Chapter 1

Introduction to Linear Programming.

This chapter introduces notations, terminologies and formulations of linear programming. Examples will be given to show how real-life problems can be modeled as linear programs. The graphical approach will be used to solve some simple linear programming problems.

What is Linear Programming?

A typical optimization problem is to *find the best element from a given set.*
In order to compare elements, we need a criterion, which we call an **objective function** $f(\mathbf{x})$.
The given set is called the **feasible set** which is usually defined by

$$\{\mathbf{x} \in \mathbf{R}^n \mid g_i(\mathbf{x}) \leq 0,\ i = 1, \ldots, m\}$$

Such an optimization problem can be formulated as

$$\begin{aligned} &\text{Maximize } \ f(\mathbf{x}) \\ &\text{Subject to } \ g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m. \end{aligned}$$

In this course, we study a special optimization problem in which f and g_i are all linear functions, so called **linear programming**.

Why do we study Linear Programming?

- It is simple, thus can be efficiently solved.
- It is the basis for the development of solution algorithms of other (more complex) types of Operations research (OR) models, including integer, nonlinear, and stochastic programming.

1.1 General Linear Programming problems.

In this section, the general linear programming problem is introduced followed by some examples to help us familiarize with some basic terminology used in LP.

Notation

1. For a matrix $\mathbf{A}$, we denote its transpose by $\mathbf{A}^T$.
2. An n-dimensional vector $\mathbf{x} \in \mathbf{R}^n$ is denoted by a column vector
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}.$$
3. For vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$, the following denotes the matrix multiplication:
$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

In a **general linear programming problem**, a cost vector $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$ is given. The objective is to minimize or maximize a linear cost function $\mathbf{c}^T\mathbf{x} = \sum_{i=1}^{n} c_i x_i$ over all vectors

$$\mathbf{x} = (x_1, x_2, \cdots, x_n)^T,$$

subject to a finite set of linear equality and inequality constraints. This can be summarized as follows:

$$\begin{array}{ll}
\text{Minimize} & \mathbf{c}^T\mathbf{x} \\
\text{(Or maximize)} & \\
\text{Subject to} & \mathbf{a}_i^T\mathbf{x} \geq b_i, \quad i \in M_+, \\
 & \mathbf{a}_i^T\mathbf{x} \leq b_i\ , \quad i \in M_-, \\
 & \mathbf{a}_i^T\mathbf{x} = b_i\ , \quad i \in M_0, \\
 & x_j \geq 0\ , \qquad j \in N_+, \\
 & x_j \leq 0\ , \qquad j \in N_-,
\end{array}$$

where $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in})^T$ is a vector in $\mathbf{R}^n$ and b_i is a scalar.

$$\mathbf{a}_i^T\mathbf{x} = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$$

Terminology

1. Variables x_i are called **decision variables**. There are n of them.
2. Each constraint is either an equation or an inequality of the form $\leq$ or $\geq$. Constraints of the form $\mathbf{a}_i^T\mathbf{x}(\leq, =, \geq)b_i$ are sometimes known as **functional constraints**.
3. If j is in neither N_+ nor N_-, there are no restrictions on the sign of x_j. The variable x_j is said to be **unrestricted in sign** or a **unrestricted variable**.
4. A vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ satisfying all of the constraints is called a **feasible solution** or **feasible vector**. The set of all feasible solutions is called the **feasible set** or **feasible region**.
5. The function $\mathbf{c}^T\mathbf{x}$ is called the **objective function** or **cost function**.

6. A feasible solution $\mathbf{x}^*$ that **minimizes** (repectively **maximizes**) the objective function, i.e. $\mathbf{c}^T\mathbf{x}^* \leq \mathbf{c}^T\mathbf{x}$ (respectively $\mathbf{c}^T\mathbf{x}^* \geq \mathbf{c}^T\mathbf{x}$) for all feasible vectors $\mathbf{x}$) is called an **optimal feasible solution** or simply, an **optimal solution**. The value of $\mathbf{c}^T\mathbf{x}^*$ is then called the **optimal cost** or **optimal objective value**.
7. For a minimization (respectively maximization) problem, the cost is said to be **unbounded** or the optimal cost is $-\infty$ (repectively the optimal cost is ∞) if for every real number K we can find a feasible solution $\mathbf{x}$ whose cost is less than K (respectively whose cost is greater than K).
8. Maximizing $\mathbf{c}^T\mathbf{x}$ is equivalent to minimizing $-\mathbf{c}^T\mathbf{x}$. More precisely,

$$\max \mathbf{c}^T\mathbf{x} = -\min -\mathbf{c}^T\mathbf{x}.$$

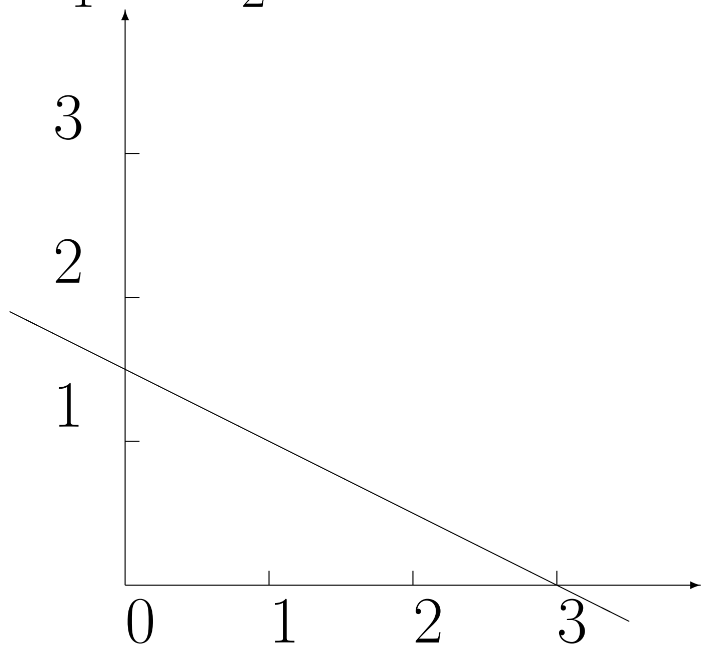
E.g., if $\mathbf{c}^T\mathbf{x} \in [1, 5]$, then

$$\max \mathbf{c}^T\mathbf{x} = 5$$
$$\min -\mathbf{c}^T\mathbf{x} = -5.$$

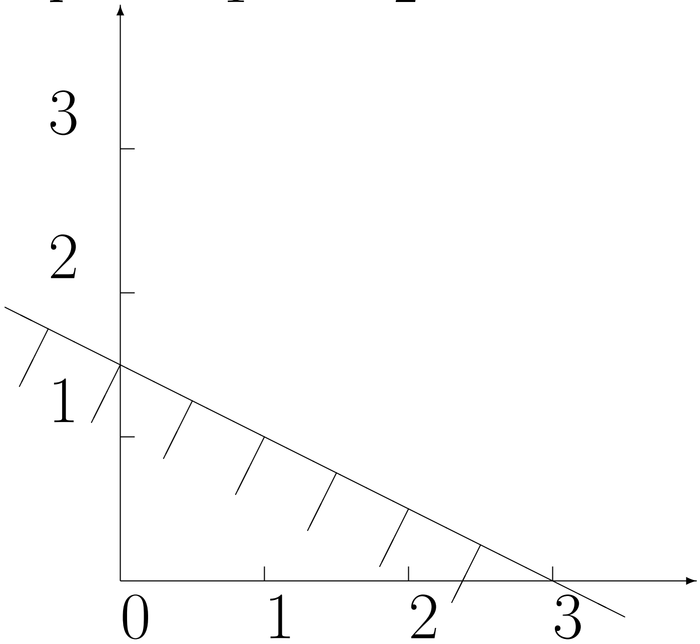
Graphical representation

In $\mathbf{R}^2$, the equation $\mathbf{a}_i^T\mathbf{x} = b_i$ describes a line perpendicular to $\mathbf{a}_i$, whereas in $\mathbf{R}^3$, the equation $\mathbf{a}_i^T\mathbf{x} = b_i$ describes a plane whose normal vector is $\mathbf{a}_i$. In $\mathbf{R}^n$, the equation $\mathbf{a}_i^T\mathbf{x} = b_i$ describes a hyperplane whose normal vector is $\mathbf{a}_i$. Moreover, $\mathbf{a}_i$ corresponds to the direction of increasing value of $\mathbf{a}_i^T\mathbf{x}$. The inequality $\mathbf{a}_i^T\mathbf{x} \leq b_i$ represents a half space. A set of inequalities represents the intersection of the half spaces.

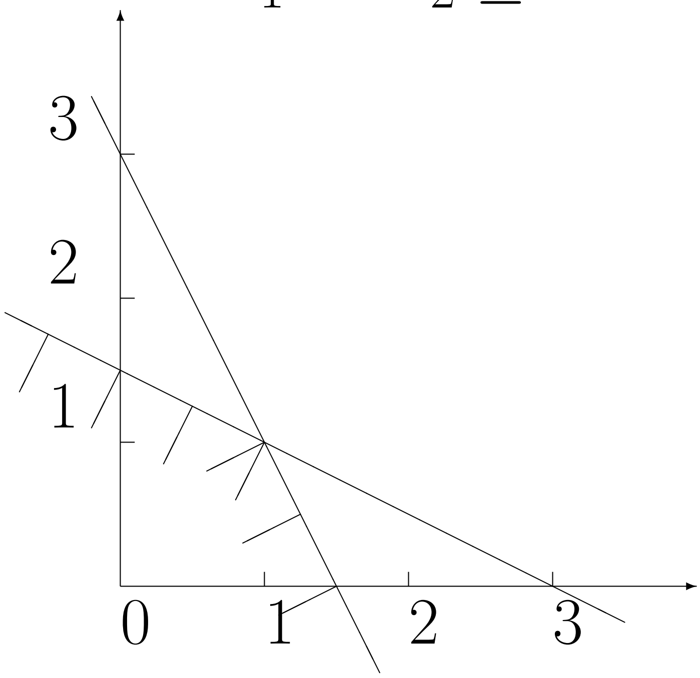
Line $x_1 + 2x_2 = 3$:



Half space $x_1 + 2x_2 \leq 3$:



Intersection $x_1 + 2x_2 \leq 3$ and $2x_1 + x_2 \leq 3$:

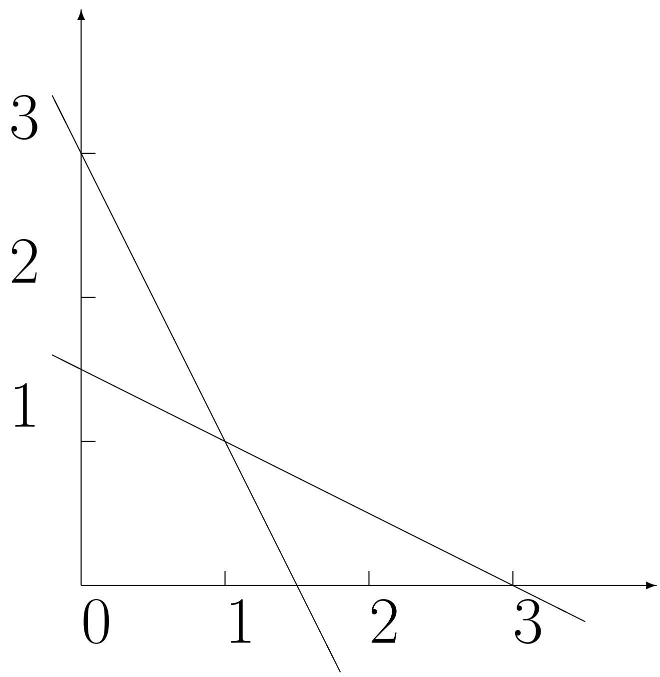


Example 1.1 (2-variables)
Consider the following LP problem:

$$\begin{array}{rrcrl} \text{minimize} & -x_1 & - & x_2 & \\ \text{subject to} & x_1 & + & 2x_2 & \leq 3 \\ & 2x_1 & + & x_2 & \leq 3 \\ & x_1, x_2 \geq 0. & & & \end{array}$$

(a) Sketch the feasible region and find an optimal solution of the LP graphically.

(b) If the cost function is changed to $-x_1 + 2x_2$, what is the optimal solution?



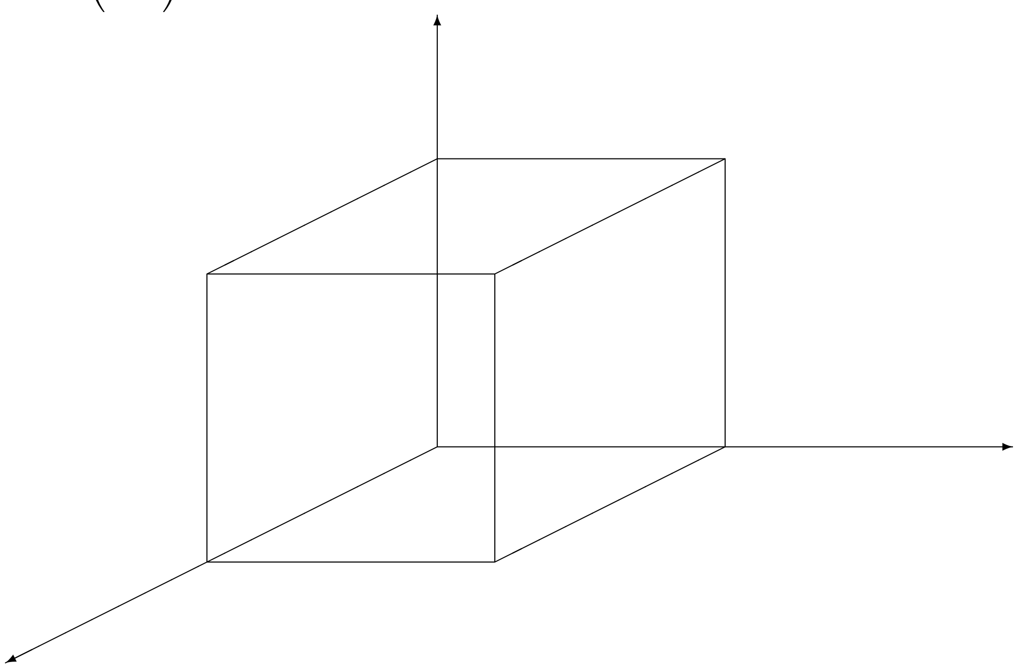
Observations

1. For any given scalar z, the set of points $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $\mathbf{c}^T\mathbf{x} = z$ is described by the line with equation $z = -x_1 - x_2$. This line is perpendicular to the vector $\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. (WHY?)
2. Different values of z lead to different lines, parallel to each other. Sketch lines corresponding to $z = 1$, and $z = -1$.
3. Increasing z corresponds to moving the line $z = -x_1 - x_2$ along the direction of the vector $\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Thus, to minimize z, the line is moved as much as possible in the direction of the vector $-\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (i.e. the opposite direction of the vector $\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$) within the feasible region.
4. The optimal solution $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a corner of the feasible set.

Example 1.2 (3-variable)
Consider the following LP problem:

$$\begin{array}{rl} \text{minimize} & -x_1 - x_2 - x_3 \\ \text{subject to} & x_i \leq 1, \quad i = 1, 2, 3, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

The feasible set is the unit cube, described by $0 \leq x_i \leq 1, i = 1, 2, 3$, and $\mathbf{c} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$. Then the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an optimal solution.



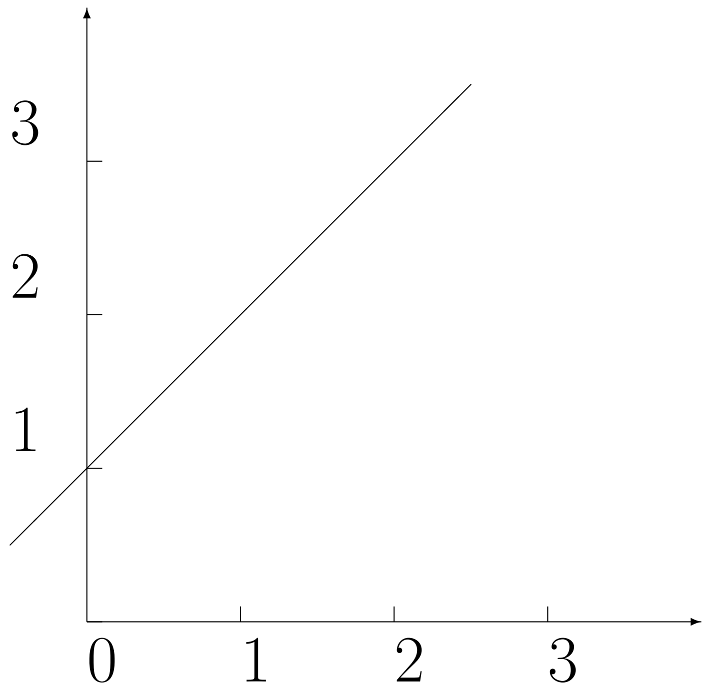
Example 1.3 (4-variable)

$$
\begin{array}{lrcrcrcrcl}
\text{Minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & & \\
\text{Subject to} & x_1 & + & x_2 & & & + & x_4 & \geq & 2 \\
 & & & 3x_2 & - & x_3 & & & = & 5 \\
 & & & & & x_3 & + & x_4 & \leq & 3 \\
 & x_1 & & & & & & & \geq & 0 \\
 & & & & & x_3 & & & \leq & 0
\end{array}
$$

We cannot present it graphically. How to solve it?

Example 1.4 Consider the feasible set in $\mathbf{R}^2$ of the linear programming problem.

$$\begin{array}{rrcl} & \text{minimize } \mathbf{c}^T\mathbf{x} & & \\ \text{Subject to} & -x_1 + x_2 & \leq & 1 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0. \end{array}$$



(a) For the cost vector $\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, there is a unique optimal solution $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) For $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, there are multiple optimal solutions $\mathbf{x}$ of the form $\mathbf{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ where $0 \leq x_2 \leq 1$. The set of optimal solutions is bounded.

(c) For $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, there are multiple optimal solutions $\mathbf{x}$ of the form $\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ where $x_1 \geq 0$. The set of optimal solutions is unbounded (some $\mathbf{x}$ is of arbitrarily large magnitude).

(d) For $\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, every feasible solution is not optimal. The optimal cost is unbounded or the optimal cost is $-\infty$.

(e) Imposing additional constraint $x_1 + x_2 \leq -2$, there is no feasible solution.

This example illustrates the following possibilities for a Linear Programming problem.

(a) There is a unique optimal solution.

(b) There are multiple optimal solutions. The set of optimal solutions is bounded or unbounded.

(c) The optimal cost is $-\infty$ and no feasible solution is optimal.

(d) The feasible set is empty. The problem is infeasible.

1.2 Formulation of LP problems.

The crux of formulating an LP model is:

Step 1 Identify the unknown variables to be determined (decision variables) and represent them in terms of algebraic symbols.

Step 2 Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of the decision variables.

Step 3 Identify the objective or criterion and represent it as a linear function of the decision variables, which is to be maximized or minimized.

Example 2.1 The diet problem

Green Farm uses at least 800 kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	kg per kg of feedstuff		
Feedstuff	Protein	Fiber	Cost ($ per kg)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The dietary requirements of the total feed stipulate at least 30% protein and at most 5% fiber. Green Farm wishes to determine the daily minimum-cost feed mix.

Formulate the problem as an LP problem.

Solution

Decision variables:

x_1 = kg of corn in the daily mix

x_2 = kg of soybean meal in the daily mix

Constraints:

Daily amount requirement: $x_1 + x_2 \geq 800$

Dietary requirements:

- Protein: $0.09x_1 + 0.60x_2 \geq 0.3(x_1 + x_2)$
- Fiber: $0.02x_1 + 0.06x_2 \leq 0.05(x_1 + x_2)$

Objective: minimize $0.3x_1 + 0.9x_2$

Thus, the complete model is

$$\begin{array}{rr} \text{minimize} & 0.3x_1 + 0.9x_2 \\ \text{subject to} & x_1 + x_2 \geq 800 \\ & -0.21x_1 + 0.3x_2 \geq 0 \\ & -0.03x_1 + 0.01x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{array}$$

Example 2.2 (The Reddy Mikks Company)
The Reddy Mikks Company owns a small paint factory that produces both interior and exterior house paints for wholesale distribution. Two basic raw materials, A and B, are used to manufacture the paints. The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirement of the raw materials per ton of the interior and exterior paints are summarized in the following table:

Tons of raw material per ton of paint

Raw Material	Exterior	Interior	Maximum Availability(tons)
A	1	2	6
B	2	1	8

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also shows that the maximum demand for interior paint is limited to 2 tons daily. The wholesale price per ton is \$ 3000 for exterior paint and \$ 2000 for interior paint.

How much interior and exterior paints should the company produce daily to maximize gross income?

Solution

Decision variables:

x_1 = number of tons of exterior paint produced daily

x_2 = number of tons of interior paint produced daily

Constraints:

Use of material A daily: $x_1 + 2x_2 \leq 6$

Use of material B daily: $2x_1 + x_2 \leq 8$

Daily Demand: $x_2 \leq x_1 + 1$

Maximum Demand: $x_2 \leq 2$.

Objective: maximize $3000x_1 + 2000x_2$

Thus, the complete LP model is:

$$\begin{aligned} \text{maximize} \quad & 3000x_1 + 2000x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Example 2.1* The diet problem

Suppose that there are n different foods and m different nutrients, and that we are given the following table with the nutritional content of a unit of each food:

	food 1	$\cdots$	food n
nutrient 1	a_{11}	$\cdots$	a_{1n}
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
nutrient m	a_{m1}	$\cdots$	a_{mn}

Let b_i be the requirements of an 'ideal food', nutrient i.

Given the cost c_j per unit of Food j, $j = 1, 2, \cdots, n$. The problem of mixing nonnegative quantities of available foods to synthesize the ideal food at minimal cost is an LP problem.

Let x_j, $j = 1, 2, \cdots, n$, be the quantity of Food j to synthesize the ideal food. The formulation of the

LP is as follows:

$$\begin{aligned} \text{Minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{Subject to} \quad & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \\ & \qquad\qquad\qquad\qquad i = 1, 2, \cdots, m, \\ & x_j \geq 0, \quad j = 1, 2, \cdots, n. \end{aligned}$$

A variant of this problem: Suppose b_i specify the minimal requirements of an adequate diet. Then $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ is replaced by

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i.$$

Example 2.2* A production problem

A firm produces n different goods using m different raw materials.

Let b_i, $i = 1, 2, \cdots, m$, be the available amount of ith raw material.

The jth good, $j = 1, 2, \cdots, n$, requires a_{ij} units of the ith raw material and results in a revenue of c_j per unit produced. The firm faces the problem of deciding how much of each good to produce in order to maximize its total revenue.

Let x_j, $j = 1, 2, \cdots, n$, be the amount of the jth good. The LP formulation becomes:

$$\begin{array}{rl} \text{Maximize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{Subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i, \\ & \qquad\qquad\qquad\qquad i = 1, 2, \cdots, m, \\ & x_j \geq 0, \quad j = 1, 2, \cdots, n. \end{array}$$

Example 2.3 Bank Loan Policy (cf. Taha p. 39)

The ABC bank is in the process of formulating a loan policy involving a total of \$12 million. Being a full-service facility, the bank is obliged to grant loans to different clientele. The following table provides the types of loans, the interest rate charged by the bank, and the probability of bad debt from past experience:

Type of loan	Interest rate	Probability of Bad Debt
Personal	0.140	0.10
Car	0.130	0.07
Home	0.120	0.03
Farm	0.125	0.05
Commercial	0.100	0.02

Bad debts are assumed unrecoverable and hence no interest revenue. Competition with other financial institutions in the area requires that the bank allocate at least 40% of the total funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car and home loans. The bank also has a stated policy specifying that the overall ratio for bad debts on all loans may not exceed 0.04. How should funds be allocated to these types of loans to maximize the net rate of return?

Solution Let x_1, x_2, x_3, x_4, x_5 (in million dollars) be the amount of funds allocated to Personal loan, Car loan, Home loan, Farm loan and Commercial loan respectively.
Net return:

- Personal: $(0.9x_1)(0.14) - 0.1x_1 = 0.026x_1$.
- Car: $(0.93x_2)(0.130) - 0.07x_2 = 0.0509x_2$.
- Home: $(0.97x_3)(0.120) - 0.03x_3 = 0.0864x_3$.
- Farm: $(0.95x_4)(0.125) - 0.05x_4 = 0.06875x_4$.
- Commercial: $(0.98x_5)(0.100) - 0.02x_5 = 0.078x_5$.

Total Fund:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$

Competition:

$$\frac{x_4 + x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \geq 0.4$$

$$\iff 0.4x_1 + 0.4x_2 + 0.4x_3 - 0.6x_4 + 0.6x_5 \leq 0$$

Housing industry:

$$x_3 \geq 0.5(x_1 + x_2 + x_3) \iff 0.5x_1 + 0.5x_2 - 0.5x_3 \leq 0$$

Overall bad debt:

$$\frac{0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.02x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \leq 0.04$$

$$\iff 0.06x_1 + 0.03x_2 - 0.01x_3 + 0.01x_4 - 0.02x_5 \leq 0$$

The LP forumulation:

$$\begin{array}{lr} \text{maximize} & 0.026x_1 + 0.0509x_2 + 0.0864x_3 + 0.06875x_4 + 0.078x_5 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 + x_5 \leq 12 \\ & 0.4x_1 + 0.4x_2 + 0.4x_3 - 0.6x_4 + 0.6x_5 \leq 0 \\ & 0.5x_1 + 0.5x_2 - 0.5x_3 \leq 0 \\ & 0.06x_1 + 0.03x_2 - 0.01x_3 + 0.01x_4 - 0.02x_5 \leq 0 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Example 2.4 (Work Scheduling Problem)
A post office requires different numbers of full-time employees on different days of the weeks. The number of full-time employees required on each day is given below:

	Number of Employees
Day 1 = Monday	17
Day 2 = Tuesday	13
Day 3 = Wednesday	15
Day 4 = Thursday	19
Day 5 = Friday	14
Day 6 = Saturday	16
Day 7 = Sunday	11

Union rules state that each full-time employee must work five consecutive days and then receive two days off. The post office wants to meet its daily requirements with only full-time employees. Formulate an LP that the post office can use to minimize the number of full-time employees that must be hired.

Let x_j be the number of employees starting their week on Day j. The formulation of the LP becomes:

$$
\begin{array}{lccccccccccccccc}
\text{Minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & & \\
\text{Subject to} & x_1 & & & & & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 17 \\
& x_1 & + & x_2 & & & & & + & x_5 & + & x_6 & + & x_7 & \geq & 13 \\
& x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & + & x_7 & \geq & 15 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & + & x_7 & \geq & 19 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & & & & \geq & 14 \\
& & & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & & \geq & 16 \\
& & & & & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 11 \\
& x_j & \geq & 0, & & x_j & \multicolumn{3}{l}{\text{integer.}} & & & & & & &
\end{array}
$$

Note The additional constraint that x_j must be an integer gives rise to a linear *integer programming* problem. Finding optimal solutions to general integer programming problems is typically difficult.

1.3 Compact form and Standard form of a general linear programming problem.

Compact form of a general linear programming problem

In a general linear programming problem, note that each linear constraint, be it an equation or inequality, can be expressed in the form $\mathbf{a}_i^T\mathbf{x} \geq b_i$.

1. $\mathbf{a}_i^T\mathbf{x} = b_i \iff \mathbf{a}_i^T\mathbf{x} \geq b_i$ and $\mathbf{a}_i^T\mathbf{x} \leq b_i$.
2. $\mathbf{a}_i^T\mathbf{x} \geq b_i \iff -\mathbf{a}_i^T\mathbf{x} \leq -b_i$.
3. Constraints $x_j \geq 0$ or $x_j \leq 0$ are special cases of constraints of the form $\mathbf{a}_i^T\mathbf{x} \geq b_i$, where $\mathbf{a}_i$ is a unit vector and $b_i = 0$.

Thus, the feasible set in a general linear programming problem can be expressed exclusively in terms of inequality constraints of the form $\mathbf{a}_i^T\mathbf{x} \geq b_i$.

Suppose all linear constraints are of the form $\mathbf{a}_i^T\mathbf{x} \geq b_i$ and there are m of them in total. We may index these constraints by $i = 1, 2, \cdots, m$.

Let $\mathbf{b} = (b_1, b_2, \cdots, b_m)^T$, and $\mathbf{A}$ be the $m \times n$ matrix whose rows are $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_m^T$, i.e.

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}_m^T \end{bmatrix}.$$

Then the constraints $\mathbf{a}_i^T \mathbf{x} \geq b_i,\ i = 1, 2, \cdots, m$, can be expressed compactly in the form $\mathbf{Ax} \geq \mathbf{b}$. ($\mathbf{Ax} \geq \mathbf{b}$ denotes for each i, the i component of $\mathbf{Ax}$ is greater than or equal to the ith component of $\mathbf{b}$.)

The general linear programming problem can be written compactly as:

$$\begin{aligned} &\text{minimize } \mathbf{c}^T \mathbf{x} \\ &\text{(or maximize)} \\ &\text{subject to } \mathbf{Ax} \geq \mathbf{b} \end{aligned}$$

A linear programming problem of this form is said to be in **compact form**.

Example 3.1 Express the following LP problem in Example 1.3 in compact form.

$$\begin{array}{lrcrcrcrcr}
\text{Minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & & \\
\text{Subject to} & x_1 & + & x_2 & & & + & x_4 & \geq & 2 \\
& & & 3x_2 & - & x_3 & & & = & 5 \\
& & & & & x_3 & + & x_4 & \leq & 3 \\
& x_1 & & & & & & & \geq & 0 \\
& & & & & x_3 & & & \leq & 0
\end{array}$$

Rewrite the above LP as

$$\begin{array}{lrcrcrcrcr}
\text{Minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & & \\
\text{Subject to} & x_1 & + & x_2 & & & + & x_4 & \geq & 2 \\
& & & 3x_2 & - & x_3 & & & \geq & 5 \\
& & - & 3x_2 & + & x_3 & & & \geq & -5 \\
& & & & - & x_3 & - & x_4 & \geq & -3 \\
& x_1 & & & & & & & \geq & 0 \\
& & & & - & x_3 & & & \geq & 0
\end{array}$$

which is in the compact form with

$$\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ -5 \\ -3 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Standard Form Linear Programming Problem

A linear programming problem of the form

$$\begin{aligned} \text{minimize } & \mathbf{c}^T \mathbf{x} \\ \text{(or maximize)} & \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

is said to be in **standard form**.

Note Two optimization problems are said to be **equivalent** if an optimal solution to one problem can be constructed from an optimal solution to another.

A general linear programming problem can be transformed into an equivalent problem in standard form by performing the following steps when necessary:

1. *Elimination of nonpositive variable and free variables.*

   Replace nonpositive variable $x_j \leq 0$ by $\bar{x}_j = -x_j$, where $\bar{x}_j \geq 0$.

