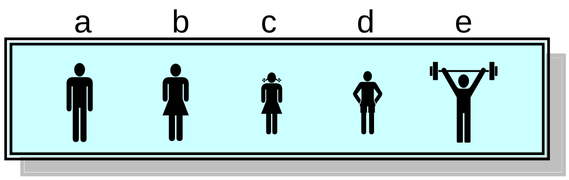
A1. Basic Reviews

PERMUTATIONS and COMBINATIONS...
or *"HOW TO COUNT"*

➤ **Question 1:** Suppose we wish to arrange $n = 5$ people {a, b, c, d, e}, standing side by side, for a portrait. How many such distinct portraits ("permutations") are possible?

Example:



Here, every different ordering counts as a distinct permutation. For instance, the ordering (a,b,c,d,e) is distinct from (c,e,a,d,b), etc.

Solution: There are 5 possible choices for which person stands in the first position (either a, b, c, d, or e). For each of these five possibilities, there are 4 possible choices left for who is in the next position. For each of these four possibilities, there are 3 possible choices left for the next position, and so on. Therefore, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ distinct permutations. *See* **Table 1**. □

This number, $5 \times 4 \times 3 \times 2 \times 1$ (or equivalently, $1 \times 2 \times 3 \times 4 \times 5$), is denoted by the symbol "5!" and read "5 factorial", so we can write the answer succinctly as $5! = 120$.

In general,

FACT 1: The number of distinct PERMUTATIONS of n objects is "n factorial", denoted by

$$n! = 1 \times 2 \times 3 \times \ldots \times n, \text{ or equivalently,}$$
$$= n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

Examples:

$$6! = 6 \times \underbrace{5 \times 4 \times 3 \times 2 \times 1}_{}$$
$$= 6 \times 5!$$
$$= 6 \times 120 \quad \text{(by previous calculation)}$$
$$= 720$$

$$3! = 3 \times 2 \times 1 = 6$$

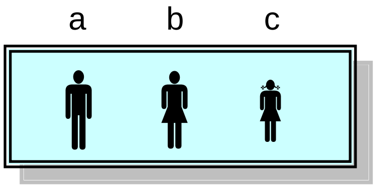
$$2! = 2 \times 1 = 2$$

$$1! = 1$$

$0! = 1$, BY CONVENTION (It may not be obvious why, but there are good mathematical reasons for it.)

➢ **Question 2:** Now suppose we start with the same $n = 5$ people {a, b, c, d, e}, but we wish to make portraits *of only* $k = 3$ *of them at a time*. How many such distinct portraits are possible?

Example:



Again, as above, every different ordering counts as a distinct permutation. For instance, the ordering (a,b,c) is distinct from (c,a,b), etc.

Solution: By using exactly the same reasoning as before, there are $5 \times 4 \times 3 = 60$ permutations. □

☞ *See* **Table 2** *for the explicit list!*

Note that this is technically NOT considered a factorial (since we don't go all the way down to 1), but we can express it as a *ratio* of factorials:

$$\mathbf{5 \times 4 \times 3} \quad = \quad \frac{\mathbf{5 \times 4 \times 3} \times (2 \times 1)}{(2 \times 1)} \quad = \quad \frac{5!}{2!}.$$

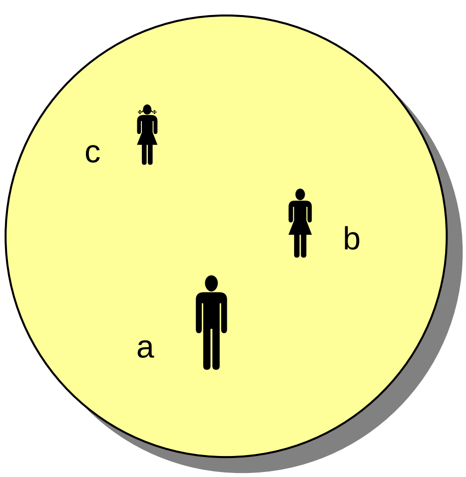
In general,

FACT 2: The number of distinct PERMUTATIONS of n objects, taken k at a time, is given by the ratio

$$\frac{n!}{(n-k)!} \quad = \quad \overbrace{n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)}.$$

➢ **Question 3:** Finally suppose that instead of portraits ("permutations"), we wish to form committees ("combinations") of $k = 3$ people from the original $n = 5$. How many such distinct committees are possible?

