0.1 Linear Programming

0.1.1 Objectives

By the end of this unit you will be able to:

- formulate simple linear programming problems in terms of an objective function to be maximized or minimized subject to a set of constraints.
- find feasible solutions for maximization and minimization linear programming problems using the graphical method of solution.
- solve maximization linear programming problems using the simplex method.
- construct the Dual of a linear programming problem.
- solve minimization linear programming problems by maximizing their Dual.

0.1.2 Introduction

One of the major applications of linear algebra involving systems of linear equations is in finding the maximum or minimum of some quantity, such as profit or cost. In mathematics the process of finding an extreme value (maximum or minimum) of a quantity (normally called a function) is known as **optimization** . **Linear programming (LP)** is a branch of Mathematics which deals with modeling a decision problem and subsequently solving it by mathematical techniques. The problem is presented in a form of a linear function which is