Replace unrestricted variable x_j by $x_j^+ - x_j^-$, and where new variables $x_j^+ \geq 0$ and $x_j^- \geq 0$.

2. *Elimination of inequality constraints.*

An inequality constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ can be converted to an equality constraint by introducing a **slack variable** s_i and the standard form constraints

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, \quad s_i \geq 0.$$

For example, $x_1 + 2x_2 \leq 3$ is converted to $x_1 + 2x_2 + S_1 = 3, \ \ S_1 \geq 0$.

An inequality constraint $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ can be converted to an equality constraint by introducing a **surplus variable** s_i and the standard form constraints

$$\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i, \quad s_i \geq 0.$$

For example, $3x_1+4x_2 \geq 1$ is converted to $3x_1+4x_2 - S_1 = 1, S_1 \geq 0$.

Example 3.2 Express the following LP problem in Example 1.3 in standard form.

$$\begin{array}{lrcrcrcrcl}
\text{Minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & & \\
\text{Subject to} & x_1 & + & x_2 & & & + & x_4 & \geq & 2 \\
& & & 3x_2 & - & x_3 & & & = & 5 \\
& & & & & x_3 & + & x_4 & \leq & 3 \\
& x_1 & & & & & & & \geq & 0 \\
& & & & & x_3 & & & \leq & 0
\end{array}$$

Replace $x_2 = x_2^+ - x_2^-$, $x_3 = -x_3'$, and $x_4 = x_4^+ - x_4^-$.

Add a surplus variable S_1 to the $\geq$-constraint, and add a slack variable S_2 to the $\leq$-constraint.

$$\begin{array}{lrcrcrcrcrcrcrcl}
\text{Minimize} & 2x_1 & - & x_2^+ & + & x_2^- & - & 4x_3' & & & & & & & & \\
\text{Subject to} & x_1 & + & x_2^+ & - & x_2^- & & & + & x_4^+ & - & x_4^- & - & S_1 & = & 2 \\
& & & 3x_2^+ & - & 3x_2^- & + & x_3' & & & & & & & = & 5 \\
& & & & & & - & x_3' & + & x_4^+ & - & x_4^- & + & S_2 & = & 3
\end{array}$$

$$x_1, x_2^+, x_2^-, x_3', x_4^+, x_4^-, S_1, S_2 \geq 0$$

which is in the standard form with

$$\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3' \\ x_4^+ \\ x_4^- \\ S_1 \\ S_2 \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 3 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

Remark (Why do we need different forms of LP problems?)

1. The general (compact) form $\mathbf{Ax} \geq \mathbf{b}$ is often used to develop the theory of linear programming.
2. The standard form $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ is computationally convenient when it comes to algorithms such as simplex methods.

1.4 Piecewise linear convex objective functions.

Piecewise linear convex function

The notation $\max_{i=1,\cdots,m}\{a_i\}$ or $\max\{a_1,\ldots,a_m\}$ denotes the maximum value among $a_1, a_2, \cdots, a_m$.

A function of the form $\max_{i=1,\cdots,m}(\mathbf{c}_i^T\mathbf{x} + d_i)$ is called **a piecewise linear convex function**.

Example 4.1

(a) Sketch the graph of $y = \max(2x, 1-x, 1+x)$.

(b) Express the absolute value function $f(x) = |x|$ as a piecewise linear convex function.

The following problem is not a formulation of an LP problem.

$$\begin{array}{ll} \text{Minimize} & \max\,(x_1, x_2, x_3) \\ \text{Subject to} & 2x_1 + 3x_2 \le 5 \\ & x_2 - 2x_3 \le 6 \\ & x_3 \le 7 \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

However, it can be converted to an equivalent LP problem by the next proposition.

Proposition

The minimization problem

$$(I)\;\begin{array}{c} \text{minimize } \max_{i=1,\cdots,m}(\mathbf{c}_i^T \mathbf{x} + d_i) \\ \text{subject to } \mathbf{A}\mathbf{x} \ge \mathbf{b}. \end{array}$$

is equivalent to the linear programming problem

$$(II)\;\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & z \ge \mathbf{c}_i^T \mathbf{x} + d_i, \quad i = 1, \cdots, m. \\ & \mathbf{A}\mathbf{x} \ge \mathbf{b}. \end{array}$$

where the decision variables are z and $\mathbf{x}$.

Proof. Note:

$\max_{i=1,\cdots,m}\{a_i\} = \min\{u \,|u \geq a_i,\ i = 1, 2, \cdots, m\}$,

the smallest upper bound of the set
$\{a_i \mid i = 1, 2, \cdots, m\}$.
Thus

$$(I) \begin{array}{l} \text{minimize } \max_{i=1,\cdots,m}(\mathbf{c}_i^T \mathbf{x} + d_i) \\ \text{subject to } \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}$$

is equivalent to

$$\begin{array}{l} \text{minimize } \min\{z \,|z \geq (\mathbf{c}_i^T \mathbf{x} + d_i),\ i = 1, 2, \cdots, m\} \\ \text{subject to } \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}$$

which is in turn equivalent to

$$(II) \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & z \geq \mathbf{c}_i^T \mathbf{x} + d_i,\ i = 1, 2, \cdots, m \\ & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}$$

□

Corollary
The following maximization problems are equivalent:

$$
(I')\quad \begin{array}{ll} \text{maximize} & \min_{i=1,\cdots,m}(\mathbf{c}_i^T\mathbf{x} + d_i) \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}
$$

$$
(II')\quad \begin{array}{ll} \text{maximize} & z \\ \text{subject to} & z \leq \mathbf{c}_i^T\mathbf{x} + d_i, \quad i = 1, \cdots, m. \\ & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}
$$

Example 4.2 Express the following as an LP problem.

$$
\begin{array}{ll} \text{Minimize} & \max\,(3x_1 - x_2, x_2 + 2x_3) \\ \text{Subject to} & 2x_1 + 3x_2 \leq 5 \\ & x_2 - 2x_3 \leq 6 \\ & x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{array}
$$

Example 4.3

A machine shop has one drill press and 5 milling machines, which are to be used to produce an assembly consisting of two parts, 1 and 2. The productivity of each machine for the two parts is given below:

Production Time in Minutes per Piece		
Part	Drill	Mill
1	3	20
2	5	15

It is desired to maintain a balanced loading on all machines such that no machine runs more than 30 minutes per day longer than any other machine (assume that the milling load is split evenly among all five milling machines). Assuming an 8-hour working day, formulate the problem as a linear programming model so as to obtain the maximum number of completed assemblies.

Solution

$x_i =$ Number of part i to be produced.

$$\begin{aligned}
\max\ & \min\{x_1, x_2\} \\
\text{s.t.}\ & 3x_1 + 5x_2 \le 8 \times 60 \\
& 20x_1 + 15x_2 \le 8 \times 60 \times 5 \\
& \left| (3x_1 + 5x_2) - \frac{20x_1 + 15x_2}{5} \right| \le 30 \\
& x_1, x_2 \ge 0,\ \text{integer.}
\end{aligned}$$

$$\begin{aligned}
\max\ & z \\
\text{s.t.}\ & z \le x_1 \\
& z \le x_2 \\
& 3x_1 + 5x_2 \le 480 \\
& 4x_1 + 3x_2 \le 480 \\
& -x_1 + 2x_2 \le 30 \\
& -x_1 + 2x_2 \ge -30 \\
& x_1, x_2 \ge 0,\ \text{integer.}
\end{aligned}$$

Chapter 2

Development of the Simplex Method.

The Simplex Method is a method for solving linear programming problems. This chapter develops basic properties of the simplex method. We begin with geometry of linear programming to show that an optimal solution of a linear program is a corner point of the feasible set of the linear program. We characterize corner points geometrically and algebraically. Finally, we present conditions for optimal solutions of a linear program, which are the foundation for development of the simplex method.

2.1 Geometry of Linear Programming.

In this section, we consider the compact form of a general LP,

$$\begin{aligned} \text{Minimize } & \mathbf{c}^T\mathbf{x} \\ \text{Subject to } & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{aligned}$$

We characterize corner points of the feasible set $\{\mathbf{x}|\mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ geometrically (via extreme points and vertices) and algebraically (via basic feasible solution).

The main results state that a nonempty polyhedron has at least one corner point if and only if it does not contain a line, and if this is the case, the search for optimal solutions to linear programming problems can be restricted to corner points.

2.1.1 Extreme point, vertex and basic feasible solution.

A **polyhedron** or **polyhedral set** is a set that can be described in the form $\{\mathbf{x} \in \mathbf{R}^n \, |\mathbf{A}\mathbf{x} \geq \mathbf{b}\}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector in $\mathbf{R}^m$.

Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

where the i-th row of $\mathbf{A}$ is $\mathbf{a}_i^T = (a_{i1}, a_{i2}, \cdots, a_{in})$, $i = 1, 2, \cdots, m$. Then, the polyhedron

$$\begin{aligned} P &= \{\mathbf{x} \in \mathbf{R}^n \,|\mathbf{A}\mathbf{x} \geq \mathbf{b}\} \\ &= \cap_{i=1}^m \{\mathbf{x} \in \mathbf{R}^n \,|\mathbf{a}_i^T \mathbf{x} \geq b_i\}. \end{aligned}$$

Geometrically, a polyhedron is a finite intersection of half spaces $\mathbf{a}_i^T \mathbf{x} \geq b_i$.

The feasible set of a linear programming problem is a polyhedron.

Three Definitions of corner point.

(a) A vector $\mathbf{x}^* \in P$ is an **extreme point** of P if we cannot find two vectors $\mathbf{y}, \mathbf{z} \in P$, and a scalar $\lambda \in (0, 1)$, such that $\mathbf{x}^* = \lambda \mathbf{y} + (1 - \lambda)\mathbf{z}$.

(b) A vector $\mathbf{x}^* \in P$ is a **vertex** of P if we can find $\mathbf{v} \in \mathbf{R}^n$ such that $\mathbf{v}^T\mathbf{x}^* < \mathbf{v}^T\mathbf{y}$ for all $\mathbf{y} \in P - \{\mathbf{x}^*\}$.

(c) A vector $\mathbf{x}^* \in P$ is a **basic feasible solution** if we can find n linearly independent vectors in the set $\{\mathbf{a}_i \mid \mathbf{a}_i^T\mathbf{x}^* = b_i\}$.

Definitions

1. If a vector $\mathbf{x}^* \in \mathbf{R}^n$ satisfies $\mathbf{a}_i^T \mathbf{x}^* = b_i$ for some $i = 1, 2, \cdots, m$, the corresponding constraint $\mathbf{a}_i^T \mathbf{x} \geq b_i$ is said to be **active** (or **binding**) at $\mathbf{x}^*$.

2. A vector $\mathbf{x}^* \in \mathbf{R}^n$ is said to be of **rank** k with respect to P, if the set $\{\mathbf{a}_i \,|\mathbf{a}_i^T \mathbf{x}^* = b_i\}$ contains k, but not more than k, linearly independent vectors. In other words, the span of $\{\mathbf{a}_i \,|\mathbf{a}_i^T \mathbf{x}^* = b_i\}$ has dimension k.

   • Thus, a vector $\mathbf{x}^* \in P$ is a basic feasible solution if and only if it has rank n.

3. A vector $\mathbf{x}^* \in \mathbf{R}^n$ (not necessary in P) is a **basic solution** if there are n linearly independent vectors in the set $\{\mathbf{a}_i \mid \mathbf{a}_i^T \mathbf{x}^* = b_i\}$. Moreover, every equality constraint (if any) must be satisfied at a basic solution.

4. Constraints $\mathbf{a}_i^T \mathbf{x} \geq b_i, i \in I$ are said to be **linearly independent** if the corresponding vectors $\mathbf{a}_i, i \in I$, are linearly independent.

Example 1.1 Consider the following LP problem:

$$\begin{array}{lrcrl} \text{minimize} & -x_1 & - & x_2 & \\ \text{subject to} & x_1 & + & 2x_2 & \leq 3 \\ & 2x_1 & + & x_2 & \leq 3 \\ & x_1, x_2 \geq 0. & & & \end{array}$$

(a) The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic feasible solution.

(b) The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a feasible solution with only one active constraint $x_1 = 0$. Thus, it has rank 1.

(c) The vector $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is a feasible solution with no active constraint. Thus, it has rank 0.

(d) The vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is not a basic feasible solution. It is not feasible. Note that there are two linearly independent active constraints. Thus, it has rank 2. It is a basic solution.

Note

Given a finite number of linear inequality constraints, there can only be a finite number of basic solutions and hence a finite number of basic feasible solutions.

Example 1.2 Consider the polyhedron P defined by

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & + & x_4 & \geq & 2 \\ & & 3x_2 & - & x_3 & & & \geq & 5 \\ & & & & x_3 & + & x_4 & \geq & 3 \\ & & x_2 & & & & & \geq & 0 \\ & & & & x_3 & & & \geq & 0 \end{array}$$

Determine whether each of the following is a basic feasible solution.

(a) $\mathbf{x}_a = (x_1, x_2, x_3, x_4)^T = (0, 2, 0, 3)^T$.
(b) $\mathbf{x}_b = (x_1, x_2, x_3, x_4)^T = (0, 4, 7, -4)^T$.
(c) $\mathbf{x}_c = (x_1, x_2, x_3, x_4)^T = (-8/3, 5/3, 0, 3)^T$.

Solution Note $\mathbf{x} \in \mathbf{R}^4$.

(a)

constraint	satisfied?	active?
1	Yes, $>$	No
2	Yes, $>$	No
3	Yes, $=$	Yes
4	Yes, $>$	No
5	Yes, $=$	Yes

All constraints are satisfied at $\mathbf{x}_a$, it is feasible with two active constraints. Rank cannot be 4. Therefore $\mathbf{x}_a$ is not a basic feasible solution.

(b) The first constraint is not satisfied at $\mathbf{x}_b$. Thus it is not a basic feasible solution.

(c) Check that all constraints are satisfied and 4 constraints are active at $\mathbf{x}_c$ (Excercise).

Rank at $\mathbf{x}_c$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Thus rank at $\mathbf{x}_c$ is 4.
The vector $\mathbf{x}_c$ is a basic feasible solution.

Remarks

The two geometric definitions, extreme point and vertex , are not easy to work with from the algorithmic point of view. It is desirable to have an algebraic definition, basic feasible solution, that relies on a representation of a polyhedron in terms of linear constraints and which reduces to an algebraic test.

The three definitions namely extreme point, vertex and basic feasible solution, are equivalent as proven in the next theorem. Therefore the three terms can be used interchangeably.

Theorem 1
Let P be a nonempty polyhedron and let $\mathbf{x}^* \in P$. Then the following are equivalent:

(a) $\mathbf{x}^*$ is a vertex;

(b) $\mathbf{x}^*$ is an extreme point;

(c) $\mathbf{x}^*$ is a basic feasible solution.

Appendix

**Proof of Theorem 1. (We shall prove (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a).)*

(a) $\Longrightarrow$ (b): Vertex $\Longrightarrow$ Extreme point. *(We prove this by contradiction.)*

Suppose $\mathbf{x}^*$ is a vertex.
Then there exists $\mathbf{v} \in \mathbf{R}^n$ such that $\mathbf{v}^T\mathbf{x}^* < \mathbf{v}^T\mathbf{y}$ for every $\mathbf{y} \in P - \{\mathbf{x}^*\}$.

Suppose on the contrary that $\mathbf{x}^*$ is not an extreme point. Then there exist two vectors $\mathbf{y}_0, \mathbf{z}_0 \in P$ and a scalar $\lambda \in (0, 1)$, such that $\mathbf{x}^* = \lambda\mathbf{y}_0 + (1 - \lambda)\mathbf{z}_0$.

However, we have $\mathbf{v}^T\mathbf{x}^* < \mathbf{v}^T\mathbf{y}_0$ and $\mathbf{v}^T\mathbf{x}^* < \mathbf{v}^T\mathbf{z}_0$. Thus,

$$\begin{aligned}
\mathbf{v}^T\mathbf{x}^* &= \mathbf{v}^T(\lambda\mathbf{y}_0 + (1 - \lambda)\mathbf{z}_0) \\
&= \lambda\mathbf{v}^T\mathbf{y}_0 + (1 - \lambda)\mathbf{v}^T\mathbf{z}_0 \\
&> \lambda\mathbf{v}^T\mathbf{x}^* + (1 - \lambda)\mathbf{v}^T\mathbf{x}^* = \mathbf{v}^T\mathbf{x}^*
\end{aligned}$$

which gives rise to a contradiction. Thus, $\mathbf{x}^*$ is an extreme point.

(b) $\Longrightarrow$ (c): Extreme point $\Longrightarrow$ basic feasible solution.

(We shall prove the contrapositive statement: not basic feasible solution $\Longrightarrow$ not extreme point.)

Suppose $\mathbf{x}^* \in P$ is not a basic feasible solution. Then the rank of $\mathbf{x}^*$ is k, $k < n$.

(To show that $\mathbf{x}^*$ is not an extreme point, we shall construct two vectors $\mathbf{y}_0, \mathbf{z}_0 \in P$ such that $\mathbf{x}^* = \lambda \mathbf{y}_0 + (1 - \lambda)\mathbf{z}_0$ for some $\lambda \in (0, 1)$.)

Let $I = \{i | \mathbf{a}_i^T \mathbf{x}^* = b_i\}$. The set $\{\mathbf{a}_i | \mathbf{a}_i^T \mathbf{x}^* = b_i\}$ has k linearly independent vectors ($k < n$). Hence the linear system of equations $\mathbf{a}_i^T \mathbf{x} = 0$, $i \in I$, has infinitely many solutions. Choose a nonzero solution $\mathbf{d}$, i.e. $\mathbf{a}_i^T \mathbf{d} = 0$, for $i \in I$.

If $\mathbf{a}_j^T\mathbf{d} = 0$ for every $j \notin I$, then $\mathbf{a}_i^T\mathbf{d} = 0$, for every $i = 1, 2, \cdots, m$. Thus, we let $\mathbf{y}_0 = \mathbf{x}^* + \mathbf{d}$ and $\mathbf{z}_0 = \mathbf{x}^* - \mathbf{d}$. Both $\mathbf{y}_0$ and $\mathbf{z}_0$ are in P. (Exercise.)

If $\mathbf{a}_j^T\mathbf{d} \neq 0$ for some $j \notin I$, then, by Lemma A, we can find $\lambda_0 > 0$ such that $\mathbf{x}^* + \lambda_0\mathbf{d} \in P$ and $\mathbf{x}^* - \lambda_0\mathbf{d} \in P$. Thus, we let $\mathbf{y}_0 = \mathbf{x}^* + \lambda_0\mathbf{d}$ and $\mathbf{z}_0 = \mathbf{x}^* - \lambda_0\mathbf{d}$.

Note that $\mathbf{x}^* = \frac{1}{2}\mathbf{y}_0 + \frac{1}{2}\mathbf{z}_0$, i.e. $\mathbf{x}^*$ is not an extreme point.

(c) $\Longrightarrow$ (a): Basic feasible solution $\Longrightarrow$ vertex. *(We prove this directly.)*

Suppose $\mathbf{x}^*$ be a basic feasible solution. Let $I = \{i | \mathbf{a}_i^T\mathbf{x}^* = b_i\}$. The set $\{\mathbf{a}_i | \mathbf{a}_i^T\mathbf{x}^* = b_i\}$ has n linearly independent vectors. Hence the linear system of equations $\mathbf{a}_i^T\mathbf{x} = b_i$, $i \in I$, has a unique solution which is $\mathbf{x}^*$.

We form a vector $\mathbf{v} = \Sigma_{i \in I}\ \mathbf{a}_i$, and shall prove that $\mathbf{v}^T\mathbf{x}^* < \mathbf{v}^T\mathbf{y}$ for $\mathbf{y} \in P - \{\mathbf{x}^*\}$.

Let $\mathbf{y} \in P - \{\mathbf{x}^*\}$. Then $\mathbf{a}_i^T\mathbf{y} \geq b_i, i = 1, 2, \cdots, m$ and hence

$$\mathbf{v}^T\mathbf{y} = \sum_{i \in I} \mathbf{a}_i^T\mathbf{y} \geq \sum_{i \in I} b_i = \sum_{i \in I} \mathbf{a}_i^T\mathbf{x}^* = \mathbf{v}^T\mathbf{x}^*.$$

If $\mathbf{v}^T\mathbf{y} = \mathbf{v}^T\mathbf{x}^* = \Sigma_{i \in I}\ b_i$, then we must have $\mathbf{a}_i^T\mathbf{y} = b_i, i \in I$ because $\mathbf{a}_i^T\mathbf{y} \geq b_i$,for each i. Thus $\mathbf{y}$ is a solution to the linear system $\mathbf{a}_i^T\mathbf{x} = b_i, i \in I$.

From the uniqueness of the solution, we must have $\mathbf{y} = \mathbf{x}^*$, contradicting $\mathbf{y} \in P - \{\mathbf{x}^*\}$.

Therefore, $\mathbf{v}^T\mathbf{y} > \mathbf{v}^T\mathbf{x}^*$ and this proves that $\mathbf{x}^*$ is a vertex. QED

Lemma A
Let P be a nonempty polyhedron defined by $\{\mathbf{x} \mid \mathbf{a}_i^T\mathbf{x} \geq b_i, i = 1, 2, \cdots, m\}$.
Let $\mathbf{x}^* \in P$ be of rank k, $k < n$.
Denote $I = \{i \mid \mathbf{a}_i^T\mathbf{x}^* = b_i\}$.
Suppose there exists a nonzero vector $\mathbf{d}$ such that $\mathbf{a}_i^T\mathbf{d} = 0$ for every $i \in I$, and $\mathbf{a}_j^T\mathbf{d} \neq 0$ for some $j \notin I$.
Then there exists $\lambda_0 > 0$ such that $\mathbf{x}^* + \lambda\mathbf{d} \in P$ for every $\lambda \in [-\lambda_0, \lambda_0]$.
Moreover, there exists λ_* such that $\mathbf{x}^* + \lambda_*\mathbf{d} \in P$ with rank at least $k + 1$.

Remark A non-zero vector $\mathbf{d}$ such that $\mathbf{x}^* + \lambda\mathbf{d} \in P$ for some $\lambda > 0$ is said to be a feasible direction.

**Proof.*
How to find a suitable a suitable $\lambda_0 > 0$ *such that the conclusion of the lemma holds?*

Note that:

$$\mathbf{x}^* + \lambda\mathbf{d} \in P \iff \mathbf{a}_j^T \cdot (\mathbf{x}^* + \lambda\mathbf{d}) \geq b_j \ \forall\ j$$
$$\iff \mathbf{a}_j^T\mathbf{x}^* + \lambda\mathbf{a}_j^T\mathbf{d} \geq b_j \ \forall\ j.$$

Denote $\mathbf{a}_j^T\mathbf{x}^* + \lambda\mathbf{a}_j^T\mathbf{d} \geq b_j$ by (*).

If $\mathbf{a}_j^T\mathbf{d} = 0$, then (*) holds since

$$\mathbf{a}_j^T\mathbf{x}^* + \lambda\mathbf{a}_j^T\mathbf{d} = \mathbf{a}_j^T\mathbf{x}^* \geq b_j \text{ for } \lambda \in \mathbf{R}.$$

If $\mathbf{a}_j^T\mathbf{d} > 0$, then (*) holds whenever

$$\frac{\mathbf{a}_j^T\mathbf{x}^* - b_j}{-\mathbf{a}_j^T\mathbf{d}} \leq \lambda, \text{ i.e. } \lambda \geq \frac{\mathbf{a}_j^T\mathbf{x}^* - b_j}{-|\mathbf{a}_j^T\mathbf{d}|}.$$

If $\mathbf{a}_j^T\mathbf{d} < 0$, then (*) holds whenever

$$\frac{\mathbf{a}_j^T\mathbf{x}^* - b_j}{-\mathbf{a}_j^T\mathbf{d}} \geq \lambda, \text{ i.e. } \lambda \leq \frac{\mathbf{a}_j^T\mathbf{x}^* - b_j}{|\mathbf{a}_j^T\mathbf{d}|}.$$

Thus, for $\mathbf{a}_j^T\mathbf{x}^* + \lambda\mathbf{a}_j^T\mathbf{d} \geq b_j \ \forall\ j$, we must have

$$\frac{\mathbf{a}_j^T\mathbf{x}^* - b_j}{-|\mathbf{a}_j^T\mathbf{d}|} \leq \lambda \leq \frac{\mathbf{a}_j^T\mathbf{x}^* - b_j}{|\mathbf{a}_j^T\mathbf{d}|} \text{ whenever } \mathbf{a}_j^T\mathbf{d} \neq \mathbf{0}.$$

Therefore we choose

$$\lambda_0 = \min\{\frac{\mathbf{a}_j^T \mathbf{x}^* - b_j}{|\mathbf{a}_j^T \mathbf{d}|} \mid \mathbf{a}_j^T \mathbf{d} \neq 0\}.$$

For $-\lambda_0 \leq \lambda \leq \lambda_0$,

$$\mathbf{a}_i^T(\mathbf{x}^* + \lambda \mathbf{d}) \geq b_j, \quad \forall j = 1, 2, \cdots, m.$$

Hence, $\mathbf{x}^* + \lambda \mathbf{d} \in P$.

To prove the last part of the lemma.

The set $\{j|\mathbf{a}_j^T \mathbf{d} \neq 0\}$ is finite, thus, $\lambda_0 = \dfrac{\mathbf{a}_{j_*}^T \mathbf{x}^* - b_{j_*}}{|\mathbf{a}_{j_*}^T \mathbf{d}|}$, for some $j_* \notin I$. Let

$$\lambda_* = \begin{cases} \lambda_0 & \text{if } \mathbf{a}_{j_*}^T \mathbf{d} < 0 \\ -\lambda_0 & \text{if } \mathbf{a}_{j_*}^T \mathbf{d} > 0. \end{cases}$$

and $\hat{\mathbf{x}} = \mathbf{x}^* + \lambda_* \mathbf{d}$. Then $\mathbf{a}_{j_*}^T \hat{\mathbf{x}} = b_{j_*}$ and $\mathbf{a}_i^T \hat{\mathbf{x}} = \mathbf{a}_i^T(\mathbf{x}^* + \lambda_* \mathbf{d}) = b_i$, for every $i \in I$.

Since $\mathbf{a}_i^T \mathbf{d} = 0$, for all $i \in I$, and $\mathbf{a}_{j_*}^T \mathbf{d} \neq \mathbf{0}$, $\mathbf{a}_{j^*}$ is not a linear combination of $\mathbf{a}_i, i \in I$.

Therefore, the set $\{\mathbf{a}_j \mid \mathbf{a}_j^T\, \hat{\mathbf{x}} = b_j\}$ contains at least $k+1$ linearly independent vectors.

Hence, $\hat{\mathbf{x}}$ has rank $\geq k+1$. QED.

2.1.2 Existence of extreme points.

Geometrically, a polyhedron containing an infinite line does not contain an extreme point. As an example, the polyhedron $P = \{\begin{pmatrix} x \\ 0 \end{pmatrix} | x \in \mathbf{R}\} \subset \mathbf{R}^2$ does not have an extreme point. In $\mathbf{R}^3$, $\mathbf{x}^* + \lambda\mathbf{d}, \lambda \in \mathbf{R}$ describes a line which is parallel to $\mathbf{d}$ and passes through $\mathbf{x}^*$.

A polyhedron $P \subset \mathbf{R}^n$ **contains a line** if there exists a vector $\mathbf{x}^* \in P$ and a nonzero vector $\mathbf{d} \in \mathbf{R}^n$ such that $\mathbf{x}^* + \lambda\mathbf{d} \in P$ for all $\lambda \in \mathbf{R}$.

Theorem 2

Suppose that the polyhedron $P = \{\mathbf{x} \in \mathbf{R}^n \,|\mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ is nonempty.
Then P does not contain a line if and only if P has a basic feasible solution.

Appendix

**Proof of Theorem 2.*
($\Longrightarrow$) Suppose P does not contain a line.
(Our aim is to show there is a basic feasible solution.)

Since P is nonempty, we may choose some $\mathbf{x}_0 \in P$.

Case rank of $\mathbf{x}_0 = n$.
Then $\mathbf{x}_0$ is a basic feasible solution.

Case rank of $\mathbf{x}_0 = k < n$.
Let $I = \{i | \mathbf{a}_i^T \mathbf{x}_0 = b_i\}$. The set $\{\mathbf{a}_i \; | \mathbf{a}_i^T \mathbf{x}_0 = b_i\}$ contains k, but not more than k, linearly independent vectors, where $k < n$. The linear system of equations $\mathbf{a}_i^T \mathbf{x} = 0$, $i \in I$, has infinitely many solutions. Choose a nonzero solution $\mathbf{d}$, i.e. $\mathbf{a}_i^T \mathbf{d} = 0$, for $i \in I$.

Claim: $\mathbf{a}_j^T \mathbf{d} \neq 0$ for some $j \notin I$.
Proof. Suppose $\mathbf{a}_i^T \mathbf{d} = 0 \;\; \forall j \notin I$, Then $\mathbf{a}_i^T \mathbf{d} = 0$

for every $i = 1, 2, \cdots, m$.
For $\lambda \in \mathbf{R}$, note that $\mathbf{a}_i^T(\mathbf{x}_0 + \lambda\mathbf{d}) = \mathbf{a}_i^T\mathbf{x}_0 \geq b_i$.
Therefore, we have $\mathbf{x_0} + \lambda\mathbf{d} \in P$, i.e. P contains the line $\mathbf{x}_0 + \lambda\mathbf{d}$, a contradiction.
Thus, $\mathbf{a}_j^T\mathbf{d} \neq 0$ for some $j \notin I$.

By Lemma A, we can find $\mathbf{x}_1 = \mathbf{x}_0 + \lambda_*\mathbf{d} \in P$ and the rank of $\mathbf{x}_1$ is at least $k + 1$.

By repeating the same argument to $\mathbf{x}_1$ and so on, as many times as needed, we will obtain a point $\mathbf{x}^*$ with rank n, i.e. $\{\mathbf{a}_i \mid \mathbf{a}_i^T\ \mathbf{x}^* = b_i\}$ contains n linearly independent vectors. Thus, there is at least one basic feasible solution.

($\Longleftarrow$) Suppose P has a basic feasible solution $\mathbf{x}^*$. Then there exist n linearly independent row vectors, say $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$ of $\mathbf{A}$ such that $\mathbf{a}_i^T \mathbf{x}^* = b_i, i = 1, 2, \cdots, n$.

Suppose, on the contrary, that P contains a line, say $\hat{\mathbf{x}} + \lambda \mathbf{d}$, where $\mathbf{d} \neq \mathbf{0}$.

