#418138

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

$(5i)\left(-\dfrac{3}{5}i\right)$

Solution

$(5i)\left(\dfrac{-3}{5}i\right)=\left(5\times\dfrac{-3}{5}\right)\times(i\times i)$

$=-3i^2=-3(-1)=3, \quad [\because i^2=-1]$

Which is clearly of the form $a+ib$ where, $a=3$ and $b=0$

#418139

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

i^9+i^{19}

Solution

$i^9+i^{19}=i^{4\times2+1}+i^{4\times4+3}$

$=\left(i^4\right)^2.\,i+\left(i^4\right)^4.\,i^3$

$=1\times i+1\times(-i), \quad [\because i^4=1, i^2=-1]$

$=i+(-i)=0=a+ib$, where, $a=0, b=0$

#418140

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

i^{-39}

Solution

$i^{-39}=i^{4\times(-9)-3}=\left(i^4\right)^{-9}.\,i^{-3}$

$=(1)^{-9}.\,i^{-3}, \quad [\because i^4=1]$

$=\dfrac{1}{i^3}=\dfrac{1}{-i}, \quad [\because i^3=-i]$

$=\dfrac{-1}{i}\times\dfrac{i}{i}$

$=\dfrac{-i}{i^2}=\dfrac{-i}{-1}=i, \quad [\because i^2=-1]$

$=a+ib$, where $a=0, b=1$

#418144

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

$3(7+7i)+i(7+i7)$

Solution

Given, $3\left(7+7i\right)+i\left(7+i7\right)$

$=21+21i+7i+7i^2$

$=21+28i+7\times\left(-1\right),\quad\left[\because i^2=-1\right]$

$=14+28i=a+ib$

where $a=14, b=28$

#418147

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

$(1-i)-(-1+i6)$

Solution

Given, $(1-i)-(-1+i6)$

$(1-i)-(-1+i6)=1-i+1-6i=2-7i$

$=a+ib$ where $a=2$ and $b=-7$

#418152

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

Solution

$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

$=\frac{1}{5}+\frac{2}{5}i-4-\frac{5}{2}i$

$=\left(\frac{1}{5}-4\right)+i\left(\frac{2}{5}-\frac{5}{2}\right)$

$=\frac{-19}{5}+i\left(\frac{-21}{10}\right)$

$=\frac{-19}{5}-\frac{21}{10}i$

$=a+ib,$ where $a=-\frac{19}{5}, b=-\frac{21}{10}$

#418157

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

Solution

$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(\frac{-4}{3}+i\right)$

$=\frac{1}{3}+\frac{7}{3}i+4+\frac{1}{3}i+\frac{4}{3}-i$

$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)$

$=\frac{17}{3}+i\frac{5}{3}=a+ib$

where, $a=\frac{17}{3}$ and $b=\frac{5}{3}$

#418165

Topic: Operations on Complex Numbers

Express the given complex number in the form $a+ib$:

$(1-i)^4$

Solution

$(1-i)^4 = \left[(1-i)^2\right]^2$
$= \left[1^2 + i^2 - 2i\right]^2$
$= [1 - 1 - 2i]^2$
$= (-2i)^2$
$= (-2i) \times (-2i)$
$= 4i^2 = -4, \quad [\because i^2 = -1]$
$= a + ib$, where $a = -4$, and $b = 0$

#418167
Topic: Operations on Complex Numbers

Express the given complex number in the form $a + ib$:
$\left(\frac{1}{3} + 3i\right)^3$

Solution

Using $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we have,
$\left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3} + 3i\right)$
$= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3} + 3i\right)$
$= \frac{1}{27} + 27(-i) + i + 9i^2, \quad [\because i^3 = -i]$
$= \frac{1}{27} - 27i + i - 9, \quad [\because i^2 = -1]$
$= \left(\frac{1}{27} - 9\right) + i(-27 + 1)$
$= \frac{-242}{27} - 26i = a + ib$
where $a = \frac{-242}{27}$ and $b = -26$

#418168
Topic: Operations on Complex Numbers

Express the given complex number in the form $a + ib$:
$\left(-2 - \frac{1}{3}i\right)^3$

