin 2x + \sin 4x + \sin 6x = 0$

or $(\sin 6x + \sin 2x) + \sin 4x = 0$

or $2\sin 4x \cos 2x + \sin 4x = 0$

or $\sin 4x\,[2\cos 2x + 1] = 0$

$$\therefore \quad \sin 4x = 0 \quad \text{or} \quad \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad 4x = n\pi \quad \text{or} \quad 2x = 2n\pi \pm \frac{2\pi}{3},\ n \in Z$$

$$x = \frac{n\pi}{4} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{3} \quad n \in Z$$

## LET US SUM UP

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$,

  $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

  $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

  $\cot(A + B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$, $\cot(A - B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

  $2\cos A\sin B = \sin(A + B) - \sin(A - B)$

  $2\cos A\cos B = \cos(A + B) - \cos(A - B)$

  $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

- $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

  $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

  $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

  $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

- $\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B$

  $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$

- $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

- $\sin^2 A = \frac{1-\cos 2A}{2}$, $\cos^2 A = \frac{1+\cos 2A}{2}$, $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

- $\sin 3A = 3\sin A - 4\sin^3 A$, $\cos 3A = 4\cos^3 A - 3\cos A$

  $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

- $\sin^3 A = \frac{3\sin A - \sin 3A}{4}$, $\cos^3 A = \frac{3\cos A + \cos 3A}{4}$

- $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

  $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

- $\sin\theta = \sin\alpha \quad \Rightarrow \quad \theta = n\pi + (-1)^n\alpha, \quad n \in Z$

- $\cos\theta = \cos\alpha \quad \Rightarrow \quad \theta = 2n\pi \pm \alpha, \quad n \in Z$

- $\tan\theta = \tan\alpha \quad \Rightarrow \quad \theta = n\pi + \alpha, \quad n \in Z$