Then, $\mathbf{a}_i^T \mathbf{d} \neq 0$ for some $i = 1, 2, \cdots, n$. (If not, $\mathbf{a}_i^T \mathbf{d} = 0$ for all $i = 1, 2, \cdots, n$ and hence $\mathbf{d} = \mathbf{0}$, since $\mathbf{a}_i^T, i = 1, 2, \cdots, n$, are linearly independent.)

Without loss of generality, we may assume $\mathbf{a}_1^T \mathbf{d} \neq 0$.

Replacing $\mathbf{d}$ by $-\mathbf{d}$ if necessary, we may further assume $\mathbf{a}_1^T \mathbf{d} > 0$.

However, $\hat{\mathbf{x}} + \lambda \mathbf{d} \notin P$ for $\lambda < \dfrac{b_1 - \mathbf{a}_1^T \hat{\mathbf{x}}}{\mathbf{a}_1^T \mathbf{d}}$, since $\mathbf{a}_1^T (\hat{\mathbf{x}} + \lambda \mathbf{d}) < b_1$.

This contradicts the assumption that P contains the line $\hat{\mathbf{x}} + \lambda \mathbf{d}$. (QED)

Example 1.3 The polyhedron P defined by

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & + & x_4 & \geq & 2 \\ & & 3x_2 & - & x_3 & & & \geq & 5 \\ & & & & x_3 & + & x_4 & \geq & 3 \\ & & x_2 & & & & & \geq & 0 \\ & & & & x_3 & & & \geq & 0 \end{array}$$

contains a basic feasible solution, namely, $\mathbf{x}^* = \begin{pmatrix} -\frac{8}{3} \\ \frac{5}{3} \\ 0 \\ 3 \end{pmatrix}$ (see Example 1.2). Thus, by Theorem 2, P does not contain a line.

A polyhedron $P = \{\mathbf{x} \in \mathbf{R}^n \, | \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ is said to be **bounded** if there exists a positive number K such that $|x_i| \leq K$ for all $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in P$.

A nonempty bounded polyhedron cannot contain a line, thus it must have a basic feasible solution.

2.1.3 Optimality at some extreme point.

Geometrically, if an LP problem has a corner point and an optimal solution, then an optimal solution occurs at some corner point. The next theorem justifies this geometrical insight. So, in searching for optimal solutions, it suffices to check on all corner points.

Theorem 3 Consider the linear programming problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron P. Suppose that P has at least one extreme point and that there exists an optimal solution. Then there exists an optimal solution which is an extreme point of P.

Proof. We denote the optimal cost by v.
Let $Q = \{\mathbf{x} \in P | \mathbf{c}^T\mathbf{x} = v\}$ be the set of optimal solutions. Then Q is a nonempty polyhedron.

Step 1 Q has an extreme point $\mathbf{x}^*$.

Since P has at least one extreme point, P does not contain a line, by Theorem 2. Hence Q, being a subset of P, does not contain a line. By Theorem 2, Q has an extreme point, say $\mathbf{x}^*$.

Step 2 $\mathbf{x}^*$ is also an extreme point of P.

Suppose $\mathbf{x}^*$ is not an extreme point of P.
Then there exists $\lambda \in (0, 1)$ and $\mathbf{y}, \mathbf{z} \in P$ such that $\mathbf{x}^* = \lambda \mathbf{y} + (1 - \lambda)\mathbf{z}$.

Suppose either $\mathbf{c}^T\mathbf{y} > v$ or $\mathbf{c}^T\mathbf{z} > v$.
Then, we have $\mathbf{c}^T\mathbf{x}^* = \mathbf{c}^T(\lambda\mathbf{y} + (1 - \lambda)\mathbf{z}) > v$, contradicting $\mathbf{c}^T\mathbf{x}^* = v$.

Therefore, both $\mathbf{c}^T\mathbf{y} = v$ and $\mathbf{c}^T\mathbf{z} = v$; thus, $\mathbf{y}, \mathbf{z} \in Q$ and $\mathbf{x}^* = \lambda\mathbf{y} + (1 - \lambda)\mathbf{z}$. This contradicts $\mathbf{x}^*$ being an extreme point of Q.

Thus, $\mathbf{x}^*$ is an extreme point of P and it is optimal (since $\mathbf{c}^T\mathbf{x}^* = v$). QED.

The simplex method is based fundamentally on the fact that the optimum solution occurs at a corner point of the solution space. It employs an iterative process that starts at a basic feasible solution, and then attempts to find an adjacent basic feasible solution that will improve the objective value.

Three tasks:

1. How to construct a basic feasible solution?

2. In which direction can we move to an adjacent basic feasible solution?

3. In which direction can we improve the objective value?

2.2 Constructing Basic Feasible Solutions.

In the rest of this chapter, we consider the standard form of a LP,

$$\begin{array}{rl} \text{Minimize} & \mathbf{c}^T\mathbf{x} \\ \text{Subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \quad \text{or} \quad \begin{array}{rl} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{Subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Assume $\mathbf{A}$ is an $m \times n$ matrix and rank $(A) = m$. Thus, row vectors $\mathbf{a}_i^T, i = 1, 2, \cdots, m$, of $\mathbf{A}$ are linearly independent and $m \leq n$. The ith column of $\mathbf{A}$ is denoted by $\mathbf{A}_i$.

Let $P = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Note that if $P \neq \phi$, then P has an extreme point since it does not contain a line. Therefore, either the optimal value is unbounded or there exists an optimal solution which can be found among the finite set of extreme points.

Recall from the previous section, the following definition of a basic solution.

A vector $\mathbf{x}^* \in \mathbf{R}^n$ (not necessary in P) is a **basic solution** if there are n linearly independent vectors in the set $\{\mathbf{a}_i \mid \mathbf{a}_i^T \mathbf{x}^* = b_i\}$. Moreover, every equality constraint (if any) must be satisfied at a basic solution.

Suppose $\mathbf{x}^*$ is a basic solution of the standard form LP. Then $\mathbf{A}\mathbf{x}^* = \mathbf{b}$, which consists m linearly independent equality (thus, active) constraints. Since a basic solution has n linearly independent active constraints, there are $n-m$ linearly independent active constraints from $\mathbf{x} \geq \mathbf{0}$. Therefore we have $n-m$ zero variables $x_i^* = 0$, where x_i^* is the i-component of $\mathbf{x}^*$. So, there are indices $B(1), B(2), \cdots, B(m)$ such that

$$x_i^* = 0 \text{ for } i \neq B(1), B(2), \cdots, B(m).$$

and

$$\sum_{i=1}^{m} \mathbf{A}_{B(i)} x_{B(i)}^* = \mathbf{b}.$$

A basic solution must have n linearly independent active constraints. The following lemma summarizes several conditions for checking linear independence of n vectors in $\mathbf{R}^n$.

Lemma. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be column vectors in $\mathbf{R}^n$. Then the following statements are equivalent.

1. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbf{R}^n$ are linearly independent.
2. $\det(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \neq 0$.
3. The matrix $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ and its transpose are nonsingular.
4. The equation system $\Sigma_{i=1}^n y_i \mathbf{a}_i = \mathbf{0}$ has the unique solution $\mathbf{y} = \mathbf{0}$.
5. The equation system $\mathbf{a}_i^T \mathbf{x} = 0$, $i = 1, \ldots, n$, has the unique solution $\mathbf{x} = \mathbf{0}$.

If there are $m(> n)$ n-dimensional vectors, then we should check linear independence of any subset of n vectors.

Throughout this course, we use the following notations: We denote

$$\mathbf{B} = \{B(1), B(2), \ldots, B(m)\},$$

which is a subset of $\{1, 2, \ldots, n\}$. We denote by $\mathbf{A_B}$ an $m \times m$ sub-matrix of $\mathbf{A}$ obtained by arranging the m columns with indices in $\mathbf{B}$ next to each other. A sub-vector $\mathbf{x}_B$ of $\mathbf{X}$ can be defined in the same way. Thus,

$$\mathbf{A}_B = \left[\, \mathbf{A}_{B(1)} \ \mathbf{A}_{B(2)} \ \cdots \ \mathbf{A}_{B(m)} \, \right],$$

$$\mathbf{x}_B = \begin{bmatrix} x_{B(1)} \\ \cdot \\ \cdot \\ \cdot \\ x_{B(m)} \end{bmatrix}.$$

The following theorem is a useful characterization of a basic solution. It allows us to construct a basic solution in a systematic way.

Theorem
Consider the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ and assume that the $m \times n$ matrix $\mathbf{A}$ has linearly independent rows. A vector $\mathbf{x}^* \in \mathbf{R}^n$ is a basic solution if and only if $\mathbf{Ax}^* = \mathbf{b}$ and there exist a set of indices $B = \{B(1), B(2), \cdots, B(m)\}$ such that

(a) The columns $\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \cdots, \mathbf{A}_{B(m)}$ are linearly independent; and

(b) $x_i^* = 0$ for $i \neq B(1), B(2), \cdots, B(m)$.

Proof.
($\Longleftarrow$) Suppose $\mathbf{x}^* \in \mathbf{R}^n$ satisfies $\mathbf{Ax}^* = \mathbf{b}$ and there exist indices $B(1), B(2), \cdots, B(m)$ such that (a) and (b) are satisfied.

Aim: To show that there are n linearly independent active constraints from:

$$\begin{cases} \mathbf{Ax} = \mathbf{b} & (1) \\ x_i = 0 \text{ for } i \neq B(1), B(2), \cdots, B(m), & (2) \end{cases}$$

Denote

$$\mathbf{B} = \{B(1), B(2), \ldots, B(m)\}$$
$$\mathbf{N} = \{1, 2, \ldots, n\} \setminus \mathbf{B}.$$

and denote by $|\mathbf{B}|$ the number of elements in $\mathbf{B}$. Then $|\mathbf{B}| = m$ and $|\mathbf{N}| = n - m$.

Now, (1) and (2) can be equivalently written as

$$\begin{pmatrix} \mathbf{A_B} & \mathbf{A_N} \\ \mathbf{0} & \mathbf{I_N} \end{pmatrix} \begin{pmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}, \tag{3}$$

where $\mathbf{I_N}$ denotes the $(n-m) \times (n-m)$ identity matrix.

By (a), $\mathbf{A_B}$ is nonsingular, thus the coefficient matrix of equation (3) is nonsingular. Hence there are n linearly independent active constraints from (1) and (2). We thus conclude that $\mathbf{x}^*$ is a basic solution.

($\Longrightarrow$) Suppose $\mathbf{x}^*$ is a basic solution. By the definition of a basic solution, all equality constraints must be satisfied, thus, we have $\mathbf{Ax}^* = \mathbf{b}$.

There are n linearly independent active constraints at $\mathbf{x}^*$ from constraints

$$\mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0},$$

m active constraints from $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $n-m$ active constraints from $x_j^* = 0$, (there may be more than $n-m$ "$x_j^* = 0$"). Therefore, there exists an index set

$$\mathbf{N} = \{N(1), \ldots, N(n-m)\} \subseteq \{1, \ldots, n\}$$

such that $x_j^* = 0 \, \forall \, j \in \mathbf{N}$ and the matrix $\begin{pmatrix} A \\ \mathbf{e}_{N(1)}^T \\ \vdots \\ \mathbf{e}_{N(n-m)}^T \end{pmatrix}$ is nonsingular.

Denote $\mathbf{B} = \{1, \ldots, n\} \setminus \mathbf{N}$. Then, $x_j^* = 0$ for $i \notin \mathbf{B}$, ((b) is satisfied).

We can write

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_B} & \mathbf{A_N} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_{N(1)}^T \\ \vdots \\ \mathbf{e}_{N(n-m)}^T \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I_N} \end{pmatrix}.$$

Then $\begin{pmatrix} \mathbf{A_B} & \mathbf{A_N} \\ \mathbf{0} & \mathbf{I_N} \end{pmatrix}$ is nonsingular. This implies that

$\mathbf{A_B}$ is nonsingular. Hence, columns vectors

$$\{\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \cdots, \mathbf{A}_{B(m)}\}$$

are linearly independent, ((a) is satisfied). QED.

Terminology

Suppose $\mathbf{x}$ is a basic solution with the **basis**

$$\mathbf{B} = \{B(1), B(2), \cdots, B(m)\}$$

as given in the above theorem.

1. Variables $x_{B(1)}, x_{B(2)}, \cdots, x_{B(m)}$ are called **basic variables**.
2. Variables $x_i = 0$ for $i \notin \mathbf{B}$, are called **nonbasic variables**.
3. The $m \times m$ matrix

   $$\mathbf{A_B} = \begin{pmatrix} \mathbf{A}_{B(1)} & \mathbf{A}_{B(2)} & \cdots & \mathbf{A}_{B(m)} \end{pmatrix}$$

   is called a **basis matrix**. A vector $\mathbf{x}_B$ can also be defined with the values of the basic variables. Note that $\mathbf{A}_B$ is invertible and $\mathbf{A}_B\mathbf{x}_B = \mathbf{b}$ so that $\mathbf{x}_B$ is the unique solution given by

   $$\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}.$$

From the last theorem, all basic solutions to a standard form polyhedron can be constructed according to the following procedure.

Procedure for constructing basic solution.

1. Choose m linearly independent columns
$$\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \cdots, \mathbf{A}_{B(m)}.$$
2. Let $x_i = 0$ for $i \neq B(1), B(2), \cdots, B(m)$.
3. Solve the system of m linear equations
$$\mathbf{A}_B \mathbf{x}_B = \mathbf{b}$$
for the unknowns $x_{B(1)}, \cdots, x_{B(m)}$.

Remark A basic solution $\mathbf{x}$ constructed according to the above procedure is a basic feasible solution if and only if $\mathbf{x} \geq \mathbf{0}$, i.e. $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} \geq 0$.

Example 2.1 For the following constraints

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

$$\mathbf{x} \geq 0.$$

(a) Find the basic solution associated with linearly independent columns $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$. Is it a basic feasible solution?

(b) Show that columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are linearly independent. Find the basis matrix $\mathbf{A}_B$ and the associated basic solution. Is it feasible?

(c) Do columns $\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5$ form a basis matrix? If so, what is the associated basic solution?

Solution

(a) Note that $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$ are linearly independent. Thus, we may proceed to find the associated basic solution.

We have $\mathbf{A}_B = [\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7] = I_4$ which is called a basis matrix.

Non-basic variables: $x_1 = 0, x_2 = 0, x_3 = 0$.
Solve for basic variables x_4, x_5, x_6, x_7:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_B = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix} \text{ where } \mathbf{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}.$$

We have

$$\mathbf{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix} \geq \mathbf{0}. \text{ (Feasible)}$$

Thus we obtain a basic feasible solution, namely,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 12 \\ 4 \\ 6 \end{bmatrix} \geq \mathbf{0}.$$

(b) Check $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are linearly independent:

$$[\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 6 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are linearly independent and $\mathbf{A}_B = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4]$ is a basis matrix.
Nonbasic variables: $x_5 = 0, x_6 = 0, x_7 = 0$.
To find values of basic variables x_1, x_2, x_3, x_4:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}_B = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix} \quad \text{where} \quad \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

$$\text{Solving yields:} \qquad \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ -6 \end{bmatrix}.$$

Thus, the associated basic solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $x_4 < 0$, the basic solution is not feasible.

(c) Check for linear independence of $\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5$:

$$[\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 6 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Columns $\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5$ are not linearly independent (WHY?). Thus, they do not form a basis matrix. (No need to proceed to find solution.)

Exercise Show $\mathbf{A}_B = [\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7]$ is a basis matrix, and $\mathbf{x}_B = (4, -12, 4, 6)^T$.

Adjacency and degeneracy.

Geometrically, adjacent basic feasible solutions are extreme points which are adjacent. The simplex method attempts to find an adjacent basic feasible solution that will improve the objective value.

Definition

Two distinct basic solutions to a set of linear constraints in $\mathbf{R}^n$ are said to be **adjacent** if and only if the corresponding bases share all but one basic column, i.e. there are $n-1$ linearly independent constraints that are active at both of them.

In the standard form ($\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq 0$), two basic solutions are adjacent if they have $n-m-1$ common $x_j = 0$, equivalently, if their basic variables differ by one component.

Example 3.1 Refer to the constraints in Example 2.1.

Basic solution	Basic columns	Basic variables
$(0, 0, 0, 8, 12, 4, 6)^T$	$\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$	x_4, x_5, x_6, x_7
$(0, 0, 4, 0, -12, 4, 6)^T$	$\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$	x_3, x_5, x_6, x_7

The above basic solutions are adjacent.

Definition

A basic solution $\mathbf{x} \in \mathbf{R}^n$ is said to be **degenerate** if it has more than n active constraints, i.e. the number of active constraints at $\mathbf{x}$ is greater than the dimension of $\mathbf{x}$.

Geometrically, a degenerate basic solution is determined by more than n active constraints (overdetermined).

In standard form, a basic solution $\mathbf{x}$ is **degenerate** if some basic variable $x_{B(i)} = 0$, i.e. more than $n - m$ components of $\mathbf{x}$ are zero.

Example 3.2 For the following constraints

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

$$\mathbf{x} \geq 0.$$

The basic feasible solution $\mathbf{x} = (0, 0, 4, 0, 0, 0, 6)^T$, associated with basis $\mathbf{A}_B = [\, \mathbf{A}_3 \ \mathbf{A}_4 \ \mathbf{A}_5 \ \mathbf{A}_7 \,]$ is degenerate because there are 9 active constraints at $\mathbf{x}$ which the dimension of $\mathbf{x}$ is 7.

Let $\mathbf{x}^*$ be a basic feasible solution with the set $\{B(1), \cdots, B(m)\}$ of basic indices, so that

$$\mathbf{A}_B = \left[\, \mathbf{A}_{B(1)} \; \mathbf{A}_{B(2)} \; \cdots \; \mathbf{A}_{B(m)} \,\right]$$

$$\mathbf{x}_B^* = \begin{bmatrix} x_{B(1)}^* \\ \cdot \\ \cdot \\ \cdot \\ x_{B(m)}^* \end{bmatrix} = \mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}.$$

When we move from $\mathbf{x}^*$ to an adjacent basic solution (may or may not be feasible) $\mathbf{x}'$, a nonbasic variable x_j of $\mathbf{x}^*$ becomes a basic variable of $\mathbf{x}'$. There is an exchange of a basic variable and nonbasic variable. In the next lemma, we shall determine the feasible direction moving away from $\mathbf{x}^*$ so that the variable x_j becomes a basic variable.

Lemma A

Fix an index $j \notin \mathbf{B} = \{B(1), \cdots, B(m)\}$. Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)^T$ with $d_j = 1$ and $d_i = 0$, for every index $i \notin \mathbf{B}$ and $i \neq j$. Then $\mathbf{x}' = \mathbf{x}^* + \theta \mathbf{d}$ with $\theta > 0$ is a feasible solution if and only if

$$\mathbf{d}_B = -\mathbf{A}_B^{-1} \mathbf{A}_j$$

and

$$\mathbf{A}_B^{-1} \mathbf{b} - \theta \mathbf{A}_B^{-1} \mathbf{A}_j \geq \mathbf{0}.$$

Proof In order to maintain feasibility of solution, we must have

$$\mathbf{A}(\mathbf{x}') = \mathbf{b} \ \text{ and } \mathbf{x}' \geq 0.$$

i.e. $\mathbf{A}(\mathbf{x}^* + \theta \mathbf{d}) = \mathbf{b}$ and $\mathbf{x}^* + \theta \mathbf{d} \geq 0$.
However, $\mathbf{A}\mathbf{x}^* = \mathbf{b}$ and $\theta > 0$ so that $\mathbf{A}(\mathbf{x}^* + \theta \mathbf{d}) = \mathbf{b}$ implies $\mathbf{A}\mathbf{d} = \mathbf{0}$. Thus,

$$\mathbf{0} = \sum_{i=1}^{n} \mathbf{A}_i d_i = \sum_{i=1}^{m} \mathbf{A}_{B(i)} d_{B(i)} + \mathbf{A}_j = \mathbf{A}_B \mathbf{d}_B + \mathbf{A}_j.$$

Therefore, $\mathbf{A}_B \mathbf{d}_B = -\mathbf{A}_j$ and hence $\mathbf{d}_B = -\mathbf{A}_B^{-1} \mathbf{A}_j$.

Note that for $i \notin \mathbf{B}$, $x_i^* + \theta d_i = 0$ $(i \neq j)$ or $= \theta$ $(i = j)$.
Thus, $\mathbf{x}^* + \theta \mathbf{d} \geq \mathbf{0}$ is equivalent to $\mathbf{x}_B^* + \theta \mathbf{d}_B \geq \mathbf{0}$, i.e. $\mathbf{A}_B^{-1} \mathbf{b} - \theta \mathbf{A}_B^{-1} \mathbf{A}_j \geq \mathbf{0}$. [QED.]

In summary, we have obtained the vector $\mathbf{d} = (d_1, d_2, \cdots, d_n)^T$ where

$$\begin{cases} d_j &= 1, \\ d_i &= 0 \text{ for every non basic index } i \neq j, \text{ and} \\ \mathbf{d}_B &= -\mathbf{A}_B^{-1}\mathbf{A}_j. \end{cases}$$

Notes

1. If $i \notin \{B(1), B(2), \cdots, B(m)\}$ and $i \neq j$, then the i-component of $\mathbf{x}'$ is $x'_i = 0$ since $x^*_i = 0$, and $d_i = 0$. The j-component of $\mathbf{x}'$ is $x'_j = \theta$ since $d_j = 1$.

2. The point $\mathbf{x}'$ is obtained from $\mathbf{x}^*$ by moving in the direction $\mathbf{d}$. It is obtained from $\mathbf{x}^*$ by selecting a nonbasic variable x_j (i.e. $j \notin \{B(1), \cdots, B(m)\}$) and increasing it to a positive value θ, while keeping the remaining nonbasic variables x_i at zero, i.e. $\mathbf{x}' = \mathbf{x}^* + \theta\mathbf{d}$, where $\mathbf{d} = (d_1, d_2, \cdots, d_n)^T$ with $d_j = 1$ and $d_i = 0$, for every nonbasic index $i, i \neq j$.

Lemma B
(a) If $\mathbf{A}_B^{-1}\mathbf{A}_j \leq \mathbf{0}$, then the polyhedron is unbounded in the x_j-direction.

(b) If $\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k > 0$ for some k, then $\theta \leq \frac{\left(\mathbf{A}_B^{-1}\mathbf{b}\right)_k}{\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k}$.

Proof.
(a) Since $\mathbf{x}_B^* \geq \mathbf{0}$ and $\mathbf{A}_B^{-1}\mathbf{A}_j \leq \mathbf{0}$, we have from Lemma A, $\mathbf{x}'_B = \mathbf{x}_B^* - \theta\mathbf{A}_B^{-1}\mathbf{A}_j \geq \mathbf{0}$ whenever $\theta > 0$. Thus $x'_j = \theta$ is unbounded.

(b) If $\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k > 0$ for some component k, then $\left(\mathbf{A}_B^{-1}\mathbf{b}\right)_k - \theta\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k \geq 0$ yields

$$\theta \leq \frac{\left(\mathbf{A}_B^{-1}\mathbf{b}\right)_k}{\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k}.$$

[QED.]

Remark
Suppose $\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k > 0$ for some k-th component. Let

$$\theta^* = \min\{\frac{\left(\mathbf{A}_B^{-1}\mathbf{b}\right)_k}{\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k} \mid \left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k > 0\}.$$

Then for some l,

$$\theta^* = \frac{\left(\mathbf{A}_B^{-1}\mathbf{b}\right)_l}{\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_l}.$$

The feasible solution $\mathbf{x'} = \mathbf{x}^* + \theta^*\mathbf{d}$ is a basic feasible solution which is adjacent to $\mathbf{x}^*$, with associated basic variables

$$\{x_{B(1)}, \cdots, x_{B(l-1)}, x_{B(l+1)}, \cdots, x_{B(m)}, x_j\}.$$

Remark
If $\mathbf{x}^*$ is nondegenerate, then we always have $\theta^* > 0$. If $\mathbf{x}^*$ is degenerate, then θ^* may be zero.

Example 3.3 Consider the LP problem

$$\begin{array}{rrrrrrrrrl} \text{minimize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & c_4x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & 2x_1 & & & + & 3x_3 & + & 4x_4 & = & 2 \\ & & & & & \multicolumn{5}{l}{x_1, x_2, x_3, x_4 \geq 0} \end{array}$$

Since columns $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{A}$ are linearly independent, we choose x_1 and x_2 as basic variables. Then

$$\mathbf{A}_B = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Set $x_3 = x_4 = 0$, we obtain $x_1 = 1$ and $x_2 = 1$. The basic feasible solution $\mathbf{x}^* = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is nondegenerate. (Thus $\mathbf{d}$ is a feasible direction.)

We construct a feasible direction corresponding to an increase in the nonbasic variable x_3 by setting $d_3 = 1$ and $d_4 = 0$. It remains to find d_1 and d_2, i.e. $\mathbf{d}_B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Now,

$$\begin{aligned} \mathbf{d}_B &= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\mathbf{A}_B^{-1}\mathbf{A}_3 \\ &= -\begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

Thus, $\mathbf{d} = \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$.

From $\mathbf{A}_B^{-1}\mathbf{A}_3 = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$, only $\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_1 > 0$. Thus we have

$$\begin{aligned} \theta^* &= \min\{\frac{\left(\mathbf{A}_B^{-1}\mathbf{b}\right)_k}{\left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k} | \ \left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_k > 0\} \\ &= \min\{\frac{1}{3/2}\} = \frac{2}{3}. \end{aligned}$$

At the adjacent basic feasible solution $\mathbf{x}'$ where x_3 enters as a basic variable, we will have $x_3 = 2/3$ and $x_1 = 0$, i.e. x_1 becomes a non basic variable. This adjacent basic feasible solution is

$$\mathbf{x}' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2/3 \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

2.4 Optimality Conditions.

In this section, we obtain optimality conditions to check whether a basic feasible solution is optimal. This is useful in the development of Simplex Method. The optimality conditions also provide a clue for searching a direction to improve the objective value in a neighbourhood of a basic feasible solution.

For the objective function $\mathbf{c}^T\mathbf{x}$, moving from $\mathbf{x}^*$ to $\mathbf{x}' = \mathbf{x}^* + \theta\mathbf{d}$, the change on the objective function is

$$\mathbf{c}^T\mathbf{x}' - \mathbf{c}^T\mathbf{x}^* = \theta\mathbf{c}^T\mathbf{d}.$$

With $\mathbf{d_B} = -\mathbf{A}_B^{-1}\mathbf{A}_j$, we obtain the rate of change in the objective value with respect to x_j is $\mathbf{c}^T\mathbf{d}$, since $d_j = 1$.

Lemma C

Suppose $\mathbf{d}$, with $\mathbf{d_B} = -\mathbf{A}_B^{-1}\mathbf{A}_j$, is the feasible direction obtained as above. Then

$$\mathbf{c}^T\mathbf{d} = c_j - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_j,$$

where $\mathbf{c}_B = (c_{B(1)}, c_{B(2)}, \cdots, c_{B(m)})^T$.

Proof. Since $d_j = 1$, we have

$$\begin{aligned}\mathbf{c}^T\mathbf{d} &= \sum_{i=1}^{n} c_i d_i = \sum_{i=1}^{m} c_{B(i)} d_{B(i)} + c_j \\ &= \mathbf{c}_B^T \mathbf{d}_B + c_j = c_j - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_j.\end{aligned}$$

[QED.]

Definition (**reduced cost**)
Let $\mathbf{x}^*$ be a basic solution, with associated basis matrix $\mathbf{A}_B$. Let $\mathbf{c}_B$ be vector of the costs of the basic variables. For each j, $j = 1, 2, \cdots, n$, the **reduced cost** $\bar{c}_j$ of the variable x_j is defined according to the formula:

$$\bar{c}_j = c_j - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_j.$$

Remark
Using the reduced costs, we can determine whether moving to an adjacent basic feasible solution improves the objective values. If the $\bar{c}_j < 0$ (respec-

tively $\bar{c}_j > 0$), then moving from $\mathbf{x}^*$ to $\mathbf{x}' = \mathbf{x}^* - \theta \mathbf{A}_B^{-1}\mathbf{A}_j$ would decrease (respectively increases) the objective value by $\theta \bar{c}_j$.

Lemma D

For each basic variable $x_{B(i)}$, $i = 1, 2, \cdots, m$, the reduced cost $\bar{c}_{B(i)} = 0$.

Proof Note that $\mathbf{A}_B^{-1} \left[\, \mathbf{A}_{B(1)} \;\; \mathbf{A}_{B(2)} \;\; \cdots \;\; \mathbf{A}_{B(m)} \, \right] = I_m$.

Thus $\mathbf{A}_B^{-1}\mathbf{A}_{B(i)} = \mathbf{e}_i$, the ith column of I_m.

Hence, $\mathbf{c}_B^T \mathbf{A}_B^{-1}\mathbf{A}_{B(i)} = c_{B(i)}$, the ith component of $\mathbf{c}_B$. Thus,

$$\bar{c}_{B(i)} = c_{B(i)} - \mathbf{c}_B^T \mathbf{A}_B^{-1}\mathbf{A}_{B(i)} = 0.$$

[QED.]

Example 4.1 Consider the LP problem (refer to Example 3.3),

$$
\begin{array}{rl}
\text{minimize} & x_1 - x_2 + 3x_3 - 4x_4 \\
\text{subject to} & x_1 + x_2 + x_3 + x_4 = 2 \\
& 2x_1 + 3x_3 + 4x_4 = 2 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

(a) For the basic feasible solution $\mathbf{x}^* = (1, 1, 0, 0)^T$, the rate of cost change along the feasible direction (with x_3 enters as a basic variable) $\mathbf{d} = \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$ is

$$\bar{c}_3 = 1(-3/2) + (-1)(1/2) + 3(1) + (-4)(0) = 1.$$

Note that $\mathbf{c}^T\mathbf{d} = \bar{c}_3$. The rate of change along this direction is 1.

(b) For each variable x_j, the reduced cost $\bar{c}_j = c_j - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_j$ are computed as follows:

For x_1: $\bar{c}_1 = c_1 - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_1 = 0.$

For x_2: $\bar{c}_2 = c_2 - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_2 = 0$.

For x_3: $\bar{c}_3 = c_3 - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_3 = 1$.

For x_4: $\bar{c}_4 = c_4 - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_4 = -7$.

Theorem (Sufficient conditions for Optimality.) Consider a basic feasible solution $\mathbf{x}$ associated with a basis matrix $\mathbf{A}_B$, let $\bar{\mathbf{c}}$ be the corresponding vector of reduced costs.

(a) For a minimization problem, if $\bar{\mathbf{c}} \geq \mathbf{0}$, then $\mathbf{x}$ is optimal.

(b) For a maximization problem, if $\bar{\mathbf{c}} \leq \mathbf{0}$, then $\mathbf{x}$ is optimal.

Proof. (a) Assume $\bar{\mathbf{c}} \geq \mathbf{0}$. Let $\mathbf{y}$ be an arbitrary feasible solution.