Solution

$\left(-2 - \frac{1}{3}i\right)^3 = (-1)^3\left(2 + \frac{1}{3}i\right)^3$
$= -\left[2^3 + \left(\frac{i}{3}\right)^3 + 3(2)\left(\frac{i}{3}\right)\left(2 + \frac{i}{3}\right)\right]$
$= -\left[8 + \frac{i^3}{27} + 2i\left(2 + \frac{i}{3}\right)\right]$
$= -\left(8 - \frac{i}{27} + 4i + \frac{2i^2}{3}\right) \quad (\because i^3 = -i)$
$= -\left(8 - \frac{i}{27} + 4i - \frac{2}{3}\right) \quad (\because i^2 = -1)$
$= -\left(\frac{22}{3} + \frac{107i}{27}\right)$
$= -\frac{22}{3} - \frac{107}{27}i$

#418169
Topic: Operations on Complex Numbers

Find the multiplicative inverse of the complex number $4 - 3i$

Solution

Let $z = 4 - 3i$

Then, $\bar{z} = 4 + 3i$ and $|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$

Therefore, the multiplicative inverse of $4 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

#418170

Topic: Operations on Complex Numbers

Find the multiplicative inverse of the complex number $\sqrt{5} + 3i$

Solution

Let $z = \sqrt{5} + 3i$

Then, $\bar{z} = \sqrt{5} - 3i$ and $|z|^2 = \left(\sqrt{5}\right)^2 + 3^2 = 5 + 9 = 14$

Therefore, the multiplicative inverse of $\sqrt{5} + 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

#418171

Topic: Operations on Complex Numbers

Find the multiplicative inverse of the complex number $-i$

Solution

Let $z = -i$

Then, $\bar{z} = i$ and $|z|^2 = 1^2 = 1$

Therefore, the multiplicative inverse of $-i$ is,

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i$$

#418172

Topic: Operations on Complex Numbers

Express the following expression in the form $a + ib$:

$$\frac{\left(3 + i\sqrt{5}\right)\left(3 - i\sqrt{5}\right)}{\left(\sqrt{3} + \sqrt{2}i\right) - \left(\sqrt{3} - i\sqrt{2}\right)}$$

Solution

Given, $\dfrac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$

$= \dfrac{(3)^2-\left(i\sqrt{5}\right)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i} \quad \left[(a+b)(a-b)=a^2-b^2\right]$

$= \dfrac{9-5i^2}{2\sqrt{2}i}$

$= \dfrac{9-5(-1)}{2\sqrt{2}i} \quad (\because i^2=-1)$

$= \dfrac{9+5}{2\sqrt{2}i} \times \dfrac{i}{i}$

$= \dfrac{14i}{2\sqrt{2}i^2}$

$= \dfrac{14i}{2\sqrt{2}(-1)}$

$= \dfrac{-7i}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{-7\sqrt{2}i}{2}$

$= a+ib$ where $a=0, b=-\dfrac{7\sqrt{2}}{2}$

#418208

Topic: Euler Form of Complex Number

Find the modulus and the argument of the complex number $z=-1-i\sqrt{3}$

Solution

$z=-1-i\sqrt{3}$

Let $r\cos\theta=-1 \quad and \quad r\sin\theta=-\sqrt{3}$

On squaring and adding, we obtain

$(r\cos\theta)^2+(r\sin\theta)^2=(-1)^2+(-\sqrt{3})^2$

$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=1+3$

$\Rightarrow r^2=4 \quad \left[\cos^2\theta+\sin^2\theta=1\right]$

$\Rightarrow r=\sqrt{4}=2$(Conventionally,r>0)

$\therefore$ Modulus of z i.e $|z|=2$

$\therefore \quad 2\cos\theta=-1 \quad and \quad 2\sin\theta=-\sqrt{3}$

Since both the values of $\sin\theta$ and $\cos\theta$ are negative and $\sin\theta$ and $\cos\theta$ are negative in III quadrant,

Argument $= -\left(\pi-\dfrac{\pi}{3}\right)=\dfrac{-2\pi}{3}$

Thus, the modules and argument of the complex number $-1-\sqrt{3i}$ are 2 and $\dfrac{-2\pi}{3}$ respectively.