Example:



Now, every different ordering does NOT count as a distinct combination. For instance, the committee {a,b,c} is the <u>same</u> as the committee {c,a,b}, etc.

Solution: This time the reasoning is a little subtler. From the previous calculation, we know that

of <u>permutations</u> of $k = 3$ from $n = 5$ is equal to $\dfrac{5!}{2!} = 60$.

But now, all the ordered *permutations* of any three people (and there are $3! = 6$ of them, by ***FACT 1***), will "collapse" into one single unordered *combination*, e.g., {a, b, c}, as illustrated. So...

of <u>combinations</u> of $k = 3$ from $n = 5$ is equal to $\dfrac{5!}{2!}$, *divided by* 3!, i.e., $60 \div 6 = 10$. □

☞ *See* **Table 3** *for the explicit list!*

This number, $\dfrac{5!}{3!\ 2!}$, is given the compact notation $\dbinom{5}{3}$, read "5 choose 3", and corresponds to the number of ways of selecting 3 objects from 5 objects, regardless of their order. Hence $\dbinom{5}{3} = 10$.

In general,

FACT 3: The number of distinct COMBINATIONS of n objects, taken k at a time, is given by the ratio

$$\frac{n!}{k!\ (n-k)!} = \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)}{k!}.$$

This quantity is usually written as $\dbinom{n}{k}$, and read "n choose k".

Examples: $\dbinom{5}{3} = \dfrac{5!}{3!\ 2!} = 10$, just done. Note that this is also equal to $\dbinom{5}{2} = \dfrac{5!}{2!\ 3!} = 10$.

$\dbinom{8}{2} = \dfrac{8!}{2!\ 6!} = \dfrac{8 \times 7 \times \cancel{6!}}{2! \times \cancel{6!}} = \dfrac{8 \times 7}{2} = 28$. Note that this is equal to $\dbinom{8}{6} = \dfrac{8!}{6!\ 2!} = 28$.

$\dbinom{15}{1} = \dfrac{15!}{1!\ 14!} = \dfrac{15 \times \cancel{14!}}{1! \times \cancel{14!}} = 15$. Note that this is equal to $\dbinom{15}{14} = 15$. Why?

$\dbinom{7}{7} = \dfrac{7!}{7!\ 0!} = 1$. (Recall that $0! = 1$.) Note that this is equal to $\dbinom{7}{0} = 1$. Why?

Observe that it is neither necessary nor advisable to compute the factorials of large numbers directly. For instance, $8! = 40320$, but by writing it instead as $8 \times 7 \times 6!$, *we can cancel* 6!, leaving only 8×7 above. Likewise, 14! cancels out of 15!, leaving only 15, so we avoid having to compute 15! , etc.

Remark: $\binom{n}{k}$ is sometimes called a "combinatorial symbol" or "binomial coefficient" (in connection with a fundamental mathematical result called the Binomial Theorem; you may also recall the related "Pascal's Triangle"). The previous examples also show that binomial coefficients possess a useful symmetry, namely, $\binom{n}{k} = \binom{n}{n-k}$. For example, $\binom{5}{3} = \frac{5!}{3!\,2!}$, but this is clearly the same as $\binom{5}{2} = \frac{5!}{2!\,3!}$. In other words, the number of ways of choosing 3-person committees from 5 people is equal to the number of ways of choosing 2-person committees from 5 people. A quick way to see this without any calculating is through the insight that every choice of a 3-person committee from a collection of 5 people *leaves behind* a 2-person committee, so the total number of *both* types of committee must be equal (10).

Exercise: List all the ways of choosing 2 objects from 5, say {a, b, c, d, e}, and check these claims explicitly. That is, match each pair with its complementary triple in the list of **Table 3**.

A Simple Combinatorial Application

Suppose you toss a coin $n = 5$ times in a row. How many ways can you end up with $k = 3$ heads?

