

CHAPTER : 11 INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

You have read in your earlier lessons that given a point in a plane, it is possible to find two numbers, called its co-ordinates in the plane. Conversely, given any ordered pair (x, y) there corresponds a point in the plane whose co-ordinates are (x, y) .

Let a rubber ball be dropped vertically in a room. The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor.

If the height of the ball above the floor is 2.5 cm and the position of the point where it strikes the ground is given by $(5, 4)$, one way of describing the position of ball in space is with the help of these three numbers $(5, 4, 2.5)$.

Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers. In this lesson, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

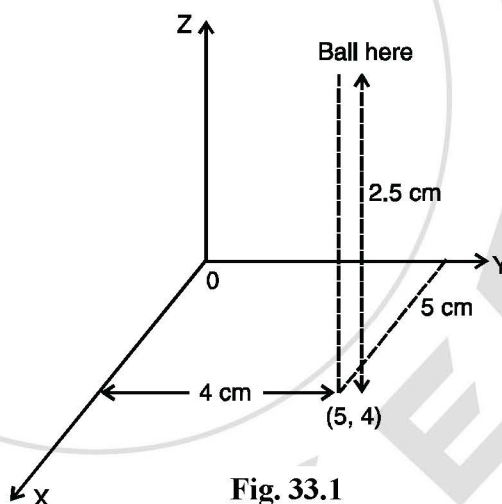


Fig. 33.1

OBJECTIVES

After studying this lesson, you will be able to :

- associate a point, in three dimensional space with given triplet and vice versa;
- find the distance between two points in space;
- find the coordinates of a point which divides the line segment joining two given points in a given ratio internally and externally;
- define the direction cosines/ratios of a given line in space;
- find the direction cosines of a line in space;

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- find the projection of a line segment on another line; and
- find the condition of perpendicularity and parallelism of two lines in space.

EXPECTED BACKGROUND KNOWLEDGE

- Two dimensional co-ordinate geometry
- Fundamentals of Algebra, Geometry, Trigonometry and vector algebra.

33.1 COORDINATE SYSTEM AND COORDINATES OF A POINT IN SPACE

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the co-ordinates of the point change. Thus, the origin can be taken arbitrarily at any point of the room.

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines $X'OX$, $Y'OY$ and $Z'OZ$ through O . The point O is called the origin of the co-ordinate system and the lines $X'OX$, $Y'OY$ and $Z'OZ$ are called the X -axis, the Y -axis and the Z -axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 33.2. The plane determined by the X -axis and the Y -axis is called XY -plane (XOY plane) and similarly, YZ -plane (YOZ -plane) and ZX -plane (ZOX -plane) can be determined. These three planes are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

Let P be any point in space. Through P draw perpendicular PL on XY -plane

meeting this plane at L . Through L draw a line LA parallel to OY cutting OX in A . If we write $OZ = x$, $AL = y$ and $LP = z$, then (x, y, z) are the co-ordinates of the point P .

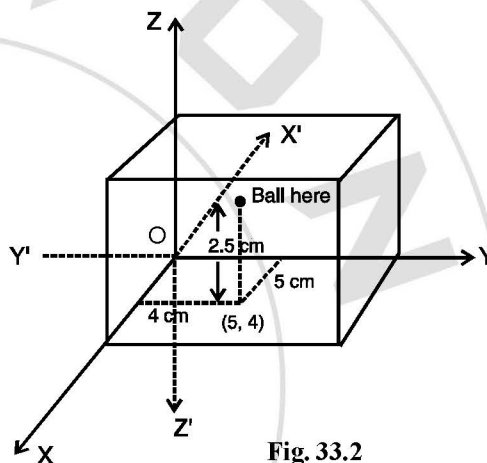


Fig. 33.2

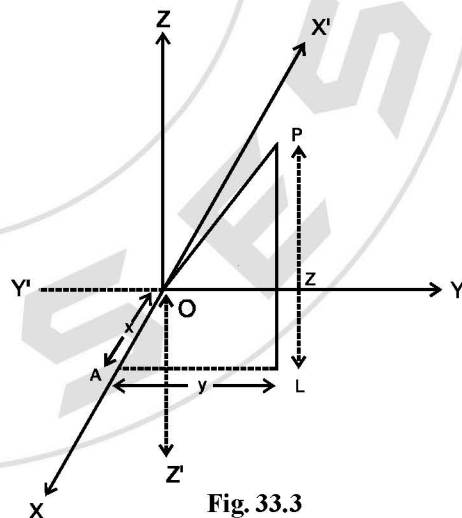


Fig. 33.3

Again, if we complete a rectangular parallelepiped through P with its three edges OA, OB and OC meeting each other at O and OP as its main diagonal then the lengths (OA, OB, OC) i.e., (x, y, z) are called the co-ordinates of the point P.