#418213

Topic: Euler Form of Complex Number

Find the modulus and the argument of the complex number $z=-\sqrt{3}+i$

Solution

Given, $z = -\sqrt{3} + i$

Let $r\cos\theta = -\sqrt{3}$and $r\sin\theta = 1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = (-\sqrt{3})^2 + 1^2$

$\Rightarrow r^2 = 3 + 1 = 4 \ (\because \cos^2\theta + \sin^2\theta = 1)$

$\Rightarrow \quad r = \sqrt{4} = 2$(Conventionally r>0)

$\therefore$ Modulus of z i.e. $|z| = 2$

$\therefore 2\cos\theta = -\sqrt{3}$and $2\sin\theta = 1$

$\Rightarrow \quad \cos\theta = \dfrac{-\sqrt{3}}{2} and \quad \sin\theta = \dfrac{1}{2}$

$\therefore \quad \theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$[As θ lies in the II quadrant]

Thus, the modulus and argument of the complex number $-\sqrt{3} + i$ are 2 and $\dfrac{5\pi}{6}$

#418218

Topic: Euler Form of Complex Number

Convert the given complex number in polar form : $1 - i$

Solution

Given, $z = 1 - i$

Let $r\cos\theta = 1 and \quad r\sin\theta = -1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + (-1)^2$

$\Rightarrow r^2 \left(\cos^2\theta + \sin^2\theta\right) = 2$

$\Rightarrow r^2 = 2$

$\Rightarrow r = \sqrt{2}$ (since,r>0)

$\therefore \sqrt{2}\cos\theta = 1$and $\sqrt{2}\sin\theta = -1$

$\therefore \quad \theta = -\dfrac{\pi}{4}$ (As θ lies in fourth quadrant.)

So, the polar form is

$\therefore 1 - i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\left(\dfrac{-\pi}{4}\right) + i\sqrt{2}\sin\left(\dfrac{-\pi}{4}\right)$

$= \sqrt{2}\left[\cos\left(\dfrac{-\pi}{4}\right) + i\sin\left(\dfrac{-\pi}{4}\right)\right]$

#418219

Topic: Euler Form of Complex Number

Convert the given complex number in polar form : $-1 + i$

Solution

Given, $z = -(1-i)$

Let $r\cos\theta = 1$ and $r\sin\theta = -1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + (-1)^2$

$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1 + 1$

$\Rightarrow \quad r^2 = 2$

$\Rightarrow \quad r = \sqrt{2}$ (Conventionally r>0)

$\therefore \sqrt{2}\cos\theta = 1$ and $\sqrt{2}\sin\theta = -1$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}}$ and $\sin\theta = -\dfrac{1}{\sqrt{2}}$

$\therefore \theta = -\dfrac{\pi}{4}$ [As θ lies in the Ii quadrant]

So, the polar form of $z = -(1-i)$ is

$\therefore -(1-i) = -(r\cos\theta + ir\sin\theta) = -\left(\sqrt{2}\cos\left(-\frac{\pi}{4}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{4}\right)\right)$

$= -\sqrt{2}\left[-\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right]$

$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

#418221

Topic: Euler Form of Complex Number

Convert the given complex number in polar form : -3

Solution

Given, $z = -3$

Let, $r\cos\theta = -3$ and $r\sin\theta = 0$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = (-3)^2$

$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 9$

$\Rightarrow r^2 = 9$

$\Rightarrow r = \sqrt{9} = 3$ (Conventionally $r > 0$)

$\therefore 3\cos\theta = -3$ and $3\sin\theta = 0$

$\Rightarrow \cos\theta = -1$ and $\sin\theta = 0$

$\therefore \theta = \pi$

So, the polar form is

$\therefore -3 = r\cos\theta + ir\sin\theta = 3\cos\pi + 3\sin\pi = 3\left(\cos\pi + i\sin\pi\right)$

#418223

Topic: Euler Form of Complex Number

Convert the given complex number in polar form : $\sqrt{3} + i$

Solution

$z = \sqrt{3} + i$

Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = (\sqrt{3})^2 + 1^2$

$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 3 + 1$

$\Rightarrow r^2 = 4$

$\Rightarrow r = \sqrt{4} = 2$ (Conventionally $r > 0$)

$\therefore 2\cos\theta = \sqrt{3}$ and $2\sin\theta = 1$

$\Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$

$\therefore \theta = \dfrac{\pi}{6}$ [As θ lies in the I quadrant]

So, the polar form is

$\therefore \quad \sqrt{3} + i = r\cos\theta + ir\sin\theta = 2\cos\dfrac{\pi}{6} + i2\sin\dfrac{\pi}{6} = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

#418224

Topic: Euler Form of Complex Number

Convert the given complex number in polar form: i

Solution

$z = i$

Let $r\cos\theta = 0$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = 0^2 + 1^2$

$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1$

$\Rightarrow r^2 = 1$

$\Rightarrow r = \sqrt{1} = 1$(Since, $r > 0$)

$\therefore \cos\theta = 0$ and $\sin\theta = 1$

$\therefore \theta = \dfrac{\pi}{2}$

So, the polar form is

$\therefore i = r\cos\theta + ir\sin\theta = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$