(*Aim: Show* $\mathbf{c}^T\mathbf{y} \geq \mathbf{c}^T\mathbf{x}$.)

Let $\mathbf{w} = \mathbf{y} - \mathbf{x}$ and note that $\mathbf{A}\mathbf{w} = \mathbf{0}$. Thus, we have

$$\mathbf{A}_B\mathbf{w}_B + \sum_{i \in N} \mathbf{A}_i w_i = \mathbf{0},$$

where N is the set of indices corresponding to the nonbasic variables.

Since $\mathbf{A}_B$ is invertible, we obtain

$$\mathbf{w}_B = -\sum_{i \in N} \mathbf{A}_B^{-1} \mathbf{A}_i w_i,$$

and

$$\begin{aligned} \mathbf{c}^T \mathbf{w} &= \mathbf{c}_B^T \mathbf{w}_B + \sum_{i \in N} c_i w_i \\ &= \sum_{i \in N} (c_i - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_i) w_i \\ &= \sum_{i \in N} \bar{c}_i w_i. \end{aligned}$$

For each nonbasic index, $i \in N$, we must have $x_i = 0$ and $y_i \geq 0$ so that $w_i \geq 0$ and hence $\bar{c}_i w_i \geq 0$.

Therefore,

$$\begin{aligned} \mathbf{c}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} &= \mathbf{c}^T (\mathbf{y} - \mathbf{x}) \\ &= \mathbf{c}^T \mathbf{w} \\ &= \sum_{i \in N} \bar{c}_i w_i \\ &\geq 0. \end{aligned}$$

Thus, $\mathbf{x}$ is optimal. [QED.]

Proposition
Consider a nondegenerate basic feasible solution $\mathbf{x}$ associated with a basis matrix $\mathbf{A}_B$, let $\bar{\mathbf{c}}$ be the corresponding vector of reduced costs.
For a minimization (respectively maximization) problem, $\mathbf{x}$ is optimal if and only if $\bar{\mathbf{c}} \geq \mathbf{0}$ (respectively $\bar{\mathbf{c}} \leq \mathbf{0}$).

Proof. (We prove the proposition for a minimization problem.)

Suppose $\mathbf{x}$ is nondegenerate feasible solution which is optimal.
(*We prove by contradiction that* $\bar{\mathbf{c}} \geq \mathbf{0}$.)

Suppose $\bar{c}_j < 0$ for some j. Then x_j must be nonbasic, by Lemma D.

Since $\mathbf{x}$ is nondegenerate, the direction $\mathbf{d}$ (obtained in Lemma A, where $\mathbf{d_B} = -\mathbf{A}_B^{-1}\mathbf{A}_j$) is a feasible direction, i.e. there is a positive scalar θ such that $\mathbf{x} + \theta\mathbf{d}$ is a feasible solution.

Since $\mathbf{c}^T\mathbf{d} = \bar{c}_j < 0$ and $\mathbf{c}^T(\mathbf{x} + \theta\mathbf{d}) = \mathbf{c}^T\mathbf{x} + \theta\mathbf{c}^T\mathbf{d} < \mathbf{c}^T\mathbf{x}$, we have a decrease in the cost at $\mathbf{x} + \theta\mathbf{d}$, contradicting $\mathbf{x}$ being optimal. QED.

We summarize the above optimality results in the following theorem.

Theorem
Let $\bar{\mathbf{c}}$ be the reduced cost at a basic feasible solution $\mathbf{x}$ with the basis $\mathbf{B}$.

(i) If $\bar{\mathbf{c}} \geq \mathbf{0}$ ($\leq \mathbf{0}$), then $\mathbf{x}$ is an optimal solution for the minimization (maximization) problem.

(ii) If some $\bar{c}_j < 0$ (> 0), then there is a direction $\mathbf{d}$ (where $d_j = 1$, $d_i = 0\ \forall j \neq i \notin \mathbf{B}$ and $\mathbf{d}_B = -\mathbf{A}_B^{-1}\mathbf{A}_j$) corresponding to the nonbasic variable x_j, and moving along $\mathbf{d}$ will result in two cases:

(a) if $\mathbf{d}_B \geq \mathbf{0}$, then the objective value $\mathbf{c}^T(\mathbf{x} + \theta\mathbf{d}) \to -\infty$ for min ($\to +\infty$ for max), as $\theta \to +\infty$;

(b) if some $d_{B(k)} < 0$, then we will obtain an adjacent basic feasible solution $\mathbf{x}' = \mathbf{x} + \theta^*\mathbf{d}$

where

$$\theta^* = \min\{\frac{x_{B(k)}}{-d_{B(k)}} \mid d_{B(k)} < 0\} \geq 0$$

which satisfies $\mathbf{c}^T\mathbf{x'} \leq (\geq)\mathbf{c}^T\mathbf{x}$, in particular, $\mathbf{c}^T\mathbf{x'} < (>)\mathbf{c}^T\mathbf{x}$ holds iff $\theta^* > 0$).

To determine whether a basic solution is optimal, we need to check for feasibility and nonnegativity (or nonpositivity) of the reduced costs. Thus we have the following definition.

Definition For a minimization (respectively maximization) problem, a basic solution $\mathbf{x}$ with basis $\mathbf{B}$ is said to be **optimal** if:

(a) $\mathbf{x_B} = \mathbf{A}_B^{-1}\mathbf{b} \geq 0$, and

(b) $\bar{\mathbf{c}}^T = \mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A} \geq \mathbf{0}$ (respectively $\bar{\mathbf{c}} \leq \mathbf{0}$).

Example 4.2 Consider the LP problem

$$\begin{array}{lrcrcrcrcl} \text{minimize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & c_4x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & 2x_1 & & & + & 3x_3 & + & 4x_4 & = & 2 \\ & & & & x_1, x_2, x_3, x_4 \geq 0 & & & & & \end{array}$$

(a) For the objective

$$\text{minimize} \;\; x_1 - x_2 + 3x_3 - 4x_4,$$

the basic feasible solution $\mathbf{x} = (1, 1, 0, 0)^T$, with

$$\mathbf{A}_B = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

The vector of reduced costs is $\bar{\mathbf{c}} = (0, 0, 1, -7)^T$ (computed in Example 2.2).

Since $\bar{\mathbf{c}}_4 < 0$ and $\mathbf{x}$ is nondegenerate, $\mathbf{x}$ is not optimal.

(b) For the objective

$$\text{minimize} \;\;\; x_1 - x_2 + 3x_3 + 4x_4$$

subject to the same constraints.

The basic feasible solution $\mathbf{x} = (1, 1, 0, 0)^T$, with the same $\mathbf{A}_B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ is an optimal solution since $\bar{\mathbf{c}} = (0, 0, 1, 1)^T \geq \mathbf{0}$.

By the definition, the basis matrix $\mathbf{A}_B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ is optimal.

Chapter 3

Implementing the Simplex Method.

Based on the theory developed in the previous chapter, the following method is proposed for solving linear programming problems.

3.1 The Simplex Method.

The simplex method is initiated with a starting basic feasible solution (guaranteed for feasible standard form problem), and continues with the following typical iteration.

1. In a typical iteration, we start with a basis consisting of the basic columns $\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \cdots,$ $\mathbf{A}_{B(m)}$, and an associated basic feasible solution $\mathbf{x}$.

2. Compute the reduced costs $\bar{c}_j = c_j - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_j$ for all nonbasic variables x_j.

   For a minimization (respectively maximization) problem, if they are all nonnegative (respectively nonpositivity) , the current basic feasible solution is optimal, and the algorithm terminates; else, choose some j_* for which $\bar{c}_{j_*} < 0$ (respectively $\bar{c}_{j_*} > 0$).

   The corresponding x_{j_*} is called the **entering variable**.

3. Compute $\mathbf{u} = \mathbf{A}_B^{-1} \mathbf{A}_{j_*}$.

4. If no component of $\mathbf{u}$ is positive, we conclude that the problem is unbounded, and the algorithm terminates.

   If some component of $\mathbf{u}$ is positive, let

$$\theta^* = \min \{ \frac{x_{B(i)}}{u_i} \mid u_i > 0 \}.$$

5. Let l be such that $\theta^* = \dfrac{x_{B(l)}}{u_l}$.

The corresponding $x_{B(l)}$ is called the **leaving variable**.

Form a new basis by replacing $\mathbf{A}_{B(l)}$ with $\mathbf{A}_{j*}$.

The entering variable x_{j*} assumes value $\theta^* = \dfrac{x_{B(l)}}{u_l}$ whereas the other basic variables assume values $x_{B(i)} - \theta^* u_i$ for $i \neq l$.

Example 1.1 We shall demonstrate the simplex iteration for the following LP problem.

$$\begin{array}{rrcrcrcrcl} \text{minimize} & x_1 & - & x_2 & + & 3x_3 & - & 4x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & 2x_1 & & & + & 3x_3 & + & 4x_4 & = & 2 \\ & & & & & & & & & \end{array}.$$

$$x_1, x_2, x_3, x_4 \geq 0$$

1. Start with the basis $\{\mathbf{A}_1, \mathbf{A}_2\}$ associated with basic feasible solution $\mathbf{x} = (1, 1, 0, 0)^T$.

2. Compute reduced costs for nonbasic variables, check for optimality and select entering variable if nonoptimal.

   For nonbasic variables x_3 and x_4, the respective reduced costs are $\bar{c}_3 = 1$ and $\bar{c}_4 = -7$.

   Since $\bar{c}_4 < 0$, choose x_4 to be the entering variable.

3. Compute the basic direction correspond to the entering variable.

   The x_4-basic direction

$$\mathbf{u} = \mathbf{A}_B^{-1}\mathbf{A}_4 = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

4. Check for positive components of $\mathbf{u}$ to select the leaving variable.

   The first component of $\mathbf{u} = u_1 = 2 > 0$, and $x_{B(1)} = x_1 = 1$. Thus

$$\theta^* = \min\left\{\frac{x_{B(i)}}{u_i} \mid u_i > 0\right\} = 1/2 = \frac{x_{B(1)}}{u_1}.$$

   Therefore, x_1 is the leaving variable.

5. Determine the new basic feasible solution and basis.

   The new basis is $\mathbf{A}_4$ and $\mathbf{A}_2$.

   The entering variable x_4 assume value $\theta^* = 1/2$, the other basic variable x_2 assumes value

$$1 - (1/2)(-1) = 3/2$$

   (from $x_{B(i)} - \theta^* u_i$, i.e. $x_2 - (1/2)u_2$) .

   New BFS:

$$\mathbf{x} = (0, \frac{3}{2}, 0, \frac{1}{2})^T.$$

3.2 Simplex Tableau Implementation.

Tableau is a convenient form for implementing the simplex method. Thus, from now on, computation and analysis will be carried out on tableaus. A generic simplex tableau looks as follows:

Basic	$\mathbf{x}$	Solution
$\bar{\mathbf{c}}$	$\bar{\mathbf{c}}^T$	$-z$
$\mathbf{x_B}$	$\mathbf{A}_B^{-1}\mathbf{A}$	$\mathbf{A}_B^{-1}\mathbf{b}$

where z is the objective value. In detail,

Basic	$\mathbf{x}$	Solution
$\bar{\mathbf{c}}$	$\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}$	$-\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{x_B}$	$\mathbf{A}_B^{-1}\mathbf{A}$	$\mathbf{A}_B^{-1}\mathbf{b}$

This tableau is obtained through following row operations:

Start with original problem

Basic	$\mathbf{x}$	Solution
$\bar{\mathbf{c}}$	$\mathbf{c}^T$	0
$\mathbf{x_B}$	$\mathbf{A}$	$\mathbf{b}$

Multiply $\mathbf{x}_B$-row with $\mathbf{A}_B^{-1}$,

Basic	$\mathbf{x}$	Solution
$\bar{\mathbf{c}}$	$\mathbf{c}^T$	0
$\mathbf{x_B}$	$\mathbf{A}_B^{-1}\mathbf{A}$	$\mathbf{A}_B^{-1}\mathbf{b}$

Then, the $\bar{\mathbf{c}}$-row is obtained row operations

$$\begin{aligned} & (\bar{\mathbf{c}}\text{-row}) - \mathbf{c}_B^T \cdot (\mathbf{x}_B\text{-row}) \\ = \; & (\mathbf{c}^T \mid 0) - \mathbf{c}_B^T(\mathbf{A}_B^{-1}\mathbf{A} \mid \mathbf{A}_B^{-1}\mathbf{b}) \\ = \; & (\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A} \mid -\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}) \end{aligned}$$

We first consider the minimization (or maximization) problem where all functional constraints are of $\leq$ type, with nonnegative right-hand side ($\mathbf{b} \geq 0$).

$$\begin{aligned} \text{Minimize } & \mathbf{c}^T\mathbf{x} \\ \text{Subject to } & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{A}$ is $m \times n$.

The corresponding standard form LP is:

$$\begin{aligned} \text{Minimize } & \mathbf{c}^T\mathbf{x} + \mathbf{0}\mathbf{S} \\ \text{Subject to } & \mathbf{A}\mathbf{x} + \mathbf{S} = \mathbf{b}, \ \text{ i.e. } [\mathbf{A}, \mathbf{I}] \left(\begin{matrix} \mathbf{x} \\ \mathbf{S} \end{matrix} \right) = \mathbf{b}, \\ & \mathbf{x}, \mathbf{S} \geq \mathbf{0}. \end{aligned}$$

For such model, each constraint is associated with a slack variable. Thus, the number of slack variables equals the number of functional constraints. The matrix $[\mathbf{A}, \mathbf{I}]$ is $m \times (n+m)$ and there are $n+m$ decision variables. Thus, a basic feasible solution $\left(\begin{matrix} \mathbf{x} \\ \mathbf{S} \end{matrix} \right)$ must satisfy $[\mathbf{A}, \mathbf{I}] \left(\begin{matrix} \mathbf{x} \\ \mathbf{S} \end{matrix} \right) = \mathbf{b}$ and there are n (nonbasic) variables from $\begin{pmatrix} \mathbf{x} \\ \mathbf{S} \end{pmatrix}$ equal to zero.

Choosing $\mathbf{x}$ to be nonbasic variables, and $\mathbf{S} = \mathbf{b}$ to be basic variables provides a starting basic feasible solution to carry out the simplex iteration.

The starting simplex tableau associated with this basis is:

Basic	x_1	$\cdots$	x_n	s_1	$\cdots$	s_m	Solution
$\bar{\mathbf{c}} = \mathbf{c}$	c_1	$\cdots$	c_n	0	$\cdots$	0	0
s_1	$\vert$		$\vert$				$\vert$
$\cdot$							
s_i	$\mathbf{A}_1$	$\cdots$	$\mathbf{A}_n$		I		$\mathbf{b}$
$\cdot$							
s_m	$\vert$		$\vert$				$\vert$

Once we obtain a basic feasible solution to a given linear programming problem, we may apply the simplex algorithm to solve the problem.

Example 2.1 The standard LP form of the following:

$$\begin{array}{rrcrcl}
\text{Minimize} & -3x_1 & - & 2x_2 & & \\
\text{Subject to} & x_1 & + & 2x_2 & \leq & 6 \\
& 2x_1 & + & x_2 & \leq & 8 \\
& -x_1 & + & x_2 & \leq & 1 \\
& & & x_2 & \leq & 2 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

is

$$\begin{array}{rrcrcrcrcrcrcl}
\text{Minimize} & -3x_1 & - & 2x_2 & + & 0s_1 & + & 0s_2 & + & 0s_3 & + & 0s_4 & & \\
\text{Subject to} & x_1 & + & 2x_2 & + & s_1 & & & & & & & = & 6 \\
& 2x_1 & + & x_2 & & & + & s_2 & & & & & = & 8 \\
& -x_1 & + & x_2 & & & & & + & s_3 & & & = & 1 \\
& & & x_2 & & & & & & & + & s_4 & = & 2 \\
& & & & & & & & & & & & & \\
\end{array}$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Step 0. A readily available starting basic feasible solution is:

basic variables: slack variables s_1, s_2, s_3, s_4,
nonbasic variables: $x_1, x_2 = 0$,
associated basis matrix $\mathbf{A}_B = \mathbf{I}$.

Starting tableau:

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
$\bar{\mathbf{c}}$	-3	-2	0	0	0	0	0
s_1	1	2	1	0	0	0	6
s_2	2	1	0	1	0	0	8
s_3	-1	1	0	0	1	0	1
s_4	0	1	0	0	0	1	2

Step 1. Check for optimality. Is there any negative value in the $\bar{\mathbf{c}}$-row?

The reduced costs of both nonbasic variables, x_1 and x_2, are negative. We choose x_1 as an entering variable. Column x_1 is the pivot column.

Step 2. Select a leaving variable from the current basic variables to be a nonbasic variable when the entering variable becomes basic.

Comparing ratios $\frac{x_{B(i)}}{u_i}$ (with positive denominators, here u_1 and u_2),

	Basic	x_1	x_2	s_1	s_2	s_3	s_4	Soln	ratio
	$\bar{\mathbf{c}}$	-3	-2	0	0	0	0	0	
	s_1	1	2	1	0	0	0	6	6
Pivot row $\rightarrow$	s_2	2	1	0	1	0	0	8	4
	s_3	-1	1	0	0	1	0	1	
	s_4	0	1	0	0	0	1	2	

s_2-row is associated with the smallest ratio. Thus s_2 is the leaving variable and s_2-row the pivot row. The entry at the pivot row and column is called the pivot entry.

Step 3. Determine the new basic solution, via row operations, by making the entering variable basic and the leaving variable nonbasic. The row operations make the pivot entry = 1 and other entries in the pivot column = 0.

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
$\bar{\mathbf{c}}$	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	0	0	12	ratio
s_1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	2	$\frac{4}{3}$
x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	4	8
s_3	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	5	$\frac{10}{3}$
s_4	0	1	0	0	0	1	2	2

Thus, the new basic feasible solution is

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (4, 0, 2, 0, 5, 2)$$

with cost -12 decreased from 0.

This completed one iteration.

Then go to Step 1 with the above new tableau and repeat Steps 1, 2 and 3 in the second iteration. From this new tableau, x_2 will be chosen as the entering variable and s_1 the leaving variable. At the end of the second iteration, we obtain the following tableau:

	Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
	$\bar{\mathbf{c}}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	$12\ \frac{2}{3}$
	x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{4}{3}$
Optimum	x_1	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{10}{3}$
	s_3	0	0	-1	1	1	0	3
	s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{2}{3}$

The solution yields

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (\frac{10}{3}, \frac{4}{3}, 0, 0, 3, 2/3)$$

with cost $-12\frac{2}{3}$ decreased from -12.

The last tableau is optimal because none of the nonbasic variables (i.e. s_1 & s_2) has a negative reduced cost in the $\bar{\mathbf{c}}$-row.

The algorithm terminates.

Graphically, the simplex algorithm starts at the origin A (**starting solution**) and moves to an **adjacent corner point** at which the objective value could be improved. At B ($x_1 = 4, x_2 = 0$), the objective value will be decreased. Thus B is a possible choice. The process is repeated to see if there is another corner point that can improve the value of the objective function. Eventually, the algorithm will stop at C (i.e. $x_1 = \frac{10}{3}, x_2 = \frac{4}{3}$) (the optimum). Hence it takes 3 iterations (A, B and C) to reach the optimum.

Putting all tableaus together:

	Basic	x_1	x_2	s_1	s_2	s_3	s_4	Soln	ratio
(0)	$\bar{\mathbf{c}}$	-3	-2	0	0	0	0	0	
	s_1	1	2	1	0	0	0	6	6
Pivot row	s_2	2	1	0	1	0	0	8	4
	s_3	-1	1	0	0	1	0	1	
	s_4	0	1	0	0	0	1	2	
(1)	$\bar{\mathbf{c}}$	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	0	0	12	ratio
	s_1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	2	$\frac{4}{3}$
x_2 enters	x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	4	8
s_1 leaves	s_3	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	5	$\frac{10}{3}$
	s_4	0	1	0	0	0	1	2	2
(2)	$\bar{\mathbf{c}}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	$12\ \frac{2}{3}$	
	x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{4}{3}$	
Optimum	x_1	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{10}{3}$	
	s_3	0	0	-1	1	1	0	3	
	s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{2}{3}$	

Example 2.2

$$
\begin{array}{lrcrcrcrcl}
\text{Maximize} & -5x_1 & + & 4x_2 & - & 6x_3 & - & 8x_4 & & \\
\text{Subject to} & x_1 & + & 7x_2 & + & 3x_3 & + & 7x_4 & \leq & 46 \\
 & 3x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & \leq & 8 \\
 & 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & \leq & 10 \\
 & & & & & x_1, & x_2, & x_3,\ x_4 & \geq & 0
\end{array}
$$

The associated standard form LP:

$$
\begin{array}{lrcrcrcrcrcrcrcl}
\text{Maximize} & -5x_1 & + & 4x_2 & - & 6x_3 & - & 8x_4 & + & 0s_1 & + & 0s_2 & + & 0s_3 & & \\
\text{Subject to} & x_1 & + & 7x_2 & + & 3x_3 & + & 7x_4 & + & s_1 & & & & & = & 46 \\
 & x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & & & + & s_2 & & & = & 8 \\
 & 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & & & & & + & s_3 & = & 10 \\
 & & & & & & & x_1, & x_2, & x_3, & x_4, & s_1, & s_2, & s_3 & \geq & 0
\end{array}
$$

Thus, the implementation of simplex method via simplex tableaus:

	Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Solution
	$\bar{\mathbf{c}}$	-5	4	-6	-8	0	0	0	0
	s_1	1	7	3	7	1	0	0	46
x_2 enters	s_2	3	-2	1	2	0	1	0	8
s_3 leaves	s_3	2	3	-1	1	0	0	1	10
	$\bar{\mathbf{c}}$	$-\frac{23}{3}$	0	$-\frac{14}{3}$	$-\frac{28}{3}$	0	0	$-\frac{4}{3}$	$-\frac{40}{3}$
	s_1	$-\frac{11}{3}$	0	$\frac{16}{3}$	$\frac{14}{3}$	1	0	$-\frac{7}{3}$	$\frac{68}{3}$
optimum	s_2	$\frac{13}{3}$	0	$\frac{1}{3}$	$\frac{8}{3}$	0	1	$\frac{2}{3}$	$\frac{44}{3}$
	x_2	$\frac{2}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{10}{3}$

Therefore the optimal solution is $(x_1, x_2, x_3, x_4) = (0, \frac{10}{3}, 0, 0)$ with optimal cost $\frac{40}{3}$.

3.3 Starting the Simplex Algorithms.

In the previous section, we considered the simple case where all constraints are of type $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{b} \geq \mathbf{0}$. In this case, a ready starting basic feasible solution is available.

However, if some constraints are of type $\geq$ or $=$, then we have to add **artificial variables** in order to obtain a basic feasible solution to the modified LP.

As artificial variables have no physical meaning, they must be forced to zero when the optimum is reached, otherwise the resulting solution is infeasible. Two (closely related) methods based on the idea of driving out the artificial variables are devised for this purpose, namely:

(a) The Two-Phase Method.

(b) The Big-M Method (or M simplex method).

How to add artificial variables?

1. For each constraint of type " $\mathbf{a}_i^T\mathbf{x} = b_i$", we add an artificial variable $y_i \geq 0$ to have the modified constraint $\mathbf{a}_i^T\mathbf{x} + y_i = b_i$.

2. For each constraint of type "$\mathbf{a}_i^T\mathbf{x} \geq b_i$", after adding a surplus variable $s_i \geq 0$, we add an artificial variable $y_i \geq 0$ to have the modified constraint $\mathbf{a}_i^T\mathbf{x} - s_i + y_i = b_i$.

Example 3.1 Consider the LP problem

$$\begin{array}{rrcrcr} \text{Minimize} & 4x_1 & + & x_2 & & \\ \text{Subject to} & 3x_1 & + & x_2 & = & 3 \\ & -4x_1 & - & 3x_2 & \leq & -6 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & & x_1, & x_2 & \geq & 0. \end{array}$$

Add artificial variables where necessary and write down the modified constraints.

Solution

1. Add artificial variable $y_1 \geq 0$ to the first constraint:
$$3x_1 + x_2 + y_1 = 3.$$
2. Multiply the second constraint by (-1) to obtain nonnegative $\mathbf{b}$: $4x_1 + 3x_2 \geq 6$. Add a surplus variable $s_1 \geq 0$ and an artificial variable $y_2 \geq 0$:
$$4x_1 + 3x_2 - s_1 + y_2 = 6.$$

(a) The Two-Phase Method.

Introduce artificial variables y_i, if necessary, and form the auxiliary LP problem, with the following modified objective and constraints:

The Auxiliary LP problem

$$
\begin{array}{rl}
\text{minimize} & \sum\limits_{i=1}^{k} y_i = y_1 + y_2 \cdots + y_k \\
\text{subject to} & \text{“Modified constraints”}, \\
 & \mathbf{x} \geq \mathbf{0}, \\
 & s_i \geq 0 \text{ for slack and surplus variables}, \\
 & y_i \geq 0 \text{ for artificial variable } y_i.
\end{array}
$$

A ready starting basic feasible solution for the auxiliary LP problem is obtained by choosing basic variables to be artificial variables y_i and slack variables s_i (nonbasic variables are $\mathbf{x}$ and surplus variables s_i, all assuming zero values), and the associated basis matrix $\mathbf{A}_B = \mathbf{I}$.

Example 3.2 For the LP problem:

$$\begin{array}{rrcrcr}
\text{Minimize} & 4x_1 & + & x_2 & & \\
\text{Subject to} & 3x_1 & + & x_2 & = & 3 \\
 & -4x_1 & - & 3x_2 & \leq & -6 \\
 & x_1 & + & 2x_2 & \leq & 4 \\
 & & x_1, & x_2 & \geq & 0.
\end{array}$$

Write down the auxiliary LP problem and a basic feasible solution for the auxiliary LP problem.

Solution

Refer to the Example 3.1, the auxiliary LP problem

$$\begin{array}{rrcrcrcrcrcrl}
\text{Minimize} & y_1 & + & y_2 & & & & & & & & \\
\text{Subject to} & 3x_1 & + & x_2 & & & & & + & y_1 & & & = 3 \\
 & 4x_1 & + & 3x_2 & - & s_1 & & & & & + & y_2 & = 6\,. \\
 & x_1 & + & 2x_2 & & & + & s_2 & & & & & = 4 \\
 & & & & & & & \multicolumn{5}{r}{x_1, x_2, s_1, s_2, y_1, y_2 \geq 0} &
\end{array}$$

A basic feasible solution to this LP problem is $(x_1, x_2, s_1, s_2, y_1, y_2) = (0, 0, 0, 4, 3, 6)$, with cost 9.

Basic variables: $s_2 = 4, y_1 = 3, y_2 = 6$

nonbasic variables: x_1, x_2, s_1.

Notes

1. The auxiliary porblem is always a minimization of $\Sigma_{i=1}^{k}\ y_i$ whether the original problem is Minimization or Maximization. (Why?)
2. If $\Sigma_{i=1}^{k}\ y_i > 0$, the original LP problem is infeasible (Why?).
3. If $\Sigma_{i=1}^{k}\ y_i = 0$, then the original LP problem has a basic feasible solution.

A complete algorithm for LP problems in standard form.

Phase I

1. Introduce artificial variables $y_1, y_2, \cdots, y_m$, wherever necessary, and apply the simplex method to the auxiliary problem with cost $\sum_{i=1}^{m} y_i$.
2. If the optimal cost in the auxiliary problem is positive, the original problem is infeasible and the algorithm terminates.
3. If the optimal cost in the auxiliary problem is zero, a basic feasible solution to the original problem has been found as follows:
   (a) If no artificial variable is in the final basis, the artificial variables and the corresponding columns are eliminated, and a feasible basis for the original problem is available.
   (b) If in the final tableau there are some artificial variables as basic variables at zero level, choose a non-artificial (nonbasic) variable to enter the basis, then an artificial (basic) variable may be

driven out of the basis. Repeat this procedure until all artificial variables are driven out of the basis.

Phase II

1. Let the final basis and tableau obtained from Phase I be the initial basis and tableau for Phase II.
2. Compute the reduced costs of all variables for the initial basis, using the cost coefficients of the original problem.
3. Apply the simplex method to the original problem.

Remark The purpose of phase 1 is to obtain a basic feasible solution to the original LP, if it exists.

Example 3.3 Use the 2-phase method to solve the LP problem:

$$\begin{array}{rrcrcr} \text{Minimize} & & & 4x_1 & + & x_2 \\ \text{Subject to} & 3x_1 & + & x_2 & = & 3 \\ & -4x_1 & - & 3x_2 & \leq & -6 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & & & x_1, \; x_2 & \geq & 0. \end{array}$$

Solution

Phase I. Auxiliary problem

$$\begin{array}{rlllllr} \text{Minimize} & y_1 + y_2 & & & & & \\ \text{Subject to} & 3x_1 + x_2 & & & + y_1 & & = 3 \\ & 4x_1 + 3x_2 & - s_1 & & & + y_2 & = 6 \\ & x_1 + 2x_2 & & + s_2 & & & = 4 \\ & & x_1, x_2, s_1, s_2, y_1, y_2 \geq 0 & & & & \end{array}$$

First, we need to compute the reduced cost $\bar{\mathbf{c}}$. With the basis $\mathbf{B} = \{y_1, y_2, s_2\}$, we have

$$\begin{aligned} \bar{\mathbf{c}} &= \mathbf{c} - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A} \\ &= \mathbf{c} - (c_{y_1}, c_{y_2}, c_{s_2}) \mathbf{A}_B^{-1} \mathbf{A} \\ &= \mathbf{c} - (1, 1, 0) \mathbf{A}_B^{-1} \mathbf{A}. \end{aligned}$$

Thus, the starting $\bar{\mathbf{c}}$-row is obtained by applying row operations:

$$\begin{aligned}
\bar{\mathbf{c}}\text{-row} &= (\mathbf{c}\text{-row}) - (y_1\text{-row}) - (y_2\text{-row}) - 0(s_2\text{-row}) \\
&= (0,0,0,1,1,0) - (3,1,0,1,0,0) \\
&\quad -(4,3,-1,0,1,0) \\
&= (-7,-4,1,0,0,0).
\end{aligned}$$

A more direct way to derive the above row operations is as follows: Note that reduced costs of basic variables must be zero. Thus, row operations are to change $\mathbf{c}_B$ to $\bar{\mathbf{c}}_B = \mathbf{0}$. These row operations are exactly as above.