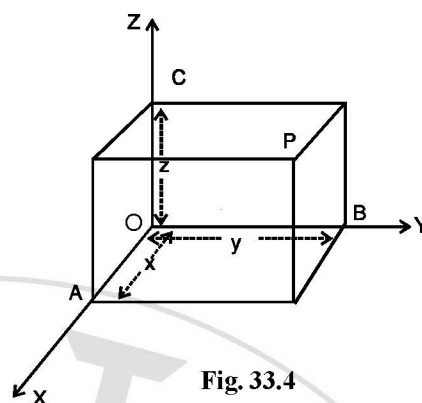


Fig. 33.4

Note : You may note that in Fig. 33.4

- (i) The x co-ordinate of P = OA = the length of perpendicular from P on the YZ-plane.
- (ii) The y co-ordinate of P = OB = the length of perpendicular from P on the ZX-plane.
- (iii) The z co-ordinate of P = OC = the length of perpendicular from P on the XY-plane.

Thus, the co-ordinates x, y, and z of any point are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

Thus, given a point P in space, to it corresponds a triplet (x, y, z) called the co-ordinates of the point in space. Conversely, given any triplet (x, y, z), there corresponds a point P in space whose co-ordinates are (x, y, z).

Remarks

1. Just as in plane co-ordinate geometry, the co-ordinate axes divide the plane into four quadrants, in three dimensional geometry, the space is divided into eight octants by the co-ordinate planes, namely OXYZ, OX'YZ, OXY'Z, OXYZ', OXY'Z', OX'YZ', OX'Y'Z and OX'Y'Z'.
2. If P be any point in the first octant, there is a point in each of the other octants whose absolute distances from the co-ordinate planes are equal to those of P. If P be (a, b, c), the other points are (-a, b, c), (a, -b, c), (a, b, -c), (a, -b, -c), (-a, b, -c), (-a, -b, c) and (-a, -b, -c) respectively in order in the octants referred in (i).
3. The co-ordinates of point in XY-plane, YZ-plane and ZX-plane are of the form (a, b, 0), (0, b, c) and (a, 0, c) respectively.
4. The co-ordinates of points on X-axis, Y-axis and Z-axis are of the form (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.
5. You may see that (x, y, z) corresponds to the position vector of the point P with reference to the origin O as the vector \overrightarrow{OP} .

Example 33.1 Name the octant wherein the given points lies :

(a) (2, 6, 8)

(b) (-1, 2, 3)

(c) (-2, -5, 1)

- (d) $(-3, 1, -2)$ (e) $(-6, -1, -2)$

Solution :

- (a) Since all the co-ordinates are positive, $\therefore (2, 6, 8)$ lies in the octant OXYZ.
 (b) Since x is negative and y and z are positive, $\therefore (-1, 2, 3)$ lies in the octant OX'YZ.
 (c) Since x and y both are negative and z is positive $\therefore (-2, -5, 1)$ lies in the octant OX'Y'Z.
 (d) $(-3, 1, -2)$ lies in octant OX'YZ'.
 (e) Since x, y and z are all negative $\therefore (-6, -1, -2)$ lies in the octant OX'Y'Z'.

33.2 DISTANCE BETWEEN TWO POINTS

Suppose there is an electric plug on a wall of a room and an electric iron placed on the top of a table. What is the shortest length of the wire needed to connect the electric iron to the electric plug ? This is an example necessitating the finding of the distance between two points in space.

Let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. With PQ as diagonal, complete the parallopiped PMSNRLKQ.

PK is perpendicular to the line KQ.

\therefore From the right-angled triangle PKQ, right angled at K,

$$\text{We have } PQ^2 = PK^2 + KQ^2$$

Again from the right angled triangle PKL right angled at L,

$$PK^2 = KL^2 + PL^2 = MP^2 + PL^2 \quad (\because KL = MP)$$

$$\therefore PQ^2 = MP^2 + PL^2 + KQ^2 \dots(i)$$

The edges MP, PL and KQ are parallel to the co-ordinate axes.