#418286

Topic: Basics of Complex Numbers

Evalaute : $\left[i^{18} + \left(\dfrac{1}{i}\right)^{25}\right]^3$

Solution

Given, $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

$=\left(i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right)^3$

$=\left(\left(i^4\right)^4.\,i^2+\frac{1}{\left(i^4\right)^6.\,i}\right)^3$

$=\left(i^2+\frac{1}{i}\right)^3 \quad (\because i^4=1)$

$=\left(-1+\frac{1}{i}\times\frac{i}{i}\right)^3 \quad (\because i^2=-1)$

$=\left(-1+\frac{i}{i^2}\right)^3$

$=(-1-i)^3$

$=(-1)^3(1+i)^3$

$=-\left[1^3+i^3+3.1.i\,(1+i)\right]$

$=-\left(1+i^3+3i+3i^2\right)$

$=-(1-i+3i-3)$

$=-(-2+2i)$

$=2-2i$

#418289

Topic: Operations on Complex Numbers

For any two complex numbers z_1 and z_2, prove that $Re\,(z_1z_2)=Rez_1Rez_2-Imz_1Imz_2$

Solution

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

$\Rightarrow z_1z_2=(x_1+iy_1)\,(x_2+iy_2)$

$=x_1x_2+i(y_1x_2+x_1y_2)+i^2y_1y_2$

$=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)$ since $(i^2=-1)$

Hence $Re(z_1z_2)=Re(z_1)Re(z_2)-Im(z_1)Im(z_2)$

#418294

Topic: Operations on Complex Numbers

Reduce $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Solution

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$$

$$=\left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left(\frac{3-4i}{5+i}\right)$$

$$=\left(\frac{1+i-2+8i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right)$$

$$=\left(\frac{-1+9i}{5-3i}\right)\left(\frac{3-4i}{5+i}\right)$$

$$=\left(\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right)$$

$$=\frac{33+31i}{28-10i}=\frac{33+31i}{2(14-5i)}$$

$=\dfrac{(33+31i)}{2(14-5i)}\times\dfrac{(14+5i)}{(14+5i)}$ [on multiplying numerator and denominator by $(14+5i)$]

$$=\frac{462+165i+434i+155i^2}{2\left[(14)^2-(5i)^2\right]}=\frac{307+599i}{2\left(196-25i^2\right)}$$

$$=\frac{307+599i}{2(221)}=\frac{307+599i}{442}=\frac{307}{442}+\frac{599i}{442}$$

which is the required standard form.

#418296

Topic: Operations on Complex Numbers

If $x-iy=\sqrt{\dfrac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2=\dfrac{a^2+b^2}{c^2+d^2}$

Solution

$$x-iy=\sqrt{\frac{a-ib}{c-id}}$$

$$=\sqrt{\frac{a-ib}{c-id}\times\frac{c+id}{c+id}}$$

$$=\sqrt{\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}}$$

$$\therefore\quad (x-iy)^2=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

$$\Rightarrow\quad x^2-y^2-2ixy=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

On comparing real and imaginary parts, we obtain

$x^2-y^2=\dfrac{ac+bd}{c^2+d^2}, -2xy=\dfrac{ad-bc}{c^2+d^2}$.....(1)

$$\therefore \left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+4x^2y^2$$

$$=\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{ad-bc}{c^2+d^2}\right)^2[using\quad 1]$$

$$=\frac{a^2c^2+b^2d^2+2acbd+a^2d^2+b^2c^2-2adbc}{\left(c^2+d^2\right)^2}$$

$$=\frac{a^2c^2+b^2d^2+a^2d^2+b^2c^2}{\left(c^2+d^2\right)^2}$$

$$=\frac{a^2\left(c^2+d^2\right)+b^2\left(c^2+d^2\right)}{\left(c^2+d^2\right)^2}$$

$$=\frac{\left(c^2+d^2\right)\left(a^2+b^2\right)}{\left(c^2+d^2\right)^2}=\frac{a^2+b^2}{c^2+d^2}$$

Hence, proved

#418307

Topic: Euler Form of Complex Number

Convert the following in the polar form:

(i) $\dfrac{1+7i}{(2-i)^2}$

(ii) $\dfrac{1+3i}{1-2i}$

Solution

(i) Here, $z=\dfrac{1+7i}{(2-i)^2}$

$=\dfrac{1+7i}{4+i^2-4i}$

$=\dfrac{1+7i}{4-1-4i}$

$=\dfrac{1+7i}{3-4i}\times\dfrac{3+4i}{3+4i}$

$=\dfrac{3+4i+21i+28i^2}{3^2+4^2}$

$=\dfrac{3+4i+21i-28}{3^2+4^2}=\dfrac{-25+25i}{25}$

$=-1+i$

Let $r\cos\theta=-1$ and $r\sin\theta=1$

On squaring and adding, we obtain

$r^2\left(\cos^2\theta+\sin^2\theta\right)=1+1$

$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=2$

$\Rightarrow r^2=2 \quad (\because \cos^2\theta+\sin^2\theta=1)$

$\Rightarrow r=\sqrt{2}$ (As r>0)

$\therefore \sqrt{2}\cos\theta=-1$ and $\sqrt{2}\sin\theta=1$

$\Rightarrow \cos\theta=\dfrac{-1}{\sqrt{2}}$ and $\sin\theta=\dfrac{1}{\sqrt{2}}$

$\therefore \theta=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$ (As θ lies in II quadrant)

$\therefore z=r\cos\theta+ir\sin\theta$

$=\sqrt{2}\cos\dfrac{3\pi}{4}+i\sqrt{2}\sin\dfrac{3\pi}{4}=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

This is the required polar form.

(ii) Here,

$z=\dfrac{1+3i}{1-2i}$

$=\dfrac{1+3i}{1-2i}\times\dfrac{1+2i}{1+2i}$

$=\dfrac{1+2i+3i-6}{1+4}$

$=\dfrac{-5+5i}{5}$

$=-1+i$

Let $r\cos\theta=-1$ and $r\sin\theta=1$

On squaring and adding, we obtain

$r^2\left(\cos^2\theta+\sin^2\theta\right)=1+1$

$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=2$

$\Rightarrow r^2=2 \quad (\because \cos^2\theta+\sin^2\theta=1)$

$\Rightarrow r = \sqrt{2}$ (As r>0)

$\therefore \sqrt{2}\cos\theta = -1$ and $\sqrt{2}\sin\theta = 1$

$\Rightarrow \cos\theta = \dfrac{-1}{\sqrt{2}}$ and $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$ (As θ lies in II quadrant)

$\therefore z = r\cos\theta + ir\sin\theta$

$= \sqrt{2}\cos\dfrac{3\pi}{4} + i\sqrt{2}\sin\dfrac{3\pi}{4} = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

This is the required polar form.

#418325

Topic: Operations on Complex Numbers

If $z_1 = 2 - i, z_2 = 1 + i$ find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$

Solution

We have $z_1 = 2 - i, z_2 = 1 + i$

$\therefore \quad \left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\dfrac{(2-i) + (1+i) + 1}{(2-i) - (1+i) + 1}\right|$

$= \left|\dfrac{4}{2-2i}\right| = \left|\dfrac{4}{2(1-i)}\right|$

$= \left|\dfrac{2}{1-i} \times \dfrac{1+i}{1+i}\right| = \left|\dfrac{2(1+i)}{1^2 - i^2}\right|$

$= \left|\dfrac{2(1+i)}{1+1}\right| = \left|\dfrac{2(1+i)}{2}\right|$

$= |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

#418332

Topic: Basics of Complex Numbers

If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \dfrac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$

Solution

$a + ib = \dfrac{(x+i)^2}{2x^2+1}$

$= \dfrac{x^2 + i^2 + 2xi}{2x^2+1}$

$= \dfrac{x^2 - 1 + 2xi}{2x^2+1}$

$= \dfrac{x^2-1}{2x^2+1} + i\left(\dfrac{2x}{2x^2+1}\right)$

On comparing real and imaginary parts, we obtain

$a = \dfrac{x^2-1}{2x^2+1} and \quad b = \dfrac{2x}{2x^2+1}$

$\therefore \quad a^2 + b^2 = \left(\dfrac{x^2-1}{2x^2+1}\right)^2 + \left(\dfrac{2x}{2x^2+1}\right)^2$

$= \dfrac{x^4 + 1 - 2x^2 + 4x^2}{\left(2x^2+1\right)^2}$

$= \dfrac{x^4 + 1 + 2x^2}{\left(2x^2+1\right)^2}$

$= \dfrac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$

$\therefore \quad a^2 + b^2 = \dfrac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$

Hence, proved.