	Basic	x_1	x_2	s_1	y_1	y_2	s_2	Soln
	$\mathbf{c}$	0	0	0	1	1	0	0
(0)	$\bar{\mathbf{c}}$	-7	-4	1	0	0	0	-9
x_1 enters	y_1	3	1	0	1	0	0	3
y_1 leaves	y_2	4	3	-1	0	1	0	6
	s_2	1	2	0	0	0	1	4
(1)	$\bar{\mathbf{c}}$	0	$-\frac{5}{3}$	1	$\frac{7}{3}$	0	0	-2
x_2 enters	x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
y_2 leaves	y_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
	s_2	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3
(2)	$\bar{\mathbf{c}}$	0	0	0	1	1	0	0
	x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
optimum	x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
	s_2	0	0	1	1	-1	1	1

At the optimum, $y_1 + y_2 = 0$, thus the original problem has a baisc feasible solution, namely $(x_1, x_2, s_1, s_2) = (3/5, 6/5, 0, 1)$ (basic variables are x_1, x_2, s_2) and we proceed to phase II.

Phase II. The artificial variables (y_1 and y_2) have now served their purpose and must be dispensed with in all subsequent computations (by setting them to be zero, i.e. $y_1 = 0, y_2 = 0$). In the simplex tableau, columns of y_1 and y_2 are removed.

With basic variables x_1, x_2, s_2, we have

$$\begin{aligned}\bar{\mathbf{c}} &= \mathbf{c} - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A} \\ &= \mathbf{c} - (c_{x_1}, c_{x_2}, c_{s_2}) \mathbf{A}_B^{-1} \mathbf{A} \\ &= \mathbf{c} - (4, 1, 0) \mathbf{A}_B^{-1} \mathbf{A}.\end{aligned}$$

The starting $\bar{\mathbf{c}}$-row for the simplex method can thus be obtained via applying row operations

$\bar{\mathbf{c}}$- row $=$ ($\mathbf{c}$- row) $- 4 \times$ (x_1- row) $- 1 \times$ (x_2-row).

(The row operations change $\mathbf{c}_B$ to $\bar{\mathbf{c}}_B = \mathbf{0}$.)

Simplex Tableau

	Basic	x_1	x_2	s_1	s_2	Soln
	$\mathbf{c}$	4	1	0	0	0
(0)	$\bar{\mathbf{c}}$	0	0	$-\frac{1}{5}$	0	$-\frac{18}{5}$
s_1 enters	x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
s_2 leaves	x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
	s_2	0	0	1	1	1
(1)	$\bar{\mathbf{c}}$	0	0	0	$\frac{1}{5}$	$-\frac{17}{5}$
	x_1	1	0	0	$-\frac{1}{5}$	$\frac{2}{5}$
optimum	x_2	0	1	0	$\frac{3}{5}$	$\frac{9}{5}$
	s_1	0	0	1	1	1

Thus, the optimal solution is $(x_1, x_2) = (\frac{2}{5}, \frac{9}{5})$ with cost $\frac{17}{5}$.

Note The artificial variables are removed in Phase II **only** when they are **nonbasic** at the end of Phase I. It is possible, however, that an artificial variable remains **basic** at zero level at the end of Phase I. In this case, provisions must be made to ensure that it never becomes positive during Phase II computations (refer to the algorithm).

Example 3.4 (Infeasible Solution.)

$$\begin{array}{rrcrcr} \text{Minimize} & -3x_1 & - & 2x_2 & & \\ \text{Subject to} & 2x_1 & + & x_2 & \leq & 2 \\ & 3x_1 & + & 4x_2 & \geq & 12 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Solution: Auxiliary LP problem:

$$\begin{array}{rl} \text{Minimize} & y \\ \text{Subject to} & 2x_1 + x_2 + s_1 = 2 \\ & 3x_1 + 4x_2 - s_2 + y = 12 \\ & x_1, x_2, s_1, s_2, y \geq 0 \end{array}$$

	Basic	x_1	x_2	s_1	s_2	y	solution
	$\mathbf{c}$	0	0	0	0	1	0
	$\bar{\mathbf{c}}$	-3	-4	0	1	0	-12
$x_2 \rightarrow$	s_1	2	1	1	0	0	2
$\leftarrow s_1$	y	3	4	0	-1	1	12
	$\bar{\mathbf{c}}$	5	0	4	1	0	-4
	x_2	2	1	1	0	0	2
	y	-5	0	-4	-1	1	4

The tableau is optimal but cost $4 \neq 0$, thus there is no feasible solution.

(b) The big-M Method

Similar to the two-phase method, the big-M method starts with the LP in the standard form, and augment an artificial variable y_i for any constraint that does not have a slack. Such variables, together with slack variables, then become the starting basic variables.

We penalize each of these variables by assigning a very *large* coefficient (M) in the objective function:

Minimize objective function $+\Sigma\, My_i$, (minimization)

or

Maximize objective function $-\Sigma\, My_i$, (maximization)

where $M > 0$.

For sufficiently large choice of M, if the original LP is feasible and its optimal value is finite, all of the artificial variables are eventually driven to zero, and we have the minimization or maximization of the original objective function.

The coefficient M is not fixed with any numerical value. It is always treated as a larger number whenever it is compared to another number. Thus the reduced costs are functions of M.

We apply simplex algorithms to the modified objective and the same constraints as in the Auxiliary LP problem in the 2-phase method.

Example 3.5 Solve the LP problem by the big-M method.

$$\begin{array}{rrcrcr} \text{Minimize} & & & 4x_1 & + & x_2 \\ \text{Subject to} & 3x_1 & + & x_2 & = & 3 \\ & 4x_1 & + & 3x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & & & x_1, \ x_2 & \geq & 0 \end{array}$$

Solution The standard form of the LP:

$$\begin{array}{rrcrcrcrcr} \text{Minimize} & 4x_1 & + & x_2 & & & & & & \\ \text{Subject to} & 3x_1 & + & x_2 & & & & & = & 3 \\ & 4x_1 & + & 3x_2 & - & s_1 & & & = & 6 \\ & x_1 & + & 2x_2 & & & + & s_2 & = & 4 \\ & & & & & x_1, \ x_2, & s_1, & s_2 & \geq & 0 \end{array}$$

We augment two artificial variables y_1 and y_2 in the 1^{st} and 2^{nd} equations, and penalize y_1 and y_2 in the objective function by adding $My_1 + My_2$.

The modified LP with its artificial variables becomes:

$$\begin{array}{rrcrcrcrcrcrcrcl}
\text{Minimize} & 4x_1 & + & x_2 & + & My_1 & + & My_2 & & & & & & & & \\
\text{Subject to} & 3x_1 & + & x_2 & & & & & & & + & y_1 & & & = & 3 \\
& 4x_1 & + & 3x_2 & - & s_1 & & & & & & & + & y_2 & = & 6 \\
& x_1 & + & 2x_2 & & & + & s_2 & & & & & & & = & 4
\end{array}$$

$$x_1, x_2, s_1, s_2, y_1, y_2 \geq 0$$

Choose artificial variables and slack variables to be basic variables.
Thus, $\mathbf{x}_B = (y_1, y_2, s_2)^T$.
Since $\mathbf{c}_B^T = (M, M, 0)$, we obtain the starting $\bar{\mathbf{c}}$-row as follows:

Starting $\bar{\mathbf{c}}$- row = ($\mathbf{c}$- row)$-M \times$(y_1- row)$-M \times$ (y_2- row) $- 0 \times$ (s_2- row).

In tableau form, we have:

	Basic	x_1	x_2	s_1	y_1	y_2	s_2	Soln
	$\mathbf{c}$	4	1	0	M	M	0	0
	$\bar{\mathbf{c}}$	$4-7M$	$1-4M$	M	0	0	0	$-9M$
x_1 enters	y_1	3	1	0	1	0	0	3
y_1 leaves	y_2	4	3	-1	0	1	0	6
	s_2	1	2	0	0	0	1	4
	$\bar{\mathbf{c}}$	0	$\frac{-1-5M}{3}$	M	$\frac{-4+7M}{3}$	0	0	$-4-2M$
x_2 enters	x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
y_2 leaves	y_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
	s_2	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3
	$\bar{\mathbf{c}}$	0	0	$-\frac{1}{5}$	$-\frac{8}{5}+M$	$\frac{1}{5}+M$	0	$-\frac{18}{5}$
s_1 enters	x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
s_2 leaves	x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
	s_2	0	0	1	1	-1	1	1
	$\bar{\mathbf{c}}$	0	0	0	$-\frac{7}{5}+M$	M	$\frac{1}{5}$	$-\frac{17}{5}$
	x_1	1	0	0	$\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$
Optimum	x_2	0	1	0	$-\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{9}{5}$
	s_1	0	0	1	1	-1	1	1

Therefore, the optimal solution is $(x_1, x_2) = (\frac{2}{5}, \frac{9}{5})$ with optimal cost $\frac{17}{5}$. Since it contains no artificial variables at positive level, the solution is feasible with respect to the original problem before the artificial variables are added. (If the problem has no feasible solution, at least one artificial variable will be positive in the optimal solution).

(A) Degeneracy

A basic feasible solution in which **one** or **more** basic variables are **zero** is called a **degenerate** basic feasible solution. A **tie** in the minimum ratio rule leads to the degeneracy in the solution. From the practical point of view, the condition reveals that the model has at least one **redundant** constraint at that basic feasible solution.

Example 4.1 (Degenerate Optimal Solution)

$$\begin{array}{rrcrcl} \text{Minimize} & -3x_1 & - & 9x_2 & & \\ \text{subject to} & x_1 & + & 4x_2 & \leq & 8 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & x_1, \ x_2 & & & \geq & 0 \end{array}$$

	Basic	x_1	x_2	S_1	S_2	Solution	
(0)	$\mathbf{c}$	-3	-9	0	0	0	ratio
x_2 enters	S_1	1	4	1	0	8	2
S_1 leaves	S_2	1	2	0	1	4	2
(1)	$\bar{\mathbf{c}}$	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18	
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2	
S_2 leaves	S_2	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	$\Longrightarrow$ degenerate BFS
(2)	$\bar{\mathbf{c}}$	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18	
	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2	
optimal	x_1	1	0	-1	2	0	$\Longrightarrow$ degenerate opt. soln

Note In iteration 2, the entering variable x_1 replaces S_2, where $S_2 = 0$ is a basic variable, hence degeneracy remains in the optimum.

Looking at the graphical solution, we see that 3 lines pass through the optimum ($x_1 = 0, x_2 = 2$). We need only 2 lines to identify a point in a two-dimensional problem hence we say that the point is *overdetermined*. For this reason, we conclude that one of the constraints is *redundant*. There are no reliable techniques for identifying redundant constraint directly from the tableau. In the absence of graphical representation, we may have to rely on other means to locate the redundancy in the model.

Theoretical Implications of Degeneracy

(a) The objective value is not improved (-18) in iterations 1 and 2. It is possible that the simplex iteration will enter a *loop* without reaching the optimal solution. This phenomenon is called "cycling", but seldom happens in practice.

(b) Both iterations 1 and 2 yield identical values:

$$x_1 = 0,\ x_2 = 2,\ S_1 = 0,\ S_2 = 0,\ z = 18$$

but with different classifications as basic and non-basic variables.

Question. Can we stop at iteration 1 (when degeneracy first appears even though it is not optimum? No, as we shall see in the next example.

Example 4.2. (Temporarily Degenerate Solution)

$$\begin{array}{rrcrl}
\text{Minimize} & -3x_1 & - & 2x_2 & \\
\text{Subject to} & 4x_1 & + & 3x_2 & \le\ 12 \\
 & 4x_1 & + & x_2 & \le\ 8 \\
 & 4x_1 & - & x_2 & \le\ 8 \\
 & x_1, & & x_2 & \ge\ 0
\end{array}$$

	Basic	x_1	x_2	S_1	S_2	S_3	Solution	
	$\mathbf{c}$	-3	-2	0	0	0	0	ratio
x_1 enters	S_1	4	3	1	0	0	12	3
S_2 leaves	S_2	4	1	0	1	0	8	2
	S_3	4	-1	0	0	1	8	2
	$\bar{\mathbf{c}}$	0	$-\frac{5}{4}$	0	$\frac{3}{4}$	0	6	
x_2 enters	S_1	0	2	1	-1	0	4	
S_1 leaves	x_1	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0	2	
	S_3	0	-2	0	-1	1	0	
	$\bar{\mathbf{c}}$	0	0	$\frac{5}{8}$	$\frac{1}{8}$	0	$\frac{17}{2}$	
	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	
optimal	x_1	1	0	$-\frac{1}{8}$	$\frac{3}{8}$	0	$\frac{3}{2}$	
	S_3	0	0	1	-2	1	4	

Note The entering variable x_2 has a negative coefficient corresponding to S_3, hence S_3 cannot be the leaving variable. Degeneracy disappears in the final optimal solution.

(B) Alternative Optima

When the objective function is **parallel** to a binding constraint, the objective function will assume the same optimal value at more than one solution point. For this reason they are called **alternative optima**.

Example 4.3.

$$
\begin{array}{rrcrcl}
\text{Minimize} & -2x_1 & - & 4x_2 & & \\
\text{Subject to} & x_1 & + & 2x_2 & \le & 5 \\
 & x_1 & + & x_2 & \le & 4 \\
 & x_1, & x_2 & & \ge & 0
\end{array}
$$

	Basic	x_1	x_2	S_1	S_2	Solution	
	$\mathbf{c}$	-2	-4	0	0	0	
x_2 enters	S_1	1	2	1	0	5	
S_1 leaves	S_2	1	1	0	1	4	
	$\bar{\mathbf{c}}$	0	0	2	0	10	
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$	optimum $(x_1, x_2) = (0, \frac{5}{2})$
S_2 leaves	S_2	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	(pt. P)
	$\bar{\mathbf{c}}$	0	0	2	0	10	
	x_2	0	1	1	-1	1	optimum $(x_1, x_2) = (3, 1)$
	x_1	1	0	-1	2	3	(pt. Q)

When the reduced cost of a nonbasic variable (here x_1) is zero, it indicates that x_1 can be an entering basic variable *without changing the cost value* , but causing a change in the values of the variables. The family of alternative optimal solutions (basic and nonbasic) is given by:
$(x_1, x_2) = \lambda(0, \frac{5}{2}) + (1 - \lambda)(3, 1)$, where $0 \leq \lambda \leq 1$.

Remark If an LP problem has k ($k \geq 2$) optimal basic feasible solutions: $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_k}$, then LP problem has infinitely many optimal solutions and the general form of an optimal solution is $\sum\limits_{i=1}^{k} \lambda_i \mathbf{x_i}$ where $\sum\limits_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, 2, \cdots, k$.

(C) Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraint and we have an **unbounded solution space**. It is not necessarily, however, that an unbounded solution space yields an unbounded value for the objective function. Unbounded objective value in a model indicates the model is poorly constructed - an infinite cost or profit!!

General rule of detecting Unboundedness

If at any iteration, the constraint coefficients $\mathbf{A}_B^{-1}\mathbf{A}_j$ of a *nonbasic* variable x_j are all *nonpositive*, the *solution space is unbounded* in that direction. If, in addition, the reduced cost $\bar{c}_j$ of that nonbasic variable is *negative* (respectively *positive*) in the minimization (respectively maximization) problem, then *the objective value is also unbounded.*

Example 4.4 (Unbounded Objective Value)

$$\begin{array}{rl} \text{Minimize} & -2x_1 - x_2 \\ \text{Subject to} & x_1 - x_2 \le 10 \\ & 2x_1 \le 40 \\ & x_1, x_2 \ge 0 \end{array}$$

Basic	x_1	x_2	S_1	S_2	Solution
$\mathbf{c}$	-2	-1	0	0	0
S_1	1	-1	1	0	10
S_2	2	0	0	1	40

Note that x_2 is a candidate for entering the solution. All the constraint coefficients in x_2-column are zero or negative implying that x_2 can be increased indefinitely without violating any of the constraints. Therefore, the solution space is unbounded in the x_2-direction and the LP has no bounded optimal solution because x_2 is a candidate of being entering variable.

Example 4.5. (Unbounded Solution Space but Finite Optimal Objective Value)

$$
\begin{array}{rrcrcl}
\text{Minimize} & -6x_1 & + & 2x_2 & & \\
\text{Subject to} & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & & & \leq & 4 \\
 & & x_1, & x_2 & \geq & 0
\end{array}
$$

	Basic	x_1	x_2	S_1	S_2	Solution
	$\mathbf{c}$	-6	2	0	0	0
$x_1 \rightarrow$	S_1	2	-1	1	0	2
$-S_1$	S_2	1	0	0	1	4
	$\bar{\mathbf{c}}$	0	-1	3	0	6
$x_2 \rightarrow$	x_1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	1
$-S_2$	S_2	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	3
	$\bar{\mathbf{c}}$	0	0	2	2	12
	x_1	1	0	0	1	4
	x_2	0	1	-1	2	6

(D) Infeasible Solution

If the constraints cannot be satisfied simultaneously, the model is said to have no feasible solution. This situation can never occur if all the constraints are of the type "$\leq$" (assuming $\mathbf{b} \geq 0$), since the slack variables always provide a feasible solution. When we have constraints of other types, we introduce artificial variables, which, by their very design, do not provide a feasible solution to the original model if some $y_i \neq 0$ in the optimal solution. From the practical point of view, an infeasible solution space shows that the model is not formulated correctly.

Example 4.6 Show that the following LP problem has no feasible solution.

$$\begin{aligned} \text{Minimize } & 3x_1 \\ \text{Subject to } & 2x_1 + \ \ x_2 \geq 6 \\ & 3x_1 + 2x_2 = 4 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

Solution. We use the Big-M method. Adding artificial variables, we obtain:

$$\begin{array}{rl} \text{Minimize} & 3x_1 + My_1 + My_2 \\ \text{Subject to} & 2x_1 + x_2 - s_1 + y_1 = 6 \\ & 3x_1 + 2x_2 + y_2 = 4 \\ & x_1, x_2, s_1, y_1, y_2 \geq 0 \end{array}$$

	Basic	x_1	x_2	s_1	y_1	y_2	R. H. S.
	$\mathbf{c}$	3	0	0	M	M	0
(0)	$\bar{\mathbf{c}}$	$3-5M$	$-3M$	M	0	0	$-10M$
x_1 enters	y_1	2	1	-1	1	0	6
y_2 leaves	y_2	3	2	0	0	1	4
	$\bar{\mathbf{c}}$	0	$\frac{(M-6)}{3}$	M	0	$\frac{(5M-3)}{3}$	$\frac{10}{3}M+4$
	y_1	0	$-\frac{1}{3}$	-1	1	$-\frac{2}{3}$	$\frac{10}{3}$
optimum	x_1	1	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$

This is an optimal tableau. However, the artificial variable $y_1 = \frac{10}{3}$, which is positive, and hence the original problem has no feasible solution.

Exercise Use the phase 1 of the 2-phase method to show that the LP problem has no feasible solution.

Summary of special cases

Consider minimization problem.
Simplex tableau at an iteration:

Basic	$\ldots \quad x_j \quad \ldots$	R. H. S.
$\bar{\mathbf{c}}$	$\ldots \quad \bar{c}_j \quad \ldots$	z
$\mathbf{x}_B$	$\ldots \quad \mathbf{u} \quad \ldots$	$\mathbf{x}_B$

<u>Observation</u>	<u>Conclusion</u>
Some $x_{B(k)} = 0$	degenerate solution
Some nonbasic $\bar{c}_j = 0$	alternative optima
$\mathbf{u} \leq \mathbf{0}$ and $\bar{c}_j < 0$	unbounded problem
Some $y_i > 0$ at optimum	no feasible solution.

Chapter 4

Duality Theory

Starting with a linear programming problem, called the primal LP, we introduce another linear programming problem, called the dual problem. Duality theory deals with the relation between these two LP problems. It is also a powerful theoretical tool that has numerous applications, and leads to another algorithm for linear programming (the dual simplex method).

Motivation

Generally speaking, if a problem (P) searches in a direction, then its dual is a problem (D) which searches in the opposite direction. Usually they meet at a point.

For example, (P) is to search for the infimum of a set $\mathbf{S}_p \subset \mathbf{R}$. A dual (D) of (P) is to search for the supremum of the set

$$\mathbf{S}_d = \{y \in \mathbf{R} \mid y \leq x, \forall x \in \mathbf{S}_p\}.$$

Note that $\inf \mathbf{S}_p = \sup \mathbf{S}_d$, i.e. the solutions of the two problems meet.

Some applications:

- Instead of solving (P), we may solve (D) which may be easier.
- For any $x \in \mathbf{S}_p$ and $y \in \mathbf{S}_d$, $x - y$ provides an upper bound on the error $|x - \inf \mathbf{S}_p|$.

Now, let us consider the standard form LP problem

$$\begin{array}{rrcl} \text{minimize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}, \end{array}$$

which we call the primal problem. Let $\mathbf{x}^*$ be an optimal solution, assumed to exist. We introduce a relaxed problem

$$\begin{array}{rl} g(\mathbf{p}) = \text{minimize} & \mathbf{c}^T\mathbf{x} + \mathbf{p}^T(\mathbf{b} - \mathbf{A}\mathbf{x}) \\ \text{subject to} & \mathbf{x} \geq \mathbf{0}. \end{array}$$

in which the constraint $\mathbf{A}\mathbf{x} = \mathbf{b}$ is replaced by a penalty $\mathbf{p}^T(\mathbf{b} - \mathbf{A}\mathbf{x})$, where $\mathbf{p}$ is a vector of the same dimension as $\mathbf{b}$.

Let $g(\mathbf{p})$ be the optimal cost for the relaxed problem, as a function of $\mathbf{p}$. Thus,

$$g(\mathbf{p}) \leq \mathbf{c}^T\mathbf{x}^* + \mathbf{p}^T(\mathbf{b} - \mathbf{A}\mathbf{x}^*) = \mathbf{c}^T\mathbf{x}^*.$$

This implies that each $\mathbf{p}$ leads to a lower bound $g(\mathbf{p})$ for the optimal cost $\mathbf{c}\mathbf{x}^*$.

The problem

$$\begin{array}{ll} \text{maximize} & g(\mathbf{p}) \\ \text{subject to} & \text{No constraints} \end{array}$$

which searches for the greatest lower bound, is known as the dual problem.

Note:

1. $g(\mathbf{p}) = \min_{\mathbf{x}\geq\mathbf{0}} \left[\mathbf{cx} + \mathbf{p}^T(\mathbf{b} - \mathbf{Ax})\right]$
   $= \mathbf{p}^T\mathbf{b} + \min_{\mathbf{x}\geq\mathbf{0}} \left[(\mathbf{c}^T - \mathbf{p}^T\mathbf{A})\mathbf{x}\right].$
2. $\min_{\mathbf{x}\geq\mathbf{0}}(\mathbf{c}^T - \mathbf{p}^T\mathbf{A})\mathbf{x} = \begin{cases} 0, & \text{if } \mathbf{c}^T - \mathbf{p}^T\mathbf{A} \geq \mathbf{0}, \\ -\infty, & \text{otherwise.} \end{cases}$

Thus, the dual problem is the same as the linear programming problem

$$\begin{array}{ll} \text{maximize} & \mathbf{p}^T\mathbf{b} \\ \text{subject to} & \mathbf{p}^T\mathbf{A} \leq \mathbf{c}^T. \end{array}$$

4.1 The dual problem.

As motivated by the observation in the previous section, we define the dual problem of a (primal) LP problem as follows.

Definition 1.1: Given a (primal) LP problem

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

the associated **dual LP problem** is

$$\begin{aligned} \text{maximize } & \mathbf{p}^T\mathbf{b} \\ \text{subject to } & \mathbf{p}^T\mathbf{A} \leq \mathbf{c}^T. \end{aligned}$$

We only define the dual problem for the standard LP problem. LP problems may appear in various forms. We will derive their dual problems in the following steps: (1) Convert the original (primal) LP problem to a standard LP problem; (2) Formulate the dual problem of the standard LP problem by using Definition 1.1; (3) Simplify the dual problem, if necessary.

Example 1.1

Consider the primal problem:

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \text{ free} \end{aligned}$$

Introducing surplus variables and replacing $\mathbf{x}$ by sign-constrained variables in the original primal problem yield the following equivalent LP:

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x}^+ - \mathbf{c}^T\mathbf{x}^- \\ \text{subject to } & \mathbf{A}\mathbf{x}^+ - \mathbf{A}\mathbf{x}^- - s = \mathbf{b}, \\ & \mathbf{x}^+ \geq \mathbf{0}, \mathbf{x}^- \geq \mathbf{0}, s \geq 0. \end{aligned}$$

By Definition 1.1, the dual problem of this standard LP is

$$\begin{aligned} \text{maximize } & \mathbf{p}^T\mathbf{b} \\ \text{subject to } & \mathbf{p}^T\mathbf{A} \leq \mathbf{c}^T \\ & -\mathbf{p}^T\mathbf{A} \leq -\mathbf{c}^T \\ & -\mathbf{p}^T\mathbf{I} \leq 0. \end{aligned}$$

Note that $\mathbf{p}^T\mathbf{A} = \mathbf{c}^T$ is equivalent to $\mathbf{p}^T\mathbf{A} \leq \mathbf{c}^T$ and $-\mathbf{p}^T\mathbf{A} \leq -\mathbf{c}^T$. Thus, the dual obtained here is equivalent to

$$\begin{aligned} \text{maximize } & \mathbf{p}^T\mathbf{b} \\ \text{subject to } & \mathbf{p} \geq \mathbf{0} \\ & \mathbf{p}^T\mathbf{A} = \mathbf{c}^T. \end{aligned}$$

The above shows the following pair of primal and dual LPs:

$$\begin{array}{rlcrl} \multicolumn{2}{c}{\textit{Primal}} & & \multicolumn{2}{c}{\textit{Dual}} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & & \text{maximize} & \mathbf{p}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b}, & & \text{subject to} & \mathbf{p} \geq \mathbf{0} \\ & \mathbf{x} \text{ free} & & & \mathbf{p}^T\mathbf{A} = \mathbf{c}^T. \end{array}$$

In general, we can show the pair of primal and dual problems are related as follows. Let $\mathbf{A}$ be a matrix with rows $\mathbf{a}_i^T$ and columns $\mathbf{A}_j$.

$$\begin{array}{rlllrlll} \min & \mathbf{c}^T\mathbf{x} & & & \max & \mathbf{p}^T\mathbf{b} & & \\ \text{s.t.} & \mathbf{a}_i^T\mathbf{x} \geq b_i, & i \in M_+, & & \text{s.t.} & p_i \geq 0, & & i \in M_+, \\ & \mathbf{a}_i^T\mathbf{x} \leq b_i, & i \in M_-, & & & p_i \leq 0, & & i \in M_-, \\ & \mathbf{a}_i^T\mathbf{x} = b_i, & i \in M_0, & & & p_i \text{ free}, & & i \in M_0, \\ & x_j \geq 0, & j \in N_+, & & & \mathbf{p}^T\mathbf{A}_j \leq c_j, & & j \in N_+, \\ & x_j \leq 0, & j \in N_-, & & & \mathbf{p}^T\mathbf{A}_j \geq c_j, & & j \in N_-, \\ & x_j \text{ free}, & j \in N_0, & & & \mathbf{p}^T\mathbf{A}_j = c_j, & & j \in N_0, \end{array}$$

Notes:

1. For each functional constraint $\mathbf{a}_i^T\mathbf{x}(\geq, \leq, =)b_i$ in the primal, we introduce a variable $p_i(\geq\ 0, \leq 0,$ free) respectively in the dual problem.
2. For each variable $x_j(\geq 0, \leq 0,$ free) in the primal problem, there is a corresponding constraint $(\leq, \geq, =)c_j$ respectively in the dual problem.

In summary:

	minimize	maximize	
constraints	$\geq$ $\leq$ $=$	≥ 0 ≤ 0 free	**variables**
variables	≥ 0 ≤ 0 free	$\leq$ $\geq$ $=$	**constraints**

Indeed, which side in the table is regarded as primal and which as dual does not matter, because we can show (exercise)

Theorem (*The dual of the dual is the primal.*) If we transform the dual into an equivalent minimization problem, and then form its dual, we obtain a problem equivalent to the original problem.

Proof: We will prove the theorem for LP in the standard form based on Definition 1.1. Consider the primal problem

$$\begin{aligned} \min\ & \mathbf{c}^T\mathbf{x} \\ s.t.\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The dual problem is

$$\begin{aligned} \max\ & \mathbf{b}^T\mathbf{p} \\ s.t.\ & \mathbf{A}^T\mathbf{p} \leq \mathbf{c} \end{aligned}$$

Now we write the dual problem in an equivalent standard form

$$\begin{aligned} \min\ & -\mathbf{b}^T\mathbf{p}^+ + \mathbf{b}^T\mathbf{p}^- \\ s.t.\ & \mathbf{A}^T\mathbf{p}^+ - \mathbf{A}^T\mathbf{p}^- + \mathbf{s} = \mathbf{c} \\ & \mathbf{p}^+, \mathbf{p}^-, \mathbf{s} \geq \mathbf{0} \end{aligned}$$

By Definition 1.1, the dual of the above problem is

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{z} \\ s.t.\ & \mathbf{A}\mathbf{z} \le -\mathbf{b} \\ & -\mathbf{A}\mathbf{z} \le \mathbf{b} \\ & \mathbf{I}\mathbf{z} \le \mathbf{0} \end{aligned}$$

Simplify the above to an equivalent problem

$$\begin{aligned} \min\ & -\mathbf{c}^T\mathbf{z} \\ s.t.\ & -\mathbf{A}\mathbf{z} = \mathbf{b} \\ & \mathbf{z} \le \mathbf{0} \end{aligned}$$

Now letting $\mathbf{x} = -\mathbf{z}$, we obtain the original primal problem. QED

Example 1.2 Consider the primal problem:

$$\begin{array}{rrcrcrcl} \text{minimize} & x_1 & + & 2x_2 & + & 3x_3 & & \\ \text{subject to} & -x_1 & + & 3x_2 & & & = & 5 \\ & 2x_1 & - & x_2 & + & 3x_3 & \ge & 6 \\ & & & & & x_3 & \le & 4 \\ & & & x_1 \ge 0 & & & & \\ & & & x_2 \le 0 & & & & \\ & & & x_3 \text{ free} & & & & \end{array}$$

(a) Write down the dual problem.

(b) Verify that the primal problem and dual of dual obtained are equivalent.

Solution: (a) The dual of the original problem is

$$\begin{array}{rrcrcrl}
\text{maximize} & 5y_1 & + & 6y_2 & + & 4y_3 & \\
\text{subject to} & -y_1 & + & 2y_2 & & & \leq 1 \\
& 3y_1 & - & y_2 & & & \geq 2 \\
& & & 3y_2 & + & y_3 & = 3 \\
& & & \multicolumn{3}{l}{y_1 \text{ free}} & \\
& & & \multicolumn{3}{l}{y_2 \geq 0} & \\
& & & \multicolumn{3}{l}{y_3 \leq 0} &
\end{array}$$

(b) Use the general primal-dual relationship to derive the dual of the LP in (a), resulting in exactly the original LP problem.