Now, the distance of the point P from the plane YOZ = x_1 and the distance of Q and M from

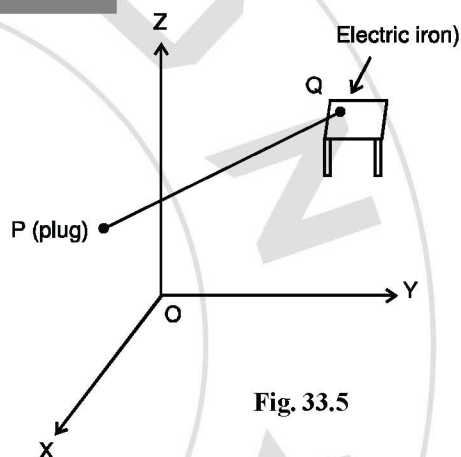


Fig. 33.5

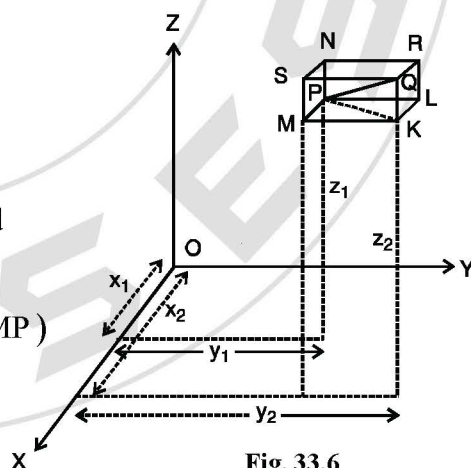


Fig. 33.6

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$$\therefore \text{MP} = |x_2 - x_1|$$

$$\text{Similarly, PL} = |y_2 - y_1| \text{ and KQ} = |z_2 - z_1|$$

$$\therefore \text{PQ}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \dots [\text{From (i)}]$$

$$\text{or } |\text{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Corollary : Distance of a Point from the Origin

If the point Q(x_2, y_2, z_2) coincides with the origin (0, 0, 0), then $x_2 = y_2 = z_2 = 0$

\therefore The distance of P from the origin is

$$\begin{aligned} |\text{OP}| &= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2} \\ &= \sqrt{x_1^2 + y_1^2 + z_1^2} \end{aligned}$$

In general, the distance of a point P(x, y, z) from origin O is given by

$$|\text{OP}| = \sqrt{x^2 + y^2 + z^2}$$

Example 33.2 Find the distance between the points (2, 5, - 4) and (8, 2, - 6).

Solution : Let P (2, 5, - 4) and Q (8, 2, - 6) be the two given points.

$$\begin{aligned} \therefore |\text{PQ}| &= \sqrt{(8 - 2)^2 + (2 - 5)^2 + (-6 + 4)^2} \\ &= \sqrt{36 + 9 + 4} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

\therefore The distance between the given points is 7 units.

Example 33.3 Prove that the points (-2, 4, - 3), (4, - 3, - 2) and (-3, - 2, 4) are the vertices of an equilateral triangle.

Solution : Let A (-2, 4, - 3), B (4, - 3, - 2) and C (-3, - 2, 4) be the three given points.

$$\begin{aligned} \text{Now } |\text{AB}| &= \sqrt{(4 + 2)^2 + (-3 - 4)^2 + (-2 + 3)^2} \\ &= \sqrt{36 + 49 + 1} = \sqrt{86} \\ |\text{BC}| &= \sqrt{(-3 - 4)^2 + (-2 + 3)^2 + (4 + 2)^2} = \sqrt{86} \\ |\text{CA}| &= \sqrt{(-2 + 3)^2 + (4 + 2)^2 + (-3 - 4)^2} = \sqrt{86} \end{aligned}$$

Since $|AB| = |BC| = |CA|$, ΔABC is an equilateral triangle.