#418335

Topic: Operations on Complex Numbers

Let $z_1 = 2 - i, z_2 = -2 + i$

Find

(i) $Re\left(\frac{z_1 z_2}{\bar{z_1}}\right)$

(ii) $Im\left(\frac{1}{z_1 \bar{z_1}}\right)$

Solution

$z_1 = 2 - i \Rightarrow \bar{z_1} = 2 + i, z_2 = -2 + i$

(i) $z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$

$\therefore \quad \frac{z_1 z_2}{\bar{z_1}} = \frac{-3 + 4i}{2 + i} = \frac{(-3 + 4i)(2 - i)}{(2 + i)(2 - i)}$

$= \frac{-6 + 3i + 8i - 4i^2}{2^2 + 1^2} = \frac{-6 + 11i - 4(-1)}{2^2 + 1^2}$

$= \frac{-2 + 11i}{5} = \frac{-2}{5} + \frac{11}{5}i$

$\therefore \mathrm{Re}\left(\frac{z_1 z_2}{\bar{z_1}}\right) = \frac{-2}{5}$

(ii) $\frac{1}{z_1 \bar{z_1}} = \frac{1}{(2 - i)(2 + i)} = \frac{1}{(2)^2 + (1)^2} = \frac{1}{5}$

$\therefore \mathrm{Im}\left(\frac{1}{z_1 \bar{z_1}}\right) = 0$

#418340

Topic: Euler Form of Complex Number

Find the modulus and argument of the complex number $\frac{1 + 2i}{1 - 3i}$

Solution

Let, $z = \dfrac{1+2i}{1-3i}$

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{1+3i+2i+6i^2}{1^2+3^2}$$

$$= \frac{1+5i+6(-1)}{1+9}$$

$$= \frac{-5+5i}{10}$$

$$= \frac{-1}{2} + \frac{1}{2}i$$

Let $z = r\cos\theta + ir\sin\theta$

$i.e., r\cos\theta = \dfrac{-1}{2}$and $r\sin\theta = \dfrac{1}{2}$

On squaring and adding, we obtain

$r^2\left(\cos^2\theta + \sin^2\theta\right) = \left(\dfrac{-1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2$

$\Rightarrow r^2 = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2}$

As $r > 0$

$\Rightarrow r = \dfrac{1}{\sqrt{2}}$

$\therefore \dfrac{1}{\sqrt{2}}\cos\theta = \dfrac{-1}{2}$ and $\dfrac{1}{\sqrt{2}}\sin\theta = \dfrac{1}{2}$

$\Rightarrow \quad \cos\theta = \dfrac{-1}{\sqrt{2}}$ and $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$[As θ lies in the II quadrant]

Therefore the modulus and argument of the given complex number are $\dfrac{1}{\sqrt{2}}$ and $\dfrac{3\pi}{4}$ respectively.

#418345

Topic: Conjugate and its Properties

Find the real numbers x and y , if $(x-iy)(3+5i)$is the conjugate of $-6-24i$.

Solution

Let $z=(x-iy)(3+5i)$

$z=3x+5xi-3yi-5yi^2=3x+5xi-3yi+5y=(3x+5y)+i(5x-3y)$

$\therefore \quad \bar{z}=(3x+5y)-i(5x-3y)$

It is given that, $\bar{z}=-6-24i$

$\therefore \quad (3x+5y)-i(5x-3y)=-6-24i$

Equating real and imaginary parts, we obtain

$3x+5y=-6.........(i)$

$5x-3y=24..........(ii)$

Multiplying equation (i) by 3 and equation (ii) by 5 and then adding them, we obtain

$9x+15y=-18$

$25x-15y=120$

$34x=102$

$\therefore \quad x=\dfrac{102}{34}=3$

Putting the value of x in equation (i) , we obtain

$3(3)+5y=-6$

$\Rightarrow \quad 5y=-6-9=-15$

$\Rightarrow \quad y=-3$

Thus, the values of x and y are 3 and -3 respectively.