4.2 The duality theorem.

Theorem (Weak duality theorem)
In a primal-dual pair, the objective value of the maximization problem is smaller than or equal to the objective value of the minimization problem.
That is, for a minimization (respectively maximization) primal LP, if $\mathbf{x}$ is a feasible solution to the primal problem and $\mathbf{p}$ is a feasible solution to the dual problem, then

$$\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x} \text{ (respectively } \mathbf{p}^T\mathbf{b} \geq \mathbf{c}^T\mathbf{x}).$$

Proof. We prove the result for primal and dual LP problems in standard form.

Suppose that $\mathbf{x}$ and $\mathbf{p}$ are primal and dual feasible solutions, respectively. Then, by Definition, they satisfy

$$\mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p}^T\mathbf{A} \leq \mathbf{c}^T.$$

Thus, we have

$$\mathbf{p}^T\mathbf{b} = \mathbf{p}^T(\mathbf{A}\mathbf{x}) = (\mathbf{p}^T\mathbf{A})\mathbf{x} \leq \mathbf{c}^T\mathbf{x}.$$

QED

Example 2.1 Consider the following linear programming problem.

Primal **Dual**

$$\begin{array}{rrcrcl} \text{Min} & -3x_1 & - & 2x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \le & 6 \\ & 2x_1 & + & x_2 & \le & 8 \\ & -x_1 & + & x_2 & \le & 1 \\ & & & x_2 & \le & 2 \\ & & x_1, x_2 \ge 0 & & & \end{array}$$

Note that $(x_1, x_2) = (1, 1)$ is a primal feasible solution, with objective value -5, whereas $(p_1, p_2, p_3, p_4) = (-1, -1, 0, 0)$ is a dual feasible solution with objective value -14.

This verifies the weak duality theorem, i.e. the objective value of the maximization problem $\le$ the objective value of the minimization problem in a primal-dual pair.

The above pair of primal and dual objective values can be used to provide a range for the optimal value of the primal (and hence the dual) problem, i.e.

$$-14 \leq \text{ the optimal objective value } \leq -5.$$

In fact, the optimal value is $-12\frac{2}{3}$.

Corollary 1
Unboundedness in one problem implies infeasibility in the other problem.
If the optimal value in the primal (respectively dual) problem is unbounded, then the dual (respectively) problem must be infeasible.

Corollary 2
Let $\bar{\mathbf{x}}$ and $\bar{\mathbf{p}}$ be primal and dual feasible solutions respectively, and suppose that $\bar{\mathbf{p}}^T\mathbf{b} = \mathbf{c}^T\bar{\mathbf{x}}$. Then, $\bar{\mathbf{x}}$ and $\bar{\mathbf{p}}$ are optimal primal and dual solutions respectively.

Theorem (Strong Duality)
If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.

Proof. Consider the standard form minimization primal problem and its dual problem:

$$\begin{array}{rlcrl} & \textit{Primal} & & & \textit{Dual} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & & \text{maximize} & \mathbf{p}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{subject to} & \mathbf{p}^T\mathbf{A} \le \mathbf{c}^T \\ & \mathbf{x} \ge \mathbf{0}, & & & \mathbf{p} \text{ free,} \end{array}$$

Let $\mathbf{x}$ be a primal optimal solution obtained from simplex method, with associated optimal basis $\mathbf{B}$. Then $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$ is the corresponding vector of basic variables and $\mathbf{c}_B$ is the vector of the costs of the basic variables.

Note that at optimal:

$$\mathbf{c}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A} \geq \mathbf{0}.$$

Now, define $\mathbf{p}^T = \mathbf{c}_B^T \mathbf{A}_B^{-1}$. (*Aim: Show:* $\mathbf{p}$ *is dual optimal.*)

Then $\mathbf{p}^T \mathbf{A} \leq \mathbf{c}^T$, showing that $\mathbf{p}$ is dual feasible.

Moreover, $\mathbf{p}^T \mathbf{b} = \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}^T \mathbf{x}$.

Thus, by Corollary 2, $\mathbf{p}$ is dual optimal, and the optimal dual cost is equal to the optimal primal cost. QED

Remark From the proof, we note that, for a standard form LP problem, if $\mathbf{x}$ is a primal optimal solution with associated basis $\mathbf{B}$ and $\mathbf{c}_B$ is the vector of the costs of the basic variables, then the dual optimal solution is given by

$$\mathbf{p}^T = \mathbf{c}_B^T \mathbf{A}_B^{-1}.$$

Example 2.1 Consider the LP problem (cf: Example 2.1 in Chapter 3)

$$
\begin{array}{rrcrcrcrcrcrcl}
\text{Minimize} & -3x_1 & - & 2x_2 & + & 0s_1 & + & 0s_2 & + & 0s_3 & + & 0s_4 & & \\
\text{Subject to} & x_1 & + & 2x_2 & + & s_1 & & & & & & & = & 6 \\
 & 2x_1 & + & x_2 & & & + & s_2 & & & & & = & 8 \\
 & -x_1 & + & x_2 & & & & & + & s_3 & & & = & 1 \\
 & & & x_2 & & & & & & & + & s_4 & = & 2 \\
 & & & & & \multicolumn{7}{c}{x_1, x_2, s_1, s_2, s_3, s_4 \geq 0} & &
\end{array}
$$

with optimal solution

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (10/3, 4/3, 0, 0, 3, 2/3),$$

where
$\mathbf{x}_B = (x_2, x_1, s_3, s_4)$.

Thus the dual optimal solution is $\mathbf{p}^T = \mathbf{c}_B^T \mathbf{A}_B^{-1}$

$$= \begin{bmatrix} -2 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & -4/3 & 0 & 0 \end{bmatrix}.$$

Alternatively, one can obtain an optimal dual solution $\mathbf{p}$ from the optimal (primal) simplex tableau readily if the starting basis matrix $\mathbf{A}_{B_0} = \mathbf{I}$.

The vector of reduced costs $\bar{\mathbf{c}}_{B_0}$ in the optimal tableau with the optimal basis $\mathbf{B}$ is

$$\bar{\mathbf{c}}_{B_0}^T = \mathbf{c}_{B_0}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_{B_0} = \mathbf{c}_{B_0}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} = \mathbf{c}_{B_0}^T - \mathbf{p}^T.$$

Thus, an optimal dual solution is $\mathbf{p}^T = \mathbf{c}_{B_0}^T - \bar{\mathbf{c}}_{B_0}^T$.

Example 2.2 (a) Consider the LP problem (cf: Example 2.1 in Chapter 3)

$$\begin{array}{rrcrcrcrcrcrcrcl}
\text{Minimize} & -3x_1 & - & 2x_2 & + & 0s_1 & + & 0s_2 & + & 0s_3 & + & 0s_4 & & \\
\text{Subject to} & x_1 & + & 2x_2 & + & s_1 & & & & & & & = & 6 \\
& 2x_1 & + & x_2 & & & + & s_2 & & & & & = & 8 \\
& -x_1 & + & x_2 & & & & & + & s_3 & & & = & 1 \\
& & & x_2 & & & & & & & + & s_4 & = & 2 \\
\end{array}$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

with optimal tableau as follows:

	Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
	$\bar{\mathbf{c}}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	$12\ \frac{2}{3}$
	x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{4}{3}$
Optimum	x_1	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{10}{3}$
	s_3	0	0	-1	1	1	0	3
	s_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{2}{3}$

With $\mathbf{x}_{B_0} = (s_1, s_2, s_3, s_4)$, the starting $\mathbf{A}_{B_0} = \mathbf{I}$, and $\mathbf{c}_{B_0}^T = (0, 0, 0, 0)$.

From the optimal tableau, $\bar{\mathbf{c}}_{B_0}^T = (\frac{1}{3}, \frac{4}{3}, 0, 0)$.

Thus, the optimal dual solution $\mathbf{p}^T = \mathbf{c}_{B_0}^T - \bar{\mathbf{c}}_{B_0}^T = (-\frac{1}{3}, -\frac{4}{3}, 0, 0)$.

Example 2.2 (b) From Example 3.5 (Chapter III), the modified LP with its artificial variables is

$$\begin{array}{rrcrcrcrcrcrcr} \text{Minimize} & 4x_1 & + & x_2 & & & & & + & My_1 & + & My_2 & & \\ \text{Subject to} & 3x_1 & + & x_2 & & & & & + & y_1 & & & = & 3 \\ & 4x_1 & + & 3x_2 & - & s_1 & & & & & + & y_2 & = & 6 \\ & x_1 & + & 2x_2 & & & + & s_2 & & & & & = & 4 \end{array}$$

$$x_1, x_2, s_1, s_2, y_1, y_2 \geq 0$$

The $\bar{\mathbf{c}}$-row in the optimal tableau is:

Basic	x_1	x_2	s_1	y_1	y_2	s_2	Soln
$\bar{\mathbf{c}}$	0	0	0	$-\frac{7}{5}+M$	M	$\frac{1}{5}$	$-\frac{17}{5}$

Starting basic variables: y_1,y_2 and s_2, with corresponding cost coefficient: M, M, and 0 respectively. Thus,

$$\mathbf{p}^T = (M, M, 0) - (-\frac{7}{5} + M, M, \frac{1}{5}) = (\frac{7}{5}, 0, -\frac{1}{5}).$$

The complementary slackness conditions in the next theorem provides a useful relation between optimal primal and dual solutions. Given an optimal solution to one problem, we can use these conditions to find the optimal solution of the other LP.

Theorem (Complementary Slackness Theorem.) Let $\mathbf{x}$ and $\mathbf{p}$ be feasible solutions to the primal problem and dual problem respectively. The vectors $\mathbf{x}$ and $\mathbf{p}$ are optimal solutions for the two respective problems if and only if

$$\begin{aligned} p_i(\mathbf{a}_i^T\mathbf{x} - b_i) &= 0, \ \ \forall i \\ (c_j - \mathbf{p}^T\mathbf{A}_j)x_j &= 0 \ \ \forall j. \end{aligned}$$

These conditions will be called Complementary slackness optimality conditions.

Proof. Assume that the primal is a minimization problem. From the general primal-dual relationship

$$
\begin{array}{llll}
\min & \mathbf{c}^T\mathbf{x} & & \\
\text{s.t.} & \mathbf{a}_i^T\mathbf{x} \geq b_i, & i \in M_+, \\
& \mathbf{a}_i^T\mathbf{x} \leq b_i, & i \in M_-, \\
& \mathbf{a}_i^T\mathbf{x} = b_i, & i \in M_0, \\
& x_j \geq 0, & j \in N_+, \\
& x_j \leq 0, & j \in N_-, \\
& x_j \text{ free}, & j \in N_0,
\end{array}
\qquad
\begin{array}{llll}
\max & \mathbf{p}^T\mathbf{b} & & \\
\text{s.t.} & p_i \geq 0, & i \in M_+, \\
& p_i \leq 0, & i \in M_-, \\
& p_i \text{ free}, & i \in M_0, \\
& \mathbf{p}^T\mathbf{A}_j \leq c_j, & j \in N_+, \\
& \mathbf{p}^T\mathbf{A}_j \geq c_j, & j \in N_-, \\
& \mathbf{p}^T\mathbf{A}_j = c_j, & j \in N_0,
\end{array}
$$

we observe

$$
\begin{aligned}
u_i &:= p_i(\mathbf{a}^T\mathbf{x} - b_i) \geq 0, \\
v_j &:= (c_j - \mathbf{p}^T\mathbf{A}_j)x_j \geq 0.
\end{aligned}
$$

Furthermore,

$$\sum_i u_i = \sum_i p_i(\mathbf{a}_i^T\mathbf{x} - b_i) = \mathbf{p}^T\mathbf{A}\mathbf{x} - \mathbf{p}^T\mathbf{b}$$

and

$$\sum_j v_j = \sum_j (c_j - \mathbf{p}^T\mathbf{A}_j)x_j = \mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{A}\mathbf{x}.$$

Adding both equalities yields the required inequality:

$$\sum_i u_i + \sum_j v_j = \mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b}.$$

By the Strong Duality Theorem, if $\mathbf{x}$ and $\mathbf{p}$ are optimal solutions for the two respective problems, then $\mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}$. Hence, $\Sigma_i u_i + \Sigma_j v_j = 0$ which implies that $u_i = 0$ and $v_j = 0$ for all i and j.

Conversely, if $u_i = 0$ and $v_j = 0$ for all i and j, then $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = 0$, i.e. $\mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}$. By Corollary 2, both $\mathbf{x}$ and $\mathbf{p}$ are optimal. QED.

Example 2.3
Consider a problem in standard and its dual:

$$\begin{array}{rrcrcrcl} \min & 13x_1 & + & 10x_2 & + & 6x_3 & & \\ \text{s.t.} & 5x_1 & + & x_2 & + & 3x_3 & = & 8 \\ & 3x_1 & + & x_2 & & & \geq & 3 \\ & & & \multicolumn{3}{c}{x_1, x_2, x_3 \geq 0} & & \end{array} \qquad \begin{array}{rrcrcl} \max & 8p_1 & + & 3p_2 & & \\ \text{s.t.} & 5p_1 & + & 3p_2 & \leq & 13 \\ & p_1 & + & p_2 & \leq & 10 \\ & 3p_1 & & & \leq & 6 \\ & & \multicolumn{4}{c}{p_1 \text{ free}, p_2 \geq 0} \end{array}$$

(a) Verify that $\mathbf{x}^* = (1, 0, 1)^T$ is a solution to the primal problem.

(b) Use Complementary Slackness Theorem to verify that $\mathbf{x}^* = (1, 0, 1)^T$ is an optimal solution to the primal problem, and obtain a dual optimal solution.

Solution
(a) $\mathbf{x}^* = (x_1, x_2, x_3)^T = (1, 0, 1)^T$ is primal feasible. (Exc.)

(b) Suppose $\mathbf{p} = (p_1, p_2)^T$ is a dual feasible solution.

By the Complementary Slackness Theorem, both $\mathbf{x}^* = (x_1, x_2, x_3)^T = (1, 0, 1)^T$ and $\mathbf{p} = (p_1, p_2)^T$ are primal and dual optimal solutions if and only if the Complementary Slackness Optimality conditions are satisfied.

We shall find (p_1, p_2) that satisfies these conditions.

These conditions are

$$\begin{aligned} p_1(5x_1 + x_2 + 3x_3 - 8) &= 0 \\ p_1(3x + x_2 - 3) &= 0 \\ x_1(13 - (5p_1 + 3p_2)) &= 0 \\ x_2(10 - (p_1 + p_2)) &= 0 \\ x_3(6 - (3p_1)) &= 0 \end{aligned}$$

For $\mathbf{x}^* = (1, 0, 1)^T$, $5x_1 + x_2 + 3x_3 - 8 = 0$, $3x + x_2 - 3 = 0$, and $x_2 = 0$. Thus, the first, second and fourth equations pose no restriction on (p_1, p_2). From the remaining equations, since $x_1 \neq 0$ and $x_3 \neq 0$, we have

$$\begin{aligned} 13 - 5p_1 - 3p_2 &= 0 \\ 6 - 3p_1 &= 0. \end{aligned}$$

Solving yields $(p_1, p_2) = (2, 1)$.

It can be verified that $(p_1, p_2) = (2, 1)$ satisfies all dual constraints, thus is dual feasible.

Since $\mathbf{x}^* = (1, 0, 1)^T$ and $\mathbf{p} = (2, 1)^T$ satisfy the complementary slackness optimality conditions and are primal and dual feasible, by the Complementary Slackness Theorem, they are optimal solutions to the two respective problems.

Example 2.4 Consider the following LP:

$$\begin{array}{lrrrrrrl}
\text{Min} & 8x_1 & + \ 6x_2 & - \ 10x_3 & + \ 20x_4 & - \ 2x_5 & & \\
\text{S.t.} & 2x_1 & + \ \ x_2 & - \ \ \ x_3 & + \ \ 2x_4 & + \ \ x_5 & = & 25 \\
& 2x_1 & & + \ \ 2x_3 & - \ \ \ x_4 & + \ 3x_5 & = & 20 \\
& & & \multicolumn{4}{r}{x_1, x_2, x_3, x_4, x_5 \geq 0} &
\end{array}$$

Is $(x_1, x_2, x_3, x_4, x_5) = (10, 5, 0, 0, 0)$ an optimal solution to the above LP?

Solution Firstly, check

$$(x_1, x_2, x_3, x_4, x_5) = (10, 5, 0, 0, 0)$$

is primal feasible. (Exc.)

Note that $u_1 = u_2 = 0$ because of equality constraints.

Associating the dual variables p_1 and p_2 to the two constraints, the dual problem is:

$$
\begin{array}{lrcrcr}
\text{Maximize} & 25p_1 & + & 20p_2 & & \\
\text{Subject to} & 2p_1 & + & 2p_2 & \leq & 8 \\
 & p_1 & & & \leq & 6 \\
 & -p_1 & + & 2p_2 & \leq & -10 \\
 & 2p_1 & - & p_2 & \leq & 20 \\
 & p_1 & + & 3p_2 & \leq & -2 \\
 & \multicolumn{3}{c}{p_1,\ p_2} & \multicolumn{2}{l}{\text{unrestricted.}}
\end{array}
$$

Suppose the feasible solution $\mathbf{x}$ is optimal and (p_1, p_2) is a dual optimal solution. By the Complementary Slackness Optimality Conditions, we must have

$$
\begin{aligned}
v_1 &= x_1(8 - (2p_1 + 2p_2)) &&= 0 \\
v_2 &= x_2(6 - p_1) &&= 0 \\
v_3 &= x_3(-10 - (-p_1 + 2p_2)) &&= 0 \\
v_4 &= x_4(20 - (2p_1 - p_2)) &&= 0 \\
v_5 &= x_5(-2 - (p_1 + 3p_2)) &&= 0.
\end{aligned}
$$

Since $x_1 = 10 > 0$ and $x_2 = 5 > 0$, we have $8 - (2p_1 + 2p_2) = 0$ and $6 - p_1 = 0$, i.e. $(p_1, p_2) = (6, -2)$.

Now we check for dual feasibility: it remains to check the last three dual constraints at $(p_1, p_2) = (6, -2)$:

$$\begin{aligned} -p_1 + 2p_2 &= -10 \\ 2p_1 - p_2 &= 14 \le 20 \\ p_1 + 3p_2 &= 0 > -2. \end{aligned}$$

Thus the last dual constraint is not satisfied, and we conclude that $(p_1, p_2) = (6, -2)$ is not dual feasible.

Therefore, there is no dual feasible solution satisfying Complementary Slackness Optimality Conditions together. Hence,

$$(x_1, x_2, x_3, x_4, x_5) = (10, 5, 0, 0, 0)$$

is not an optimal solution.

4.3 Economic interpretation of optimal dual variables.

At the **optimal** solution of both the primal and dual, there is an economic interpretation of the dual variables p_i as marginal costs for a minimization primal problem, or as marginal profits for a maximization primal problem.

Consider the standard form problem and its dual problem:

$$
\begin{array}{rlcrl}
\multicolumn{2}{c}{\textit{Primal}} & & \multicolumn{2}{c}{\textit{Dual}} \\
\text{minimize} & \mathbf{c}^T\mathbf{x} & \qquad & \text{maximize} & \mathbf{p}^T\mathbf{b} \\
\text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{subject to} & \mathbf{p}^T\mathbf{A} \le \mathbf{c}^T \\
& \mathbf{x} \ge \mathbf{0}, & & & \mathbf{p} \text{ free,}
\end{array}
$$

where A is $m \times n$ with linearly independent rows.

Let $\mathbf{x}^*$ be a nondegenerate primal optimal solution, with associated optimal basis $\mathbf{B}$ and the corresponding dual optimal solution $\mathbf{p}^*$ is given by $\mathbf{p}^{*T} = \mathbf{c}_B^T\mathbf{A}_B^{-1}$.

Let $\triangle = (\triangle_1, \cdots, \triangle_i, \cdots, \triangle_n)^T$ where each $\triangle_i$ is a small change in b_i, for each i, such that $\mathbf{A}_B^{-1}(\mathbf{b}+\triangle) \geq \mathbf{0}$ (feasibility is maintained).

Note that $\bar{\mathbf{c}}^T = \mathbf{c}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1}\mathbf{A} \geq \mathbf{0}$ remain unaffected,and hence optimality conditions are not affected. Thus, $\mathbf{A}_B^{-1}(\mathbf{b}+\triangle)$, with the same basis matrix $\mathbf{B}$, is an optimal solution to the perturbed problem (perturb means small change).

The optimal cost in the perturbed problem is

$$\begin{aligned}
\mathbf{c}_B^T \mathbf{A}_B^{-1}(\mathbf{b}+\triangle) &= \mathbf{p}^{*T}(\mathbf{b}+\triangle) \\
&= \mathbf{p}^{*T}\mathbf{b} + \mathbf{p}^{*T}\triangle \\
&= \mathbf{c}_B^T \mathbf{A}_B^{-1}\mathbf{b} + \mathbf{p}^{*T}\triangle.
\end{aligned}$$

Thus, a small change $\triangle$ in $\mathbf{b}$ results in a change of $\mathbf{p}^{*T}\triangle$ in the optimal cost. In particular, for a fixed i, if $\triangle_i = \delta$, and $\triangle_j = 0$ for $j \neq i$, then the change in the optimal objective value is

$$\mathbf{p}^{*T}\triangle = \delta p_i.$$

Therefore, each component p_i of the optimal dual vector $\mathbf{p}^*$ indicates the contribution of ith requirement b_i towards the objective function. Thus, p_i is interpreted as the *marginal cost* (or *shadow cost*) of the ith requirement b_i.

Remark

For a maximization primal problem, the component p_i of the optimal dual vector $\mathbf{p}$ is interpreted as the *marginal profit* (or *shadow price*) per unit increase of the ith requirement b_i. It is also known as the worth of the i-th resource or requirement b_i.

Dual variables p_i's can be used to **rank** the 'requirements' according to their contribution to the objective value. For example, in a minimization problem, if $p_1 < 0$, then increasing b_1 (sufficiently) will reduce the total cost. Thus, if $p_1 < 0$ and $p_2 < 0$, and we are allowed to increase only one requirement, then the requirement b_i corresponds to the most negative p_i is given a higher priority to increase.

Example 3.1 (An example to illustrate the use of p_i.)

Consider the product-mix problem in which each of three products is processed on three operations. The limits on the available time for the three operations are 430, 460, 420 minutes daily and the profits per unit of the three products are \$3, \$2 and \$5. The times in minutes per unit on the three operations are given as follows:

	Product 1	Product 2	Product 3
Operation 1	1	2	1
Operation 2	3	0	2
Operation 3	1	4	0

The LP model is written as:

$$
\begin{array}{lrcl}
\text{Max} & 3x_1 + 2x_2 + 5x_3 & & \text{(daily profit)} \\
\text{S.t.} & x_1 + 2x_2 + x_3 & \leq 430 & \text{(op. 1)} \\
 & 3x_1 \qquad\qquad + 2x_3 & \leq 460 & \text{(op. 2)} \\
 & x_1 + 4x_2 \qquad\qquad & \leq 420 & \text{(op. 3)} \\
 & x_1, x_2, x_3 \geq 0. & &
\end{array}
$$

Adding slack variables S_1, S_2 and S_3 to the three constraints, the optimal tableau is given as:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Solution
$\bar{\mathbf{c}}$	-4	0	0	-1	-2	0	-1350
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
S_3	2	0	0	-2	1	1	20

(a) Suppose an additional minute for Operation 2 costs \$1.50, is it advisable to increase the limit of available time for Operation 2?

(b) Rank the three operations in order of priority for increase in time allocation, assuming that costs per additional time for all operations are equal.

Solution. The dual prices are found to be $p_1 = 1$, $p_2 = 2$ and $p_3 = 0$. (Verify)

(a) $p_2 = 2$ implies that a unit (i.e. 1 minute) increase in the time Operation 2 causes an increase of \$2 in the objective value. Since the cost of an additional minute for Operation 2 costs \$1.50. There is a net profit of \$0.50 when we increase the time for Operation 2. It is advisable to increase the operation time for Operation 2.

(b) From the dual prices, $p_1 = 1$, $p_2 = 2$ and $p_3 = 0$, if we are to increase the limits on the available time for the three operations, we would give a higher priority to Operation 2 followed by Operation 1. Note that since $p_3 = 0$, increasing the limit on the available time for Operation 3 has no effect on the profit.

4.4 The dual Simplex Method.

The simplex method from the duality perspective

Consider the primal and dual problems

$$\begin{array}{llcll} \min & \mathbf{c}^T\mathbf{x} & \qquad & \max & \mathbf{p}^T\mathbf{b} \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{s. t.} & \mathbf{p}^T\mathbf{A} \le \mathbf{c}^T \\ & \mathbf{x} \ge \mathbf{0} & & & \end{array}$$

and the corresponding simplex tableau:

Basic	$\mathbf{x}$	Solution
$\bar{\mathbf{c}}$	$\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}$	$-\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{x_B}$	$\mathbf{A}_B^{-1}\mathbf{A}$	$\mathbf{A}_B^{-1}\mathbf{b}$

Let $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{A}_B^{-1}\mathbf{b} \\ \mathbf{0}) \end{pmatrix}$ be a primal solution and $\mathbf{p}^T = \mathbf{c}^T\mathbf{A}_B^{-1}$ a dual solution.

Observations:

- Primal and dual objective values at $\mathbf{x}$ and $\mathbf{p}$ are equal:

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b} = \mathbf{p}^T\mathbf{b}.$$

  Therefore, $\mathbf{x}$ and $\mathbf{p}$ are optimal solutions iff they are feasible.

- Primal feasibility:

$$\mathbf{A}\mathbf{x} = \mathbf{A}_B\mathbf{x}_B + \mathbf{A}_N\mathbf{x}_N = \mathbf{A}_B\mathbf{A}_B^{-1}\mathbf{b} = \mathbf{b}.$$

  Thus, $\mathbf{x}$ is feasible iff $\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$.

- Dual feasibility:

$$\mathbf{p}^T\mathbf{A} \leq \mathbf{c}^T$$

  iff

$$\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A} \leq \mathbf{c}^T$$

  iff

$$\bar{\mathbf{c}} \geq \mathbf{0}.$$

  Thus, $\mathbf{p}$ is feasible iff $\bar{\mathbf{c}} \geq \mathbf{0}$.

The simplex method is an algorithm that maintains primal feasibility ($\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$) and works towards dual feasibility ($\bar{\mathbf{c}} \geq \mathbf{0}$, i.e. primal optimality). A method with this property is generally called a *primal* algorithm. An alternative is to start with a dual feasible solution ($\bar{\mathbf{c}} \geq \mathbf{0}$) and work towards primal feasibility ($\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$). This method is called a *dual* algorithm.We shall implement the dual simplex method in terms of the simplex tableau.

An iteration of the dual simplex method.

1. For a minimization (respectively maximization) problem, a typical iteration starts with the tableau associated with a basis $\mathbf{B}$ and with all reduced costs nonnegative (respectively nonpositive).
2. Examine the components of the vectors $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$.

   If they are all nonnegative, we have an optimal basic feasible solution and the algorithm stops;

   Else, choose some l such that $x_{B(l)} < 0$.

3. Consider the l-th row (the pivot row) of the tableau, with elements

$$v_1, v_2, \cdots, v_n.$$

If $v_i \geq 0$ for all i, then the primal LP is infeasible and the algorithm stops.

Else, for each i such that $v_i < 0$, compute the ratio $|\frac{\bar{c}_i}{v_i}|$ and let j be the index of a column that corresponds to the smallest ratio. The column $\mathbf{A}_{B(l)}$ leaves the basis and the column $\mathbf{A}_j$ enters the basis. (The minimum ratio ensures that the optimality conditions is maintained.)

4. Add to each row of the tableau a multiple of the l-row (the pivot row) so that v_j (the pivot element) becomes 1 and all other entries of the pivot column become 0.

Key points to note

1. The dual simplex method is carried out on the simplex tableau of the primal problem.

2. Unlike the primal simplex method, we do not require $\mathbf{A}_B^{-1}\mathbf{b}$ to be nonnegative. Thus, $\mathbf{x}$ needs not be primal feasible.

Example 4.1 Consider the simplex tableau of a minimization problem.

Basic	x_1	x_2	x_3	x_4	x_5	Solution
$\bar{\mathbf{c}}$	2	6	10	0	0	0
x_4	-2	4	1	1	0	2
x_5	4	-2	-3	0	1	-1

1. The given basic solution $\mathbf{x}$ satisfies the optimality conditions but it not feasible. (WHY?)
2. $x_{B(2)} = x_5 < 0$: Choose the x_5-row as pivot row.
3. $v_2 = -2 < 0$ and $\bar{c}_2 = 6$: ratio $\bar{c}_2/|v_2| = 3$ (smallest), and

   $v_3 = -3 < 0$ and $\bar{c}_3 = 10$: ratio $\bar{c}_2/|v_2| = 10/3$.

   Thus, the entering variable is x_2 and the leaving variable is x_5.

Recompute the tableau:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
$\bar{\mathbf{c}}$	14	0	1	0	3	-3
x_4	6	0	-5	1	2	0
x_2	$-1/2$	1	$3/2$	0	$-1/2$	$1/2$

Note: The cost has increased to 3, and the new basic solution is optimal and feasible. An optimal solution is $\mathbf{x} = (0, 1/2, 0, 0, 0)$ with optimal cost 3.

Combine the two tableaus as follows:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
$\bar{\mathbf{c}}$	2	6	10	0	0	0
x_4	-2	4	1	1	0	2
x_5	4	-2	-3	0	1	-1
$\bar{\mathbf{c}}$	14	0	1	0	3	-3
x_4	6	0	-5	1	2	0
x_2	$-1/2$	1	$3/2$	0	$-1/2$	$1/2$

When should we use the dual simplex method?

1. A basic solution of the primal problem satisfying optimality conditions is readily available. (Equivalently a basic feasible solution of the dual problem is readily available.)

2. Most importantly, it is used in Sensitivity and Postoptimality analysis. Suppose that we have already an optimal basis for a linear programming problem, and that we wish to solve the same problem for a different choice of vector $\mathbf{b}$. The optimal basis for the original problem may be primal infeasible under the new $\mathbf{b}$. On the other hand , a change in $\mathbf{b}$ does not affect the reduced costs so that optimality conditions are satisfied. Thus, we may apply the dual simplex algorithm starting from the optimal basis for the original problem.

Example 4.2 Solve the LP problem:

$$\begin{array}{rrcrcl} \text{Minimize} & 2x_1 & + & x_2 & & \\ \text{Subject to} & 3x_1 & + & x_2 & \geq & 3 \\ & 4x_1 & + & 3x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & \leq & 3 \\ & & x_1, & x_2 & \geq & 0 \end{array}$$

Solution

Transform into standard form by multiplying each of the equations associated with the surplus variables S_1 and S_2 by -1 so that the RHS will show readily as infeasible basic solution:

$$\begin{array}{rrcrcrcr} \text{Minimize} & 2x_1 & + & x_2 & & & & \\ \text{Subject to} & -3x_1 & - & x_2 & + & S_1 & = & -3 \\ & -4x_1 & - & 3x_2 & + & S_2 & = & -6 \\ & x_1 & + & 2x_2 & + & S_3 & = & 3 \\ & \multicolumn{5}{r}{x_1,\ x_2,\ S_1,\ S_2,\ S_3} & \geq & 0 \end{array}$$

If we choose $\mathbf{B} = \{S_1, S_2, S_3\}$ as the basis in the starting solution, then $\mathbf{c}_B = \mathbf{0}$, and thus $\bar{\mathbf{c}} = \mathbf{c} \geq \mathbf{0}$. This starting solution is dual feasible but primal infeasible. Therefore, we can use the dual simplex method.