Solution: The answer can be obtained by calculating the number of ways of rearranging 3 objects among 5; it only remains to determine whether we need to use *permutations* or *combinations*. Suppose, for example, that the 3 heads occur in the first three tosses, say a, b, and c, as shown below. Clearly, rearranging these three letters in a different order would not result in a different outcome. Therefore, different orderings of the letters a, b, and c should *not* count as distinct permutations, and likewise for any other choice of three letters among {a, b, c, d, e}. Hence, there are $\binom{5}{3} = 10$ ways of obtaining $k = 3$ heads in $n = 5$ independent successive tosses.

Exercise: Let "H" denote heads, and "T" denote tails. Using these symbols, construct the explicit list of 10 combinations. (*Suggestion:* Arrange this list of H/T sequences in alphabetical order. You should see that in each case, the three H positions match up exactly with each ordered triple in the list of **Table 3**. Why?)

a b c d e

Table 1 – *Permutations* of {a, b, c, d, e}

These are the 5! = 120 ways of arranging 5 objects, in such a way that all the different orders count as being distinct.

a b c d e	b a c d e	c a b d e	d a b c e	e a b c d
a b c e d	b a c e d	c a b e d	d a b e c	e a b d c
a b d c e	b a d c e	c a d b e	d a c b e	e a c b d
a b d e c	b a d e c	c a d e b	d a c e b	e a c d b
a b e c d	b a e c d	c a e b d	d a e b c	e a d b c
a b e d c	b a e d c	c a e d b	d a e c b	e a d c b
a c b d e	b c a d e	c b a d e	d b a c e	e b a c d
a c b e d	b c a e d	c b a e d	d b a e c	e b a d c
a c d b e	b c d a e	c b d a e	d b c a e	e b c a d
a c d e b	b c d e a	c b d e a	d b c e a	e b c d a
a c e b d	b c e a d	c b e a d	d b e a c	e b d a c
a c e d b	b c e d a	c b e d a	d b e c a	e b d c a
a d b c e	b d a c e	c d a b e	d c a b e	e c a b d
a d b e c	b d a e c	c d a e b	d c a e b	e c a d b
a d c b e	b d c a e	c d b a e	d c b a e	e c b a d
a d c e b	b d c e a	c d b e a	d c b e a	e c b d a
a d e b c	b d e a c	c d e a d	d c e a b	e c d a b
a d e c b	b d e c a	c d e d a	d c e b a	e c d b a
a e b c d	b e a c d	c e a b d	d e a b c	e d a b c
a e b d c	b e a d c	c e a d b	d e a c b	e d a c b
a e c b d	b e c a d	c e b a d	d e b a c	e d b a c
a e c d b	b e c d a	c e b d a	d e b c a	e d b c a
a e d b c	b e d a c	c e d a b	d e c a b	e d c a b
a e d c b	b e d c a	c e d b a	d e c b a	e d c b a

Table 2 – *Permutations* of {a, b, c, d, e}, taken 3 at a time

These are the $\frac{5!}{2!}$ = 60 ways of arranging 3 objects among 5, in such a way that different orders of any triple count as being distinct, e.g., the 3! = 6 permutations of (a, b, c), shown below .

a b c	b a c	c a b	d a b	e a b
a b d	b a d	c a d	d a c	e a c
a b e	b a e	c a e	d a e	e a d
a c b	b c a	c b a	d b a	e b a
a c d	b c d	c b d	d b c	e b c
a c e	b c e	c b e	d b e	e b d
a d b	b d a	c d a	d c a	e c a
a d c	b d c	c d b	d c b	e c b
a d e	b d e	c d e	d c e	e c d
a e b	b e a	c e a	d e a	e d a
a e c	b e c	c e b	d e b	e d b
a e d	b e d	c e d	d e c	e d c

Table 3 – *Combinations* of {a, b, c, d, e}, taken 3 at a time

If different orders of the same triple are *not* counted as being distinct, then their six permutations are lumped as one, e.g., {a, b, c}. Therefore, the total number of combinations is $\frac{1}{6}$ *of the original* 60, or 10. Notationally, we express this as $\frac{1}{3!}$ *of the original* $\frac{5!}{2!}$, i.e., $\frac{5!}{3!\,2!}$, or more neatly, as $\binom{5}{3}$.

These $\binom{5}{3}$ = 10 combinations are listed below.

a b c
a b d
a b e
a c d
a c e
a d e
b c d
b c e
b d e
c d e