#418346

Topic: Modulus of Complex Numbers

Find the modulus of : $\dfrac{1+i}{1-i}-\dfrac{1-i}{1+i}$

Solution

Let $z=\dfrac{1+i}{1-i}-\dfrac{1-i}{1+i}=\dfrac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$

$=\dfrac{1+i^2+2i-1-i^2+2i}{1^2+1^2}=\dfrac{4i}{2}=2i$

$\therefore |z|=|2i|=\sqrt{2^2}=2$

#418348

Topic: Operations on Complex Numbers

If $(x+iy)^3=u+iv$, then show that $\dfrac{u}{x}+\dfrac{v}{y}=4\left(x^2-y^2\right)$

Solution

$(x+iy)^3=u+iv$

$\Rightarrow \quad x^3+(iy)^3+3.x.iy(x+iy)=u+iv$

$\Rightarrow \quad x^3+i^3y^3+3x^2yi+3xy^2i^2=u+iv$

$\Rightarrow \quad x^3-iy^3+3x^2yi-3xy^2=u+iv$

$\Rightarrow \quad \left(x^3-3xy^2\right)+i\left(3x^2y-y^3\right)=u+iv$

On equating real and imaginary parts, we obtain

$u=x^3-3xy^2, v=3x^2y-y^3$

$\therefore \quad \dfrac{u}{x}+\dfrac{v}{y}=\dfrac{x^3-3xy^2}{x}+\dfrac{3x^2y-y^3}{y}$

$=\dfrac{x\left(x^2-3y^2\right)}{x}+\dfrac{y\left(3x^2-y^2\right)}{y}$

$=x^2-3y^2+3x^2-y^2$

$=4x^2-4y^2$

$=4\left(x^2-y^2\right)$

$\therefore \quad \dfrac{u}{x}+\dfrac{v}{y}=4\left(x^2-y^2\right)$

Hence, proved.

#418353

Topic: Modulus of Complex Numbers

If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

Solution

Let $a = a + ib$and $\beta = x + iy$

It is given that, $|\beta| = 1 \Rightarrow \sqrt{x^2+y^2} = 1$

$\Rightarrow x^2 + y^2 = 1....(i)$

$$\therefore \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{(x+iy)-(a+ib)}{1-(a-ib)(x+iy)}\right|$$

$$= \left|\frac{(x-a)+i(y-b)}{1-(ax-aiy-ibx+by)}\right|$$

$$= \left|\frac{(x-a)+i(y-b)}{(1-ax-by)+i(bx-ay)}\right|$$

$$= \frac{|(x-a)+i(y-b)|}{|(1-ax-by)+i(bx-ay)|}$$

$$= \frac{\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(1-ax-by)^2+(bx-ay)^2}}$$

$$= \frac{\sqrt{x^2+a^2-2ax+y^2+b^2-2by}}{\sqrt{1+a^2x^2+b^2y^2-2ax+2abxy-2by+b^2x^2+a^2y^2-2abxy}}$$

$$= \frac{\sqrt{(x^2+y^2)+a^2+b^2-2ax-2by}}{\sqrt{1+a^2(x^2+y^2)+b^2(y^2+x^2)-2ac-2by}}$$

$$= \frac{\sqrt{1+a^2+b^2-2ax-2by}}{\sqrt{1+a^2+b^2-2ax-2by}} \text{ [Using (i)]}$$

$= 1$

#418362

Topic: Modulus of Complex Numbers

Find the number of non-zero integral solution of the equation $|1-i|^x = 2^x$

Solution

Given, $|1-i|^x = 2^x$

$\Rightarrow \quad \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$

$\Rightarrow \quad (\sqrt{2})^x = 2^x$

$\Rightarrow \quad 2^{\frac{x}{2}} = 2^x$

$\Rightarrow \quad \dfrac{x}{2} = x$

$\Rightarrow \quad x = 2x$

$\Rightarrow x = 0$

Thus, $x = 0$ is the only integral solution of the given equation.

Therefore, the number of non-zero integral solutions of the given equation is 0.

#418370

Topic: Operations on Complex Numbers

If $(a+ib)(c+id)(e+if)(g+ih)=A+iB$ then show that $\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right)=A^2+B^2$

Solution

$(a+ib)(c+id)(e+if)(g+ih)=A+iB$

Taking modulus both sides,

$\Rightarrow \quad |(a+ib)(c+id)(e+if)(g+ih)|=|A+iB|$

$\Rightarrow \quad |(a+ib)|\times|(c+id)|\times|(e+if)|\times|(g+ih)|=|A+iB|$

$\Rightarrow \quad \sqrt{a^2+b^2}\times\sqrt{c^2+d^2}\times\sqrt{e^2+f^2}\times\sqrt{g^2+h^2}=\sqrt{A^2+B^2}$

On squaring both sides, we obtain

$\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right)=A^2+B^2$

Hence, proved

#418380

Topic: Basics of Complex Numbers

If $\left(\dfrac{1+i}{1-i}\right)^m=1$, then find the least positive integral value of m.