	Basic	x_1	x_2	S_1	S_2	S_3	Solution	
	$\bar{\mathbf{c}}$	2	1	0	0	0	0	
	S_1	-3	-1	1	0	0	-3	
S_2 leaves	S_2	-4	-3	0	1	0	-6	
x_2 enters	S_3	1	2	0	0	1	3	
	$\bar{\mathbf{c}}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0	-2	$\leftarrow$ Always
S_1 leaves	S_1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1	remains
x_1 enters	x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2	optimal
	S_3	$-\frac{5}{3}$	0	0	$\frac{2}{3}$	1	-1	
	$\bar{\mathbf{c}}$	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0	$-\frac{12}{5}$	
	x_1	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	
optimum	x_2	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	$\frac{6}{5}$	
	S_3	0	0	-1	1	1	0	feasible

The graph:

The solution starts at point A ($x_1 = 0, x_2 = 0$ and $S_1 = -3, S_2 = -6, S_3 = 3$) with cost 0, which is infeasible with respect to the solution space. The next iteration is secured by moving to point B ($x_1 = 0, x_2 = 2$) with cost 2 which is still infeasible. Finally, we reach point C ($x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$) at which cost $\frac{12}{5}$. This is the first time we encounter a feasible solution, thus signifying the end of the iteration process. Notice that the value of cost associated with A, B, and C are 0, 2, and $\frac{12}{5}$ respectively, which explains why the solution starts at A is better than optimal (smaller than the minimum).

Note If instead we let S_1 be the leaving variable (forcing the *negative* basic variable out of the solution), then the iterations would have proceeded in the order $A \to D \to C$.

Example 4.3 Solve by the dual simplex method:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + 3x_2 \\ \text{Subject to} \quad & 2x_1 + 3x_2 \le 1 \\ & x_1 + x_2 = 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Solution. We replace the equality constraint by two inequalities to obtain:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + 3x_2 \\ \text{Subject to} \quad & 2x_1 + 3x_2 + S_1 = 1 \\ & x_1 + x_2 + S_2 = 2 \\ & -x_1 - x_2 + S_3 = -2 \\ & x_1, x_2, S_1, S_2, S_3 \ge 0 \end{aligned}$$

	Basic	x_1	x_2	S_1	S_2	S_3	Solution
	$\mathbf{c}$	2	3	0	0	0	0
S_3 leaves	S_1	2	3	1	0	0	1
x_1 enters	S_2	1	1	0	1	0	2
	S_3	-1	-1	0	0	1	-2
	$\bar{\mathbf{c}}$	0	1	0	0	2	-4
	S_1	0	1	1	0	2	-3
	S_2	0	0	0	1	1	0
	x_1	1	1	0	0	-1	2

Since $S_1 = -3$, S_1 is the leaving variable. However, and all the values in the S_1-row are nonnegative. Thus, we conclude that the primal LP is infeasible, i.e. there is no primal feasible solution.

Chapter 5

Sensitivity and Postoptimality Analysis.

Sensitivity (or postoptimality) analysis is concerned with the study of possible changes in the available optimal solution as a result of making changes in the original problem.

Why do we study Sensitivity?

- In practice, there is often incomplete knowledge of the problem data. We cannot predict changes of data, but we may wish to predict the effects of certain parameter changes, e.g. to which parameters the profit (or cost) is more (or less) sensitive.
- We may want and be able to change some input parameters. Which parameters are worth change and how much they can be changed, allowing for offset of costs?

How to analyze Sensitivity?

Consider the standard form problem

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array} \quad \text{or} \quad \begin{array}{rl} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

where A is $m \times n$ with linearly independent rows.

We shall study the dependence of the optimal objective value and the optimal solution on the coefficient matrix $\mathbf{A}$, the requirement vector $\mathbf{b}$, and the cost vector $\mathbf{c}$.

In-hand information: Suppose $\mathbf{x}^*$ is a optimal primal solution, with associated optimal basis $\mathbf{B}$. Then $\mathbf{x}_B^* = \mathbf{A}_B^{-1}\mathbf{b} > \mathbf{0}$ and the optimal cost is $\mathbf{c}^T\mathbf{x}^* = \mathbf{c}_B^T\mathbf{x}_B^* = \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$.

Given changes of A, b, c, we look for a new optimal solution.

- First, we check if the current optimal basis $\mathbf{B}$ and/or solution $\mathbf{x}^*$ is still optimal.
- If not, we compute a new optimal solution, starting from $\mathbf{x}^*$ and $\mathbf{B}$.

Conditions we need to check:

$$\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0} \text{ Feasibility}$$

$$\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A} \geq \mathbf{0} \text{ Optimality (minimization)}$$

OR

$$\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0} \text{ Feasibility}$$

$$\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A} \leq \mathbf{0} \text{ Optimality (maximization)}$$

Suppose that some entry of $\mathbf{A}$, $\mathbf{b}$ or $\mathbf{c}$ has been changed, or that a new variable is added , or that a new constraint is added. These two conditions may be affected.

We shall look for ranges of parameter changes under which current basis is still optimal. If the feasibility conditions or optimality conditions are violated, we look for algorithm that finds a new optimal solution without having to solve the new problem from scratch.

5.1 A new variable is added.

Consider the standard form problem:

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

Suppose a new variable x_{n+1}, together with a corresponding $\mathbf{A}_{n+1}$ and cost c_{n+1} is added. This yields the new problem:

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x} + c_{n+1}x_{n+1} \\ \text{subject to } & \mathbf{A}\mathbf{x} + \mathbf{A}_{n+1}x_{n+1} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, x_{n+1} \geq 0. \end{aligned}$$

Question Is $\mathbf{B}$ still optimal?

First, note that $(\mathbf{x}, x_{n+1}) = (\mathbf{x}^*, 0)$ is a basic feasible solution to the new problem with basis $\mathbf{B}$. Thus, we only need to check whether optimality conditions are satisfied. This amounts to checking whether $\bar{c}_{n+1} = c_{n+1} - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_{n+1} \geq 0$.

If $\bar{c}_{n+1} \geq 0$, then $(\mathbf{x}, x_{n+1}) = (\mathbf{x}^*, 0)$ is an optimal solution to the new problem.

If $\bar{c}_{n+1} < 0$, then $(\mathbf{x}, x_{n+1}) = (\mathbf{x}^*, 0)$ is a basic feasible solution but not necessary optimal. We add a column to the simplex tableau, associated with the new variable, and apply the primal simplex algorithm starting from current basis $\mathbf{B}$.

Remark If the primal is a maximization problem, then we check whether

$$\bar{c}_{n+1} = c_{n+1} - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_{n+1} \leq 0$$

for $\mathbf{B}$ to be optimal.

Example 1 Consider the problem

$$\begin{array}{rrrrrrrrrrr} \text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = & 10 \\ & 5x_1 & + & 3x_2 & & & + & x_4 & = & 16 \\ & & & & & & & & & \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

An optimal solution to this problem is given by $\mathbf{x} = (2, 2, 0, 0)^T$ and the corresponding optimal simplex tableau is given by

Basic	x_1	x_2	x_3	x_4	
$\bar{\mathbf{c}}$	0	0	2	7	12
x_1	1	0	−3	2	2
x_2	0	1	5	−3	2

From columns under x_3 and x_4, we have

$$\mathbf{A}_B^{-1} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}.$$

Introduce a new variable x_5 with $\mathbf{A}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $c_5 = -1$, we obtain the new problem:

$$
\begin{array}{lrcrcrcrcrcr}
\text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & & -x_5 & & \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & + & x_5 & = & 10 \\
 & 5x_1 & + & 3x_2 & & & + & x_4 & + & x_5 & = & 16 \\
\end{array}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Check $\bar{c}_5 = c_5 - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_5 \geq 0$?

$$\bar{c}_5 = -1 - [-5 \quad -1] \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -4.$$

Since $\bar{c}_5 < 0$, introducing the new variable to the basis can be beneficial.

Now, $\mathbf{A}_B^{-1} \mathbf{A}_5 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

We augment the tableau as follows with a new column associated x_5, and apply primal simplex algorithm:

Basic	x_1	x_2	x_3	x_4	x_5	
$\bar{\mathbf{c}}$	0	0	2	7	-4	12
x_1	1	0	-3	2	-1	2
x_2	0	1	5	-3	2	2
$\bar{\mathbf{c}}$	0	2	12	1	0	16
x_1	1	0.5	-0.5	0.5	0	3
x_5	0	0.5	2.5	-1.5	1	1

An optimal solution is given by

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T = (3, 0, 0, 0, 1)^T,$$

with optimal cost -16.

5.2 A new constraint is added.

Consider the standard form problem:

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

Suppose a new constraint $\mathbf{a}_{m+1}^T\mathbf{x} \leq b_{m+1}$ is added to the original problem, where b_{m+1} can be any number.

This yields the new problem:

$$\text{minimize } \mathbf{c}^T\mathbf{x} + 0x_{n+1}$$

$$\text{subject to } \begin{array}{rcl} \mathbf{A}\mathbf{x} + \mathbf{0}x_{n+1} & = & \mathbf{b} \\ \mathbf{a}_{m+1}^T\mathbf{x} + \quad x_{n+1} & = & b_{m+1} \\ \multicolumn{3}{c}{\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \geq \mathbf{0}, x_{n+1} \geq 0} \end{array}$$

If the optimal solution $\mathbf{x}^*$ satisfies the new constraint, then the solution remains optimal to the new problem.

If this constraint is violated at $\mathbf{x}^*$, then

$$b_{m+1} - \mathbf{a}_{m+1}^T\mathbf{x}^* < 0.$$

We will derive a new tableau from the original optimal tableau

Basic	$\mathbf{x}_B$	$\mathbf{x}_N$	Soln
$\bar{\mathbf{c}}$	$\mathbf{0}$	$\bar{\mathbf{c}}_N^T$	$-\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{I}$	$\mathbf{A}_B^{-1}\mathbf{A}_N$	$\mathbf{A}_B^{-1}\mathbf{b}$

Write $\mathbf{a}_{m+1}^T = (\mathbf{a}_B^T, \mathbf{a}_N^T)$. Add the new constraint

$$\mathbf{a}_B^T\mathbf{x}_B + \mathbf{a}_N^T\mathbf{x}_N + x_{n+1} = b_{m+1}$$

into the tableau, resulting in

Basic	$\mathbf{x}_B$	$\mathbf{x}_N$	x_{n+1}	Soln
$\bar{\mathbf{c}}$	$\mathbf{0}$	$\bar{\mathbf{c}}_N^T$	0	$-\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{I}$	$\mathbf{A}_B^{-1}\mathbf{A}_N$	$\mathbf{0}$	$\mathbf{A}_B^{-1}\mathbf{b}$
x_{n+1}	$\mathbf{a}_B^T$	$\mathbf{a}_N^T$	1	b_{m+1}

The above tableau is not a Simplex tableau. Perform row operations to change the last row, obtaining the following Simplex tableau

Basic	$\mathbf{x}_B$	$\mathbf{x}_N$	x_{n+1}	Soln
$\bar{\mathbf{c}}$	$\mathbf{0}$	$\bar{\mathbf{c}}_N^T$	0	$-\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{I}$	$\mathbf{A}_B^{-1}\mathbf{A}_N$	$\mathbf{0}$	$\mathbf{A}_B^{-1}\mathbf{b}$
x_{n+1}	$\mathbf{0}$	$\mathbf{a}_N^T - \mathbf{a}_B^T\mathbf{A}_B^{-1}\mathbf{A}_N$	1	$b_{m+1} - \mathbf{a}_B^T\mathbf{A}_B^{-1}\mathbf{b}$

The new reduced cost $\geq \mathbf{0}$, thus the optimality still holds.

The new basic solution $\ngeq \mathbf{0}$, because $x_{n+1} = b_{m+1} - \mathbf{a}_B^T\mathbf{A}_B^{-1}\mathbf{b} = b_{m+1} - \mathbf{a}_{m+1}^T\mathbf{x}^* < 0$, thus it is not feasible.

Hence, we have obtain an 'optimal' but infeasible basic solution to the new problem. Thus, we apply dual simplex method to the new problem.

Example 2 Consider the same LP problem as in Example 1.

$$
\begin{array}{rrcrcrcrl}
\text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = 10 \\
 & 5x_1 & + & 3x_2 & & & + & x_4 & = 16 \\
 & & & & & & & & \\
\end{array}
$$
$$x_1, x_2, x_3, x_4 \geq 0$$

with the optimal simplex tableau given by

Basic	x_1	x_2	x_3	x_4	
$\bar{\mathbf{c}}$	0	0	2	7	12
x_1	1	0	-3	2	2
x_2	0	1	5	-3	2

Consider the additional constraint

$$x_1 + x_2 \geq 5.$$

It is violated by the original optimal solution $\mathbf{x}^* = (2, 2, 0, 0)^T$.

The new problem is:

$$\begin{array}{llllllllll}
\text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & & \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = & 10 \\
& 5x_1 & + & 3x_2 & & & + & x_4 & = & 16 \\
& -x_1 & - & x_2 & & & & x_5 & = & -5 \\
& \multicolumn{9}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0.}
\end{array}$$

The tableau with the additional constraint:

Basic	x_1	x_2	x_3	x_4	x_5	
$\bar{\mathbf{c}}$	0	0	2	7	0	12
x_1	1	0	-3	2	0	2
x_2	0	1	5	-3	0	2
x_5	-1	-1	0	0	1	-5

Performing row operations, we obtain the Simplex tableau:

Basic	x_1	x_2	x_3	x_4	x_5	
$\bar{\mathbf{c}}$	0	0	2	7	0	12
x_1	1	0	-3	2	0	2
x_2	0	1	5	-3	0	2
x_5	0	0	2	-1	1	-1

Performing one iteration of the dual simplex method, we obtain

Basic	x_1	x_2	x_3	x_4	x_5	
$\bar{\mathbf{c}}$	0	0	16	0	7	5
x_1	1	0	1	0	2	0
x_2	0	1	-1	0	-3	5
x_5	0	0	-2	1	-1	1

The optimal solution to the new problem is

$$\mathbf{x} = (0, 5, 0, 1, 0)^T$$

with the objective value -5.

5.3 Changes in the requirement vector b.

Suppose that some component b_i of the requirement vector $\mathbf{b}$ is changed to $b_i + \delta$, i.e. $\mathbf{b}$ is changed to $\mathbf{b} + \delta\mathbf{e}_i$.

Our aim is to determine the range of values of δ under which the current basis remains optimal.

Optimality conditions are unaffected by the change in $\mathbf{b}$ (WHY?). It remains to examine the feasibility condition

$$\mathbf{A}_B^{-1}(\mathbf{b} + \delta\mathbf{e}_i) \geq \mathbf{0}, \quad \text{i.e. } \mathbf{x}_B^* - \delta(\mathbf{A}_B^{-1}\mathbf{e}_i) \geq \mathbf{0}.$$

This provides a range for δ to maintain feasibility (as illustrated in the next example).

However, if δ is not in the range, then feasibility condition is violated, and we apply dual simplex method starting from the basis $\mathbf{B}$.

Example 3 Consider the same LP problem in Example 1, with the optimal solution $\mathbf{x}^* = (2, 2, 0, 0)$ and optimal simplex tableau:

Basic	x_1	x_2	x_3	x_4	
$\bar{\mathbf{c}}$	0	0	2	7	12
x_1	1	0	-3	2	2
x_2	0	1	5	-3	2

(a) Find the range of b_1 so that $\mathbf{B}$ remains as an optimal basis.

(b) How is the cost affected?

Solution.

(a) Suppose b_1 is changed to $b_1 + \delta$. Then, the values of basic variables are changed:

$$\mathbf{x}_B = \mathbf{A}_B^{-1} \begin{bmatrix} 10 + \delta \\ 16 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 10 + \delta \\ 16 \end{bmatrix} = \begin{bmatrix} 2 - 3\delta \\ 2 + 5\delta \end{bmatrix}$$

For the new solution to be feasible, both $2 - 3\delta \geq 0$ and $2 + 5\delta \geq 0$, yielding $-2/5 \leq \delta \leq 2/3$.

Thus, the range for b_1 is $10-2/5 \leq b_1 \leq 10+2/3$, i.e. $9\frac{3}{5} \leq b_1 \leq 10\frac{2}{3}$ for $\mathbf{B}$ to remain as the optimal basis.

The corresponding change in the cost is

$$\delta \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{e}_1 = (-5, -1) \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 10\delta.$$

Note If $\delta > 2/3$, then $x_1 < 0$ and the basic solution becomes infeasible. We can perform the dual simplex method to remove x_1 from the basis and x_3 enters the basis.

5.4 Changes in the cost vector $\mathbf{c}$.

Consider the standard form problem:

$$\begin{aligned} \text{minimize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

Suppose that some component c_j of the cost vector $\mathbf{c}$ is changed to $c_j + \delta$.

The primal feasibility condition is not affected by the change of $\mathbf{c}$. It thus remains to examine the optimality condition

$$\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A} \geq \mathbf{0}.$$

For a nonbasic variable x_j, if c_j is changed to $c_j + \delta_j$, then, $\mathbf{c}_B$ is not affected, and only the following inequality is affected

$$(c_j + \delta_j) - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_j \geq 0,$$

i.e.

$$\bar{c}_j + \delta_j \geq 0.$$

This gives a range for δ_j, namely, $\delta_j \geq -\bar{c}_j$.

For a basic variable x_j, if c_j is changed to $c_j + \delta_j$, then, $\mathbf{c}_B$ is affected, and hence all the optimality conditions are affected.

We shall illustrate this case in the next example and also determine a range for δ_j for a basic variable.

Example 4 Consider the same LP problem as in Example 1.

$$\begin{array}{rrrrrl} \text{minimize} & -5x_1 & - \quad x_2 & + \ 12x_3 & & \\ \text{subject to} & 3x_1 & + \ 2x_2 & + \quad x_3 & & = \ 10 \\ & 5x_1 & + \ 3x_2 & & + \ x_4 & = \ 16 \\ & & & x_1, x_2, x_3, x_4 \geq 0. & & \end{array}$$

with the optimal solution $\mathbf{x}^* = (2, 2, 0, 0)^T$ and the optimal simplex tableau given by

Basic	x_1	x_2	x_3	x_4	Soln
$\bar{\mathbf{c}}$	0	0	2	7	12
x_1	1	0	-3	2	2
x_2	0	1	5	-3	2

(a) Determine the range of changes δ_3 and δ_4 of c_3 and c_4 respectively under which the basis remains optimal.

(b) Determine the range of change for δ_1 of c_1 under which the basis remains optimal.

Solution

(a) For nonbasic variables x_3 and x_4, the corresponding optimality conditions are

$$(c_3 + \delta_3) - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_3 \geq 0$$

$$(c_4 + \delta_4) - -\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_4 \geq 0,$$

i.e.

$$\bar{c}_3 + \delta_3 \geq 0 \;\text{ and }\; \bar{c}_4 + \delta_4 \geq 0.$$

Therefore, $\delta_3 \geq -\bar{c}_3 = -2$ and $\delta_4 \geq -\bar{c}_4 = -7$.

In this range, $\mathbf{x}^* = (2, 2, 0, 0)^T$ remains optimal.

(b) For basic variables x_1 and x_2, note that changes in c_1 and c_2 affect $\mathbf{c}_B$. The reduced costs of x_1 and x_2 are zero. Thus, we need to compute the reduced costs of all nonbasic variables.

The reduced cost of the nonbasic variable x_3

$$= c_3 - \left[\, c_1 + \delta_1,\ c_2 \,\right] \mathbf{A}_B^{-1} \mathbf{A}_3$$

$$= (c_3 - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_3) - \left[\, \delta_1 \ \ 0 \,\right] \mathbf{A}_B^{-1} \mathbf{A}_3$$

$$= \bar{c}_3 - \left[\, \delta_1 \ \ 0 \,\right] \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \bar{c}_3 + 3\delta_1$$

and the reduced cost of the nonbasic variable x_4 $= \bar{c}_4 - 2\delta_1$ (Check).

Thus, to maintain optimality conditions, we must have

$$\bar{c}_3 + 3\delta_1 \geq 0 \ \text{ and } \ \ \bar{c}_4 - 2\delta_1 \geq 0.$$

i.e. $\delta_1 \geq -2/3$ and $\delta_1 \leq 7/2$. Hence, in the range

$$-2/3 \leq \delta_1 \leq 7/2$$

the solution $\mathbf{x}^* = (2, 2, 0, 0)^T$ remains optimal.

5.5 Changes in a nonbasic column of A.

Suppose that some entry a_{ij} of the nonbasic column of $\mathbf{A}_j$ is changed to $a_{ij} + \delta$. We wish to determine the range of values of δ for which the old primal optimal basis matrix remains optimal.

Since $\mathbf{A}_j$ is nonbasic, the basis matrix $\mathbf{A}_B$ does not change. Hence, the primal feasibility conditions are unaffected. However, among the reduced costs, only $\bar{c}_j$ is affected. Thus in examining the optimality conditions, we only examine the jth -reduced cost:

$$\bar{c}_j = c_j - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_j.$$

If this optimality condition is violated, the old primal optimal solution is feasible but not optimal; and thus we should proceed to apply primal simplex method.

Example 5 Consider the same LP problem as in Example 1

$$\begin{array}{rrcrcrcrcl} \text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = & 10 \\ & 5x_1 & + & 3x_2 & & & + & x_4 & = & 16 \\ & & & & x_1, x_2, x_3, x_4 \geq 0. & & & & & \end{array}$$

Suppose that $\mathbf{A}_3$ is changed from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Will the optimal solution $\mathbf{x}^* = (2, 2, 0, 0)^T$ be affected?

[*Solution*] Changing $\mathbf{A}_3$ does not affect the optimality condition $\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$, and the only affected reduced cost is $\bar{c}_3$.

$$\begin{aligned} \bar{c}_3 &= c_3 - \mathbf{c}_B^T \mathbf{A}_B^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 12 - \begin{bmatrix} -5 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 9 \geq 0. \end{aligned}$$

Thus, $\mathbf{x}^* = (2, 2, 0, 0)^T$ remains as the optimal solution to the new problem.

NOTE However, if $\mathbf{A}_3$ is to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the reduced cost $\bar{c}_3 = -1 < 0$. This indicates that $\mathbf{x}^* = (2, 2, 0, 0)^T$ a basic feasible solution to the new problem but it is not optimal. Thus, we apply primal simplex method to the following simplex tableau, where the x_3-column is replaced by $\mathbf{A}_B^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$:

Basic	x_1	x_2	x_3	x_4	
$\bar{\mathbf{c}}$	0	0	-1	7	12
x_1	1	0	-4	2	2
x_2	0	1	7	-3	2

where x_3 enters and x_2 leaves the basis.

5.6 Applications.

Example 6

DeChi produces two models of electronic gadgets that use resistors, capacitors and chips. The following table summarizes the data of the situation:

Resource	Unit resource requirements Model 1 (units)	Model 2 (units)	Maximum availability (units)
Resistors	2	3	1200
Capacitors	2	1	1000
Chips	0	4	800
Unit profit ($)	3	4	

Let x_1 and x_2 be the amounts produced of Models 1 and 2, respectively. The following is the corresponding LP problem:

$$\begin{array}{lll}
\text{Maximize} & 3x_1 + 4x_2 & \\
\text{Subject to} & 2x_1 + 3x_2 \le 1200 & \text{(Resistors)} \\
& 2x_1 + x_2 \le 1000 & \text{(Capacitors)} \\
& 4x_2 \le 800 & \text{(Chips)} \\
& x_, x_2 \ge 0 &
\end{array}$$

The associated optimal simplex tableau is given as follows:

Basic	x_1	x_2	s_1	s_2	s_3	Solution
$\bar{c}$	0	0	$-\frac{5}{4}$	$-\frac{1}{4}$	0	-1750
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	450
s_3	0	0	-2	2	1	400
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	100

.

Here, s_1, s_2 and s_3 represent the slacks in the respective constraints.

Optimal Basic variables: x_1, s_3, x_2.

From the optimal simplex tableau,

$$\mathbf{A}_B^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

(a) If the available number of resistors is increased to 1300 units, find the new optimal solution.

[Solution] If the available number of resistors is increased to 1300 units, i.e. $b_1 = 1300$, the optimality conditions are not affected. We check the feasibility condition, $\mathbf{x}_B \geq 0$.

Check: $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1300 \\ 1000 \\ 800 \end{bmatrix} = \begin{bmatrix} 425 \\ 200 \\ 150 \end{bmatrix} \geq \mathbf{0}.$

Thus the basis $\mathbf{B}$ is again optimal.
The new solution is $x_1 = 450, x_2 = 150$ and the profit is $3x_1 + 4x_2 = 1875$.

(b) If the available number of chips is reduced to 350 units, will you be able to determine the new optimum solution directly from the given information? Explain.

[Solution] If the available number of chips is reduced to 350 units, note that the optimality conditions are unaffected. We check the feasibility condition.

$$\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1200 \\ 1000 \\ 350 \end{bmatrix} = \begin{bmatrix} 450 \\ -50 \\ 100 \end{bmatrix},$$

which is not feasible.

Thus, we reoptimize the problem: note that

$$\mathbf{c}_B^T \mathbf{A}_B^{-1}\mathbf{b} = 1750.$$

Basic	x_1	x_2	s_1	s_2	s_3	Solution
$\bar{c}$	0	0	$-\frac{5}{4}$	$-\frac{1}{4}$	0	-1750
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	450
s_3	0	0	-2	2	1	-50
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	100
$\bar{c}$	0	0	0	$-\frac{3}{2}$	$-\frac{5}{8}$	-1518.75
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{8}$	456.25
s_1	0	0	1	-1	$-\frac{1}{2}$	25
x_2	0	1	0	0	$\frac{1}{4}$	87.5

Thus the new optimal solution is

$$x_1 = 456.25, \quad x_2 = 87.5$$

and the profit is \$1518.75.

(c) A new contractor is offering DeChi additional resistors at 40 cents each but only if DeChi would purchase at least 500 units. Should DeChi accept the offer?

[Solution] We take $b_1 = 1200 + 500 = 1700$.

Check the feasibility conditions:

$$\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1700 \\ 1000 \\ 800 \end{bmatrix} = \begin{bmatrix} 325 \\ -600 \\ 350 \end{bmatrix},$$

which is not feasible.

Moreover, $\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b} = 2375$. So, we reoptimize:

Basic	x_1	x_2	s_1	s_2	s_3	Solution
$\bar{c}$	0	0	$-\frac{5}{4}$	$-\frac{1}{4}$	0	-2375
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	325
s_3	0	0	-2	2	1	-600
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	350
$\bar{c}$	0	0	0	$-\frac{3}{2}$	$-\frac{5}{8}$	-2000
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{8}$	400
s_1	0	0	1	-1	$-\frac{1}{2}$	300
x_2	0	1	0	0	$\frac{1}{4}$	200

.

Thus the new optimal solution is

$$x_1 = 400, \quad x_2 = 200$$

and the profit is \$2000.

Change in profit is $2000 - 1750 = 250$;

Cost for additional 500 units of resistors is $500 \times 0.4 = 200$.

Thus, there is a net profit of \$50. Hence, DeChi should accept the offer.

(d) Find the unit profit range for Model 1 that will maintain the optimality of the current solution.

[Solution] We want to find the range of c_1 that will maintain the optimality of the current solution. We should find c_1 which satisfies the optimality condition: $\bar{\mathbf{c}} = \mathbf{c}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A} \le \mathbf{0}$. From the optimal simplex tableau:

$$\mathbf{A}_B^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

$$\begin{aligned} \bar{\mathbf{c}}^T &= \mathbf{c}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A} \\ &= (c_1, 4, 0, 0, 0) - (c_1, 0, 4) \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \\ &= (0, 0, \frac{1}{4}c_1 - 2, -\frac{3}{4}c_1 + 2, 0) \le \mathbf{0}. \end{aligned}$$

This yields $\frac{8}{3} \le c_1 \le 8$. The unit profit range for Model 1 that will maintain the optimality of the current solution is between $\frac{8}{3}$ and 8.

(e) If the unit profit of model 1 is increased to \$ 6, determine the new solution.

[Solution] If the unit profit of model 1 is increased to \$ 6, this falls in the range obtained in (d). Thus, the same solution $x_1 = 450, x_2 = 100$ holds but with profit being $6 \times 450 + 4 \times 100 = 3100$.

(f) Suppose that the objective function is changed to "maximize $5x_1 + 2x_2$".

Determine the associated optimal solution of the new problem.

[Solution] Now $\mathbf{c}^T = (5, 2, 0, 0, 0)$. To check if the current solution is optimal, we check

$$\bar{\mathbf{c}}^T = (5, 2, 0, 0, 0) - (5, 0, 2) \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

$$= (0, 0, \frac{1}{4}, -\frac{11}{4}, 0),$$

which is not optimal.

The new objective value $5x_1 + 2x_2$ at the current solution $\mathbf{x} = (450, 100)^T$ is

$$5 \times 450 + 2 \times 100 = 2450.$$

Re-optimize by primal simplex algorithm:

Basic	x_1	x_2	s_1	s_2	s_3	Solution
$\bar{c}$	0	0	$\frac{1}{4}$	$-\frac{11}{4}$	0	-2450
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	450
s_3	0	0	-2	2	1	400
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	100
$\bar{c}$	0	$-\frac{1}{2}$	0	$-\frac{5}{2}$	0	-2500
x_1	1	$\frac{1}{4}$	0	$\frac{1}{2}$	0	500
s_3	0	4	0	0	1	800
s_1	0	2	1	-1	0	200

Optimal solution: $x_1 = 50, x_2 = 0$
Profit = \$ 2500.

Chapter 6

Transportation Problems.

6.1 Transportation Models and Tableaus.

The transportation model deals with determining a minimum cost plan for transporting a single commodity from a number of sources (such as factories) to a number of destinations (such as warehouses).

Basically, the model is a linear program that can be solved by the regular simplex method. However, its special structure allows the development of a solution procedure, called the **transportation algorithm**, that is computationally more efficient.

The transportation model can be depicted as a **network** with m sources and n destinations as follows:

Units of supply	Sources	Destinations	Units of demand
a_1	1	1	b_1
.	.	.	.
.	.	j	b_j
a_i	i	.	.
.	.	.	.
a_m	m	n	b_n

where c_{ij} is the unit transportation cost between source i and destination j.