Example 33.4 Verify whether the following points form a triangle or not :

(a) $A(-1, 2, 3)$ $B(1, 4, 5)$ and $C(5, 4, 0)$

(b) $(2, -3, 3)$, $(1, 2, 4)$ and $(3, -8, 2)$

Solution :

$$\begin{aligned} \text{(a) } |AB| &= \sqrt{(1+1)^2 + (4-2)^2 + (5-3)^2} \\ &= \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3} = 3.464 \text{ (approx.)} \end{aligned}$$

$$\begin{aligned} |BC| &= \sqrt{(5-1)^2 + (4-4)^2 + (0-5)^2} \\ &= \sqrt{16 + 0 + 25} = \sqrt{41} = 6.4 \text{ (approx.)} \end{aligned}$$

$$\begin{aligned} \text{and } |AC| &= \sqrt{(5+1)^2 + (4-2)^2 + (0-3)^2} \\ &= \sqrt{36 + 4 + 9} = 7 \end{aligned}$$

$$\therefore |AB| + |BC| = 3.464 + 6.4 = 9.864 > |AC|, |AB| + |AC| > |BC|$$

$$\text{and } |BC| + |AC| > |AB|.$$

Since sum of any two sides is greater than the third side, therefore the above points form a triangle.

(b) Let the points $(2, -3, 3)$, $(1, 2, 4)$ and $(3, -8, 2)$ be denoted by P, Q and R respectively,

$$\begin{aligned} \text{then } |PQ| &= \sqrt{(1-2)^2 + (2+3)^2 + (4-3)^2} \\ &= \sqrt{1 + 25 + 1} = 3\sqrt{3} \end{aligned}$$

$$\begin{aligned} |QR| &= \sqrt{(3-1)^2 + (-8-2)^2 + (2-4)^2} \\ &= \sqrt{4 + 100 + 4} = 6\sqrt{3} \end{aligned}$$

$$\begin{aligned} |PR| &= \sqrt{(3-2)^2 + (-8+3)^2 + (2-3)^2} \\ &= \sqrt{1 + 25 + 1} \\ &= 3\sqrt{3} \end{aligned}$$

In this case $|PQ| + |PR| = 3\sqrt{3} + 3\sqrt{3} = 6\sqrt{3} = |QR|$. Hence the given points do not form a triangle. In fact the points lie on a line.

Example 33.5 Show that the points $A(1, 2, -2)$, $B(2, 3, -4)$ and $C(3, 4, -3)$ form a right angled triangle.

Solution : $AB^2 = (2-1)^2 + (3-2)^2 + (-4+2)^2 = 1+1+4 = 6$

$$BC^2 = (3-2)^2 + (4-3)^2 + (-3+4)^2 = 1+1+1 = 3$$

and $AC^2 = (3-1)^2 + (4-2)^2 + (-3+2)^2 = 4+4+1 = 9$

We observe that $AB^2 + BC^2 = 6 + 3 = 9 = AC^2$

$\therefore \Delta ABC$ is a right angled triangle.

Hence the given points form a right angled triangle.

Example 33.6 Prove that the points $A(0, 4, 1)$, $B(2, 3, -1)$, $C(4, 5, 0)$ and $D(2, 6, 2)$ are vertices of a square.

Solution : Here,

$$AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

$$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$CD = \sqrt{(2-4)^2 + (6-5)^2 + (2-0)^2}$$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

and

$$DA = \sqrt{(0-2)^2 + (4-6)^2 + (1-2)^2}$$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

\therefore

$$AB = BC = CD = DA$$

Now

$$AC^2 = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$$

$$= 16+1+1 = 18$$

$$\therefore AB^2 + BC^2 = 3^2 + 3^2 = 18 = AC^2$$

$$\therefore \angle B = 90^\circ$$

$$\therefore \text{In quadrilateral } ABCD, AB = BC = CD = DA \text{ and } \angle B = 90^\circ$$

$\therefore ABCD$ is a square.

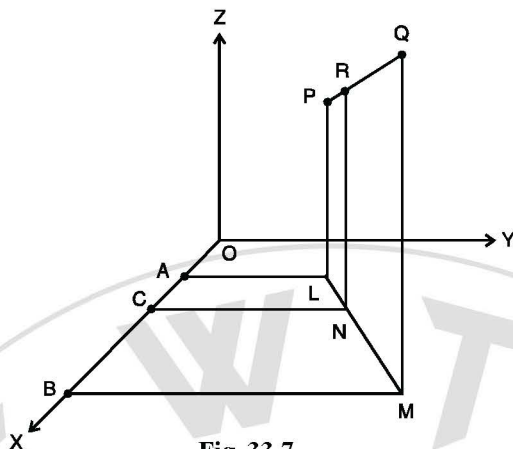


Fig. 33.7

Let the point $R(x, y, z)$ divide PQ in the ratio $l : m$ internally.

Let the co-ordinates of P be (x_1, y_1, z_1) and the co-ordinates of Q be (x_2, y_2, z_2) . From points P, R and Q , draw PL, RN and QM perpendiculars to the XY -plane.

Draw LA, NC and MB perpendiculars to OX .

Now $AC = OC - OA = x - x_1$ and $BC = OB - OC = x_2 - x$

$$\text{Also we have, } \frac{AC}{BC} = \frac{LN}{NM} = \frac{PR}{RQ} = \frac{l}{m}$$

$$\therefore \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

$$\text{or } mx - mx_1 = lx_2 - lx$$

$$\text{or } (l + m)x = lx_2 + mx_1$$

$$\text{or } x = \frac{lx_2 + mx_1}{l + m}$$

Similarly, if we draw perpendiculars to OY and OZ respectively,

$$\text{we get } y = \frac{ly_2 + my_1}{l + m} \text{ and } z = \frac{lz_2 + mz_1}{l + m}$$

$$\therefore R \text{ is the point } \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m} \right)$$

If $\lambda = \frac{l}{m}$, then the co-ordinates of the point R which divides PQ in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1} \right), \lambda + 1 \neq 0$$

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It is clear that to every value of λ , there corresponds a point of the line PQ and to every point R on the line PQ, there corresponds some value of λ . If λ is positive, R lies on the line segment PQ and if λ is negative, R does not lie on line segment PQ.

In the second case you may say the R divides the line segment PQ externally in the ratio $-\lambda : 1$.

Corollary 1 : The co-ordinates of the point dividing PQ externally in the ratio $l : m$ are

$$\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m} \right)$$

Corollary 2 : The co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Example 33.7 Find the co-ordinates of the point which divides the line segment joining the points $(2, -4, 3)$ and $(-4, 5, -6)$ in the ratio $2 : 1$ internally.

Solution : Let $A(2, -4, 3)$, $B(-4, 5, -6)$ be the two points.

Let $P(x, y, z)$ divides AB in the ratio $2 : 1$.

$$\therefore x = \frac{2(-4) + 1 \cdot 2}{2 + 1} = -2, \quad y = \frac{2 \cdot 5 + 1(-6)}{2 + 1} = 2$$

$$\text{and } z = \frac{2(-6) + 1 \cdot 3}{2 + 1} = -3$$

Thus, the co-ordinates of P are $(-2, 2, -3)$

Example 33.8 Find the point which divides the line segment joining the points $(-1, -3, 2)$ and $(1, -1, 2)$ externally in the ratio $2 : 3$.

Solution : Let the points $(-1, -3, 2)$ and $(1, -1, 2)$ be denoted by P and Q respectively.

Let R (x, y, z) divide PQ externally in the ratio $2 : 3$. Then

$$x = \frac{2(1) - 3(-1)}{2 - 3} = -5, \quad y = \frac{2(-1) - 3(-3)}{2 - 3} = -7$$

$$\text{and } z = \frac{2(2) - 3(2)}{2 - 3} = 2$$

Thus, the co-ordinates of R are $(-5, -7, 2)$.

Example 33.9 Find the ratio in which the line segment joining the points $(2, -3, 5)$ and $(7, 1, 3)$ is divided by the XY-plane.

Solution : Let the required ratio in which the line segment is divided be $l : m$.

The co-ordinates of the point are $\left(\frac{7l + 2m}{l + m}, \frac{l - 3m}{l + m}, \frac{3l + 5m}{l + m} \right)$

Since the point lies in the XY-plane, its z-coordinate is zero.

$$\text{i.e., } \frac{3l + 5m}{l + m} = 0 \text{ or } \frac{l}{m} = -\frac{5}{3}$$

Hence the XY-plane divides the join of given points in the ratio $5 : 3$ externally.

LET US SUM UP

- For a given point $P(x, y, z)$ in space with reference to rectangular co-ordinate axes, if we draw three planes parallel to the three co-ordinate planes to meet the axes (in A, B and C say), then

$OA = x$, $OB = y$ and $OC = z$ where O is the origin.

Conversely, given any three numbers, x , y and z we can find a unique point in space whose co-ordinates are (x, y, z) .

- The distance PQ between the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In particular the distance of P from the origin O is $\sqrt{x_1^2 + y_1^2 + z_1^2}$.

- The co-ordinates of the point which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $l : m$

(a) internally are $\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m} \right)$

(b) externally are $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m} \right)$

In particular, the co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$