Solution

$\left(\dfrac{1+i}{1-i}\right)^m=1$

$\Rightarrow \quad \left(\dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}\right)^m=1$

$\Rightarrow \quad \left(\dfrac{(1+i)^2}{1^2+1^2}\right)^m=1$

$\Rightarrow \quad \left(\dfrac{1^2+i^2+2i}{2}\right)^m=1$

$\Rightarrow \quad \left(\dfrac{1-1+2i}{2}\right)^m=1$

$\Rightarrow \quad \left(\dfrac{2i}{2}\right)^m=1$

$\Rightarrow \quad i^m=1$

$\therefore \quad m=4k$, where k is some integer.

Therefore, the least positive integer k is 1.

Thus the least positive integral value of m is 4.

#465305

Topic: Solving Quadratic Equation

Find the roots of the following equations:

(i) $x-\dfrac{1}{x}=3, x\neq 0$

(ii) $\dfrac{1}{x+4}-\dfrac{1}{x-7}=\dfrac{11}{30}, x\neq -4, 7$

Solution

i)

$x-\frac{1}{x}=3$

$\Rightarrow x^2-3x-1=0$

$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow a=1, b=-3, c=-1$

$\Rightarrow x=\frac{3\pm\sqrt{9+4}}{2}$

$\Rightarrow x=\frac{3\pm\sqrt{13}}{2}$

ii)

$\frac{1}{(x+4)}-\frac{1}{(x-7)}=\frac{11}{30}$

$\Rightarrow \frac{(x-7-x-4)}{(x+4)(x-7)}=\frac{11}{30}$

$\Rightarrow \frac{-11}{(x+4)(x-7)}=\frac{11}{30}$

$\Rightarrow -30=x^2-3x-28$

$\Rightarrow x^2-3x+2=0$

$\Rightarrow x^2-2x-x+2=0$

$\Rightarrow x(x-2)-1(x-2)=0$

$\Rightarrow (x-2)(x-1)=0$

$\Rightarrow x=1,2$

#465315

Topic: Nature of Roots

Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:

(i) $2x^2-3x+5=0$

(ii) $3x^2-4\sqrt{3}x+4=0$

(iii) $2x^2-6x+3=0$

Solution

i)

$x^2 - 3x + 5 = 0$
$a = 2, b = -3, c = 5$

Discriminant, $D = b^2 - 4ac = 9 - 40 = -31$

Since $D < 0,$ the roots are imaginary for the given equation.

ii)

$3x^2 - 4\sqrt{3}x + 4 = 0$
$a = 3, b = -4\sqrt{3}, c = 4$

Discriminant, $D= b^2 - 4ac = 48 - 48 = 0$

Since $D = 0,$ the roots are real and equal for the given equation.

$3x^2 - 4\sqrt{3}x + 4 = 0$
$a = 3, b = -4\sqrt{3}, c = 4$
$x = \dfrac{[-b \pm \sqrt{b^2 - 4ac}]}{2a}$
$x = \dfrac{4\sqrt{3}}{6} = \dfrac{2}{\sqrt{3}}$

iii)

$2x^2 - 6x + 3 = 0$
$a = 2, b = -6, c = 3$
Discriminant, $D = b^2 - 4ac = 36 - 24 = 12$

Since $D > 0,$ roots are real but not equal.

$x = \dfrac{[-b \pm \sqrt{b^2 - 4ac}]}{2a}$
$x = \dfrac{6 \pm \sqrt{12}}{4}$
$x = \dfrac{6 \pm 2\sqrt{3}}{4}$
$x = \dfrac{3 \pm \sqrt{3}}{2}$

#465316
Topic: Nature of Roots

Find the values of k for each of the following quadratic equations, so that they have two equal roots.

(i) $2x^2 + kx + 3 = 0$

(ii) $kx\,(x - 2) + 6 = 0$

Solution

i)

If $2x^2+kx+3=0$has equal roots, then $b^2-4ac=0$

Here, $a=2, b=k, c=3$

$x^2+kx+3=0$
$b^2-4ac=0$
$k^2-4(2)(3)=0$
$k^2=24$
$k=\pm\sqrt{24}=\pm 2\sqrt{6}$

ii)

If $kx^2-2kx+6=0$has equal roots, then $b^2-4ac=0$

Here, $a=k, b=-2k, c=6$

$kx(x-2)+6=0$
$kx^2-2kx+6=0$
$4k^2-4(k)(6)=0$
$4k^2-24k=0$
$4k(k-6)=0$
For $k=0,$ equation is not quadratic.

$k=6$