The objective of the model is to determine x_{ij} which is the amount to be transported from source i to destination j so that the total transportation cost is minimum.

It can be represented by the following LP:

$$
\begin{array}{ll}
\text{Minimize} & \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} c_{ij}x_{ij} \qquad \text{(cost)} \\
\text{Subject to} & \Sigma_{j=1}^{n} x_{ij} \le a_i, i = 1, 2, \cdots m \\
& \text{(sum of shipments from source } i \text{ cannot exceed its supply)} \\
& \Sigma_{i=1}^{m} x_{ij} \ge b_j, j = 1, 2, \cdots n \\
& \text{(sum of shipments to destination } j \text{ must satisfy its demand)} \\
& x_{ij} \ge 0, i = 1, 2, \cdots m, j = 1, 2, \cdots n
\end{array}
$$

The first two sets of constraints imply

$$
\begin{array}{lll}
\Sigma_{i=1}^{m} \Sigma_{j=1}^{n} x_{ij} & \le & \Sigma_{i=1}^{m} a_i \\
\Sigma_{j=1}^{n} \Sigma_{i=1}^{m} x_{ij} & \ge & \Sigma_{j=1}^{n} b_j
\end{array}
\Rightarrow \sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_i
$$

i.e. total supply must be at least equal to total demand.

When $\sum\limits_{i=1}^{m} a_i = \sum\limits_{j=1}^{n} b_i$, the resulting formulation is called a **balanced transportation model**. In the balanced transportation model, all constraints are equations, that is

$$\begin{aligned} \Sigma_{j=1}^{n} x_{ij} &= a_i, \ i = 1, 2, \cdots m \\ \Sigma_{i=1}^{m} x_{ij} &= b_j, \ j = 1, 2, \cdots n \end{aligned}$$

Proof: It follows from the first two sets of constraints that

$$\begin{aligned} \sum_{i=1}^{m} a_i &\geq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} \\ &\geq \sum_{j=1}^{n} b_j. \end{aligned}$$

Thus, $\sum\limits_{i=1}^{m} a_i = \sum\limits_{j=1}^{n} b_i$ implies

$$\begin{aligned} \sum_{i=1}^{m} a_i &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \\ \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} &= \sum_{j=1}^{n} b_j. \end{aligned}$$

Since $\Sigma_{j=1}^{n} x_{ij} - a_i \leq 0$ for all i,

$$\sum_{i=1}^{m}(\sum_{j=1}^{n} x_{ij} - a_i) = 0$$

implies

$$\sum_{j=1}^{n} x_{ij} - a_i = 0, \quad \forall\, i.$$

We can similarly show the other equations. QED

The transportation algorithm to be introduced works on a balanced transportation model.

When the transportation problem is not balanced, i.e. $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_i$, we can balance it by adding dummy source or a dummy destination. We shall discuss unbalanced problems in the last section.

Transportation problem as an LP problem.

Example 1.1

G Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are $1,000$, $1,500$ and $1,200$ cars. The quarterly demand at the two distribution centers are $2,300$ and $1,400$ cars.

The transportation cost per car on the different routes, rounded to the nearest dollar, are calculated as given in Table 1-1.

Table 1-1

	Denver	Miami
Los Angeles	\$80	\$215
Detroit	\$100	\$108
New Orleans	\$102	\$68

Represent the transportation problem as an LP problem.

Solution

The LP model of the problem in Table 1-1:

Minimize $80x_{11}+215x_{12}+100x_{21}+108x_{22}+102x_{31}+68x_{32}$

Subject to

$$\begin{array}{rcl}
x_{11} + x_{12} & = & 1000 \\
x_{21} + x_{22} & = & 1500 \\
x_{31} + x_{32} & = & 1200 \\
x_{11} + x_{21} + x_{31} & = & 2300 \\
x_{12} + x_{22} + x_{32} & = & 1400
\end{array}$$

$$x_{ij} \geq 0,\; i = 1,2,3; j = 1,2.$$

Note that these constraints are equations because the total supply from the three sources equals the total demand at the two destinations. This is a balanced transportation model.

Number of basic variables

Proposition 6.1.1 *The balanced transportation problem has $m+n-1$ basic variables.*

Proof: The number of basic variables equals to the number of linearly independent equality constraints.

The coefficient matrix of equality constraints is represented as follows:

$$\begin{pmatrix}
1 & 1 & \dots & 1 & & & & & & & & & \\
 & & & & 1 & 1 & \dots & 1 & & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & 1 & \dots & 1 \\
1 & & & & 1 & & & & & 1 & & & \\
 & 1 & & & & 1 & & & & & 1 & & \\
 & & \ddots & & & & \ddots & & \dots & & & \ddots & \\
 & & & 1 & & & & 1 & & & & & 1
\end{pmatrix}$$

The sum of first m rows minus the sum of last n rows equals to 0. Thus, the rank of the matrix $\leq m+n-1$.

On the other hand, we can find $m+n-1$ linearly independent columns, e.g.

$\{$ first n columns, $(2n)$-column, $(3n)$-column, $\dots$, (mn)-column $\}$.

Therefore, the rank is $m+n-1$. QED

Transportation tableau

The **transportation tableau** is used instead of the simplex tableau as illustrated in the following example.

Example 1.2 The transportation tableau of Example 1.1:

Table 1-2

	Denver	Miami	
Los Angeles	80 x_{11}	215 x_{12}	1000
Detroit	100 x_{21}	108 x_{22}	1500
New Orleans	102 x_{31}	68 x_{32}	1200
Demand	2300	1400	

Remark In the transportation tableau, the (i, j)-cell in the i-row and j-column represents the decision variable x_{ij}. We write the unit transportation cost from source i to destination j on the top right hand corner of the (i, j)-cell.

6.2 The Transportation Algorithm

The transportation algorithm works on a balanced transportation model. The steps of the transportation algorithm are exact parallels of the simplex method, namely:

Step 1 Determine a starting basic feasible solution, and go to Step 2.

Step 2 Use the optimality condition of the simplex method to determine the entering variable from among the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to Step 3.

Step 3 Use the feasibility condition of the simplex method to determine the leaving variable from among all the current basic variables, and find the new basic variable. Return to Step 2.

However, we take advantage of the special structure of the transportation model to present the algorithm in a more convenient form. Each of the steps is detailed subsequently via the following example.

Example 2.1: The Sun Ray Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table 2-1. The unit transportation costs, c_{ij}, (shown in the northeast corner of each box) are in hundreds dollars.

Table 2-1

		Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo	1	x_{11} (10)	x_{12} (2)	x_{13} (20)	x_{14} (11)	15
	2	x_{21} (12)	x_{22} (7)	x_{23} (9)	x_{24} (20)	25
	3	x_{31} (4)	x_{32} (14)	x_{33} (16)	x_{34} (18)	10
Demand		5	15	15	15	

The purpose of the model is to determine the minimum cost shipping schedule between the silos and the mills, i.e. to determining the quantity x_{ij} shipped from silo i to mill j ($i = 1, 2, 3$; $j = 1, 2, 3, 4$).

Step 1. Determine a starting basic feasible solution.

For a general transportation tableau of size $m \times n$, there are $m + n - 1$ basic variables. Three different procedures will be discussed: (1) **Northwest-corner Method**, (2) **Least-cost Method**, and (3) **Vogel's Approximation Method (VAM)**.

1. **Northwest-corner Method** starts at the northwest corner cell (x_{11}) of the tableau.

Step 1 Allocate as much as possible to the selected cell, and adjust the associated amount of supply and demand by subtracting the allocated amount.

Step 2 Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both the column and row net to zero simultaneously, cross out **one only** (either one), and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3 Move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to Step 1.

Example 2-1. NW corner method

	1	2	3	4	Supply
Source 1	10	2	20	11	15
Source 2	12	7	9	20	25
Source 3	4	14	16	18	10
Demand	5	15	15	15	

The basic variables of the starting basic solution is

$$\begin{array}{lll} x_{11} = 5 & x_{12} = 10 & \\ x_{22} = 5 & x_{23} = 15 & x_{24} = 5 \\ x_{34} = 10 & & \end{array}$$

Total cost $= 5(10) + 10(2) + 5(7) + 15(9) + 5(20) + 10(18) = 520$.

Note There are $3 + 4 - 1 = 6$ basic variables in the starting basic feasible solution.

2. **Least-cost Method** finds a better starting solution by concentrating on the cheapest routes. It starts at the cell with the smallest unit cost.

Step 1 Assign as much as possible to the variable with the smallest unit cost in the entire tableau. (Ties are broken arbitrarily.) Adjust the associated amount of supply and demand by subtracting the allocated amount.

Step 2 Cross out the satisfied row or column. As in the northwest-corner method, if a column and a row are satisfied simultaneously, cross out **one only**.

Step 3 Move to the uncrossed-out cell with the smallest unit cost. Go to Step 1.

Example 2-2. Least-cost method

		1	2	3	4	Supply
	1	10	2	20	11	15
Source	2	12	7	9	20	25
	3	4	14	16	18	10
Demand		5	15	15	15	

The basic variables of the starting basic feasible solution is

$$x_{12} = 15 \quad x_{14} = 0$$
$$x_{23} = 15 \quad x_{24} = 10$$
$$x_{31} = 5 \quad x_{34} = 5$$

and the associated cost is

$$15(2) + 0(11) + 15(9) + 10(20) + 5(4) + 5(18) = 475.$$

3. **Vogel's Approximation Method (VAM)** is an improved version of the least-cost method that generally produces better starting solutions.

Step 1 For each row (column) with strictly positive supply (demand), evaluate a penalty measure by subtracting the **smallest** cost element in the row (column) from the **next smallest** cost element in the same row (column). If more than one cost is the smallest, then the penalty = 0.

Step 2 Identify the row (column) with the largest penalty, breaking ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row (column). Adjust the supply and demand and cross out the satisfied row (column) . If a column and a row are satisfied simultaneously, crossed out the row (column) with the largest penalty and the remaining column (row) is assigned a zero demand (supply).

Step 3 Recompute the penalties for the uncrossed out rows and columns, then go to Step 2.

Remark

1. The row and column penalties are the penalties that will be incurred if, instead of shipping over the **best** route, we are forced to ship over the **second-best** route. The most serious one (largest penalty) is selected and allocate as much as possible to the variable with the smallest unit cost.

2. The variable at the selected cell must be regarded as a basic variable even if it is assigned zero amount.

Example 2-3. Vogel's method

		1	2	3	4	Supply
	1	10	2	20	11	15
Source	2	12	7	9	20	25
	3	4	14	16	18	10
Demand		5	15	15	15	

The basic variables of the starting basic feasible solution is

$$x_{12} = 15 \quad x_{22} = 0$$
$$x_{23} = 15 \quad x_{24} = 10$$
$$x_{31} = 5 \quad x_{34} = 5$$

and the associated cost is

$$15(2) + 0(7) + 15(9) + 10(20) + 5(4) + 5(18) = 475.$$

Same as the solution obtained by the least-cost method.

An example for comparing the three methods.

NW-Corner Method:

	1	2	3	4	Supply
Source 1	2	3	2	5	20
2	12	20	8	10	25
3	6	30	9	20	15
Demand	25	15	10	10	

The basic variables of the starting basic feasible solution is

and the associated cost is

Least-Cost Method:

		1	2	3	4	Supply
	1	2	3	2	5	20
Source	2	12	20	8	10	25
	3	6	30	9	20	15
Demand		25	15	10	10	

The basic variables of the starting basic feasible solution is

and the associated cost is

Vogel's Approximation Method:

		1	2	3	4	Supply
	1	2	3	2	5	20
Source	2	12	20	8	10	25
	3	6	30	9	20	15
Demand		25	15	10	10	

The basic variables of the starting basic feasible solution is

and the associated cost is

Step 2 Determine an entering variable.

After determining a basic feasible solution, we use the Method of Multipliers (or **UV method**) to compute the reduced costs of nonbasic variables x_{pq}. If the optimality conditions are satified, the basic feasible solution is optimal. Otherwise, we proceed to determine the entering variable among the current nonbasic variables.

Method of Multipliers.

Primal:

$$\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\begin{array}{lrclc}
\text{Subject to} & & & & \text{Dual variables} \\
& x_{11} + x_{12} + \cdots + x_{1n} & = & a_1 & u_1 \\
& x_{21} + x_{22} + \cdots + x_{2n} & = & a_2 & u_2 \\
& \cdots & . & . & . \\
& x_{m1} + x_{m2} + \cdots + x_{mn} & = & a_m & u_m \\
& x_{11} + x_{21} + \cdots + x_{m1} & = & b_1 & v_1 \\
& x_{12} + x_{22} + \cdots + x_{m2} & = & b_2 & v_2 \\
& \cdots & . & . & . \\
& x_{1n} + x_{2n} + \cdots + x_{mn} & = & b_n & v_n
\end{array}$$

$$x_{ij} \geq 0, \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$$

In the equality constraints, the coefficient (column vector) of x_{ij} is

$$\mathbf{A}_{ij} = \begin{pmatrix} \mathbf{e}_i \\ \mathbf{e}_j \end{pmatrix}.$$

For each x_{ij} there is an associated dual constraint

$$\mathbf{A}_{ij}^T \mathbf{p} \leq c_{ij}$$

where

$$\mathbf{A}_{ij}^T \mathbf{p} = \begin{pmatrix} \mathbf{e}_i \\ \mathbf{e}_j \end{pmatrix}^T \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \mathbf{e}_i^T \mathbf{u} + \mathbf{e}_j^T \mathbf{v} = u_i + v_j,$$

Thus, the Dual:

$$\begin{array}{ll} \text{Maximize} & \sum\limits_{i=1}^{m} a_i u_i + \sum\limits_{j=1}^{n} b_j v_j \\ \text{Subject to} & u_i + v_j \leq c_{ij} \\ & u_i, v_j \text{ unrestricted in sign} \\ & i = 1, 2, \cdots, m; j = 1, 2, \cdots, n. \end{array}$$

NOTES

1. At a basic feasible solution,with basis $\mathbf{B}$, we let
$$\mathbf{p}^T = \mathbf{c}_B^T \mathbf{B}^{-1}.$$
Thus, the reduced cost of x_{ij} is
$$\bar{c}_{ij} = c_{ij} - \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_{ij} = c_{ij} - \mathbf{A}_{ij}^T \mathbf{p} = c_{ij} - u_i - v_j.$$
2. The reduced cost of $\bar{c}_{ij}$ of a basic variable x_{ij} must be zero. Thus, we have
$$u_i + v_j = c_{ij} \text{ for each basic variable } x_{ij}$$
These give $m+n-1$ equations in $m+n$ variables $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$. Thus, we set $u_1 = 0$, and use the equations to solve for the remaining variables $u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$.
3. The reduced cost of a **nonbasic** variable x_{pq} can be computed as follows:
$$\overline{c}_{pq} = c_{pq} - (u_p + v_q).$$
4. Since the transportation problem is a minimization problem, the entering variable is a nonbasic variable with negative $\overline{c}_{pq}$.

The UV method involves computation of reduced costs of nonbasic variables via the introduction of multipliers (which are dual variables) u_i and v_j. However, the special structure of the transportation model allows simpler computations.

Summary of steps to determine an entering variable.

1. Associate the multipliers u_i and v_j with row i and column j of the transportation tableau.
2. For each basic variable x_{ij}, solve for values of u_i and v_j from the following equations:
$$u_i + v_j = c_{ij}$$
by arbitrarily setting $u_1 = 0$.
3. For each nonbasic variable x_{pq}, compute $\bar{c}_{pq} = c_{pq} - (u_p + v_q)$. If $\bar{c}_{pq} \geq 0$ for all nonbasic x_{pq}, stop and conclude that the starting feasible solution is optimal.

   Otherwise, choose x_{pq} corresponding to a negative value $\bar{c}_{pq}$ to be the entering variable.

Example 2-4 We use the staring basic feasible solution in Example 2-1, which is obtained by Northwest Corner Method:

		1	2	3	4	Supply
	1	10 (5)	2 (10)	20	11	15
Source	2	12	7 (5)	9 (15)	20 (5)	25
	3	4	14	16	18 (10)	10
Demand		5	15	15	15	

Step 3 Determine the leaving variable.

The leaving variable is determined by a loop.

Definition 6.2.1 *An ordered sequence of at least four different cells is called a* **loop** *if*

1. *Any two consecutive cells lie in either the same row or same column;*
2. *No three consecutive cells lie in the same row or column;*
3. *The last cell in the sequence has a row or column in common with the first cell in the sequence.*

An important relationship between the loop and the constraint coefficient matrix $\mathbf{A}$:

Lemma 6.2.2 *The cells corresponding to a set of variables contains a loop if and only if the corresponding columns of* $\mathbf{A}$ *are linearly dependent.*

The leaving variable is chosen from the current basic variables by the following steps.

1. Construct a loop that starts and ends at the entering variable. Each corner of the loop, with the exception of that in the entering variable cell, must coincide with a current basic variable. (Exactly one loop exists for a given entering variable.)
2. Assign the amount θ to the entering variable cell. Alternate between subtracting and adding the amount θ at the successive corners of the loop.

   (In the tableau, starting with $(-)$, indicate signs $(-)$ or $(+)$ alternatively in the south corner of each cell corresponds to a current basic variable at corners.)

3. Choose the largest possible value of $\theta > 0$ such that for each current basic variable x_{ij}, we have $x_{ij} \pm \theta \geq 0$ (according to the sign assigned in Step 2). Choose the basic variable x_{ij} corresponding to yielding this largest allowable value of θ as the leaving variable.

   (In the tableau, the leaving variable is selected among the corner basic variables of the loop labeled $(-)$ and has the **smallest value** x_{ij}.)

The next basic feasible solution.
The value of the entering variable x_{pq} is increased to θ, the maximum value found in Step 3. Each value of the corner (basic) variables is adjusted accordingly to satisfy the supply (demand). The new solution is thus obtained.

The new cost.
The transportation cost of each unit transported through the new route via the entering variable x_{pq} is changed by $\bar{c}_{pq} = c_{pq} - (u_p + v_q)$. Thus the total transportation cost associated with the new route is reduced by $\theta\bar{c}_{pq}$.

Example 2-5 In Example 2-4, we have found that the entering variable is x_{31}. Based on the same starting basic feasible solution, we form a close loop

$$x_{31} \rightarrow x_{11} \rightarrow x_{12} \rightarrow x_{22} \rightarrow x_{23} \rightarrow x_{24} \rightarrow x_{34} \rightarrow x_{31}$$

We assign a value θ to x_{31}, and alternate the signs of θ along the loop.

	1	2	3	4	Supply
1	10 (5)	2 (10)	20	11	15
Source 2	12	7 (5)	9 (15)	20 (5)	25
3	4 θ	14	16	18 (10)	10
Demand	5	15	15	15	

We proceed to compute new basic feasible solutions.

	1	2	3	4
1	10	2	20	11
2	12	7	9	20
3	4	14	16	18

	1	2	3	4
1	10	2	20	11
2	12	7	9	20
3	4	14	16	18

Optimal solution: $x_{12} = 5$, $x_{14} = 10$, $x_{22} = 10$, $x_{23} = 15$, $x_{31} = 5$, $x_{34} = 5$.
Cost =
$5(2) + 10(11) + 10(7) + 15(9) + 5(4) + 5(18) = 435.$

6.3 Unbalanced Transportation model.

The transportation algorithm works on a balanced transportation model. If the given model is not balanced, we will balance it before we carry out the transportation algorithm. A transportation model can always be **balanced** by introducing a **dummy supply (source)** or a **dummy demand (destination)** as follows:

1. If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_i$, a dummy destination is used to **absorb the surplus** $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_i$ with unit transportation cost equal to zero or stated storage costs at the various sources.
2. If $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_i$, a dummy source is used to **supply the shortage** amount by $\sum_{j=1}^{n} b_i - \sum_{i=1}^{m} a_i$ with unit transportation cost equal to zero or stated penalty costs at the various destinations for unsatisfied demands.

Example 3.1

Telly's Toy Company produces three kinds of dolls: the Bertha doll, the Holly doll, and the Shari doll in quantities of $1,000$, $2,000$ and $2,000$ per week respectively. These dolls are demanded at three large department stores: Shears, Nicholas and Words. Contract requiring $1,500$ total dolls per week are to be shipped to each store. However, Words does not want any Bertha dolls. Because of past contract commitments and size of other orders, profile vary from store to store on each kind of doll. A summary of the unit profit per doll is given below:

	Shears	Nicholas	Words
Bertha	5	4	–
Holly	16	8	9
Shari	12	10	11

(a) Set up the problem as a transportation problem.

(b) Obtain a starting basic feasible solution by the VAM and proceed to find an optimal solution.

(c) Obtain an alternative optimal solution.

Solution (a) The objective is to maximize the profit which can be converted to a minimization problem with the transportation cost being the negative of the profit.

	Shears	Nicholas	Words	Supply
Bertha	-5	-4	M	1000
Holly	-16	-8	-9	2000
Shari	-12	-10	-11	2000
Demand	1500	1500	1500	

We have assigned a value of $+M$ to the cell from Bertha to Words as 'Words does not want any Bertha dolls'. This large unit transportation cost ensures that the corresponding variable assumes zero value.

Remark In general, unacceptable transportation routes would be assigned a unit transportation cost value of $+M$.

(b) The transportation problem is not balanced. Thus, we introduce a dummy demand of 500 to form a balanced transportation model. This example does not specify the costs of transportation to the dummy demand, this means, we are indifferent in which sources supply the dummy destination. Hence, we should assign equal unit cost to each dummy cell. Here we simply assign the unit transportation cost at each dummy cell to be 0.

	S	N	W	Dummy	Supply
B	-5	-4	M	0	1000
H	-16	-8	-9	0	2000
S	-12	-10	-11	0	2000
D.	1500	1500	1500	500	

Using the UV-method iteratively:

	S	N	W	Dummy
B	-5	-4	M	0
H	-16	-8	-9	0
S	-12	-10	-11	0

	S	N	W	Dummy
B	-5	-4	M	0
H	-16	-8	-9	0
S	-12	-10	-11	0

Therefore the optimal solution:

Doll	Store	Number	Profit (\$)
Bertha	Nicholas	500	2,000
Holly	Shears	1,500	24,000
Holly	Words	500	4,500
Shari	Nicholas	1,000	10,000
Shari	Words	1,000	11,000
		Total	51,000

(c) Because the reduced cost at H-N cell is 0, we proceed to find an alternative optimal solution:

	S	N	W	Dummy
B	-5	-4	M	0
H	-16	-8	-9	0
S	-12	-10	-11	0

Doll	Store	Number	Profit ($)
Bertha	Nicholas	500	$2,000$
Holly	Shears	$1,500$	$24,000$
Holly	Nicholas	500	$4,000$
Shari	Nicholas	500	$5,000$
Shari	Words	$1,500$	$16,500$
		Total	$51,000$

6.4 Assignment problems

The assignment problem deals with the allocation (assignment) of resources (e.g. employees, machines, time slots) to activities (e.g. jobs, operators, events) on a one-to-one basis. The cost of assigning resource i to activity j is c_{ij}, and the objective is to determine how to make the assignment in order to minimize the total cost.

Example: MachineCo has four machines and four jobs to be completed. Each machine must be assigned to complete one job. The time required to complete a job is shown in the table:

		Job 1	Job 2	Job 3	Job 4
Machine	1	14	5	8	7
	2	2	12	6	5
	3	7	8	3	9
	4	2	4	6	10

MachineCo wants to minimize the total time needed to complete the four jobs.

Solution: We define (for $i, j = 1, 2, 3, 4$)

$$x_{ij} = \begin{cases} 1 & \text{if machine } i \text{ is assigned to job } j \\ 0 & \text{otherwise,} \end{cases}$$

MachineCo's problem may be formulated as a linear program:

$$\begin{aligned} \min\ z = {} & 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} \\ & +2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\ & +7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} \\ & +2x_{41} + 4x_{42} + 6x_{43} + 10x_{44} \\ s.t.\ & x_{11} + x_{12} + x_{13} + x_{14} = 1 \\ & x_{21} + x_{22} + x_{23} + x_{24} = 1 \\ & x_{31} + x_{32} + x_{33} + x_{34} = 1 \\ & x_{41} + x_{42} + x_{43} + x_{44} = 1 \\ & x_{11} + x_{21} + x_{31} + x_{41} = 1 \\ & x_{12} + x_{22} + x_{32} + x_{42} = 1 \\ & x_{13} + x_{23} + x_{33} + x_{43} = 1 \\ & x_{14} + x_{24} + x_{34} + x_{44} = 1 \\ & x_{ij} = 0 \text{ or } 1 \text{ for all } i, j. \end{aligned}$$

The first four constraints (machine constraints) ensure that each machine is assigned to a job.
The last 4 constraints (job constraints) ensure that each job is completed by a machine.

In general, the assignment problem can expressed as

LP:

$$
\begin{aligned}
\min\ z = & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\
s.t.\ & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n, \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n, \\
& x_{ij} = 0 \text{ or } 1.
\end{aligned}
$$

An assignment solution $\{x_{ij}\}$ is **feasible** if and only if exactly one from i-th row $\{x_{i1}, x_{i2}, \ldots, x_{in}\}$ equals 1 (the others equal 0), and exactly one from j-th column $\{x_{1j}, x_{2j}, \ldots, x_{nj}\}$ equals 1 (the others equal 0).

Table form:

$$
\begin{array}{c|cccccc|c}
 & 1 & 2 & \dots & j & \dots & n & \\
\hline
1 & c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1n} & 1 \\
2 & c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} & 1 \\
\vdots & & & \vdots & & & & \vdots \\
i & c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{in} & 1 \\
\vdots & & & \vdots & & & & \vdots \\
n & c_{n1} & c_{n2} & \dots & c_{nj} & \dots & c_{nn} & 1 \\
\hline
 & 1 & 1 & \dots & 1 & \dots & 1 &
\end{array}
$$

Observations:

1. This is a special case of the transportation problem ($s_i = d_j = 1$),
2. For the assignment problem to have a feasible solution, we must have $m = n$. (It is necessary to balance the problem by adding dummy jobs or machines if $m \neq n$).

The Hungarian Method for Assignment Problems

The method is based on the following theorem.

Theorem 6.4.1 *The optimal solution of an assignment problem remains the same if a constant is added to or subtracted from any row or column of the cost table.*

Proof. Suppose constants u_i and v_j are subtracted from the i-th row and j-th column, respectively. The new assignment cost c'_{ij} is

$$c'_{ij} = c_{ij} - u_i - v_j, \qquad \forall\, i, j.$$

Let z' denote the new total cost. Then

$$\begin{aligned}
z' &= \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} u_i x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} v_j x_{ij} \\
&= z - \sum_{i=1}^{n} \left(u_i \sum_{j=1}^{n} x_{ij} \right) - \sum_{j=1}^{n} \left(v_j \sum_{i=1}^{n} x_{ij} \right) \\
&= z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j \\
&= z - \text{constant}
\end{aligned}$$

Therefore, $\min z' = \min z$. QED

Idea of the Hungarian Method:

Create a new reduced cost table by subtracting constants from rows and columns so that some entries become zero and all entries are nonnegative. If there are enough zeros to constitute a feasible solution, then this solution must be optimal because the cost cannot be negative.

The procedure:

Step 1: Subtract the smallest entry in each row from that row. Subtract the smallest entry in each column from that column.

Step 2: Try to make a feasible solution with assignments only to zero entries. If such a feasible solution is obtained, stop (the solution is optimal). Otherwise, go to step 3.

Step 3: Cross out all zeros with the least number of vertical and/or horizontal lines.

Step 4: Let θ be the smallest uncrossed entry. Subtract θ from every uncrossed entry and add θ to every entry which is crossed out twice.
Return to step 2.

Explanation for Step 4: The optimal solutions remain the same if we subtract θ from whole rows which are not crossed out by horizontal lines and add θ to whole columns which are crossed out by vertical lines. This results in the following:

Entries crossed out once are unchanged.

Entries crossed out twice are increased by θ.

Example: (MachineCo)

$$(1)\quad \begin{array}{c} \\ \boxed{\begin{array}{cccc} 14 & 5 & 8 & 7 \\ 2 & 12 & 6 & 5 \\ 7 & 8 & 3 & 9 \\ 2 & 4 & 6 & 10 \end{array}} \end{array} \quad \begin{array}{c} \text{row min} \\ 5 \\ 2 \\ 3 \\ 2 \end{array}$$

Subtract row min's.

$$(2)\quad \begin{array}{cl} \boxed{\begin{array}{cccc} 9 & 0 & 3 & 2 \\ 0 & 10 & 4 & 3 \\ 4 & 5 & 0 & 6 \\ 0 & 2 & 4 & 8 \end{array}} & \\ \begin{array}{cccc} 0 & 0 & 0 & 2 \end{array} & \text{column min} \end{array}$$

Subtract column min's.

$$
(3)\ \boxed{\begin{array}{cccc} 9 & 0 & 3 & 0 \\ 0 & 10 & 4 & 1 \\ 4 & 5 & 0 & 4 \\ 0 & 2 & 4 & 6 \end{array}}
$$

— Cross out zeros.

— Subtract $\theta(=1)$ from uncrossed entries.

— Add θ to double-crossed entries.

$$
(4)\ \boxed{\phantom{\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}}}
$$

Optimal solution:

$$
\begin{aligned} x_{12} = x_{24} = x_{33} = x_{41} &= 1, \\ \text{other } x_{ij} &= 0. \end{aligned}
$$

Total cost:

$$
z = 5 + 5 + 3 + 2 = 15 \text{ (hours)}
$$

Unbalanced assignment problems

Example: (Job shop Co.)
3 new machines are to be assigned to 4 locations. Material handling costs are given in table below.

		Location 1	2	3	4
Machine	1	13	16	12	11
	2	15	–	13	20
	3	5	7	10	6

("–" indicates that machine 2 cannot be assigned to location 2.)

Objective: Assign the new machines to the available locations to minimize the total cost of material handling.

Solution:
To balance the problem, we introduce a dummy machine with assignment costs to the various locations equal to 0.

Set the cost of assigning machine 2 to location 2 to be M (very large).

The assignment problem is formulated as

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	M	13	20
	3	5	7	10	6
	4(D)	0	0	0	0

Solve the problem with the Hungarian method:

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\
\hline
13 & 16 & 12 & 11 \\
15 & \mathrm{M} & 13 & 20 \\
5 & 7 & 10 & 6 \\
0 & 0 & 0 & 0 \\
\hline
\end{array}
\begin{array}{c}
 \\ -11 \\ -13 \\ -5 \\ -0
\end{array}
\Rightarrow
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\
\hline
2 & 5 & 1 & 0 \\
2 & \mathrm{M} & 0 & 7 \\
0 & 2 & 5 & 1 \\
0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Optimal solution is to assign

machine	to	location	cost
1		4	11
2		3	13
3		1	5

(Location 2 is not used.)

Optimal cost $= 11 + 13 + 5 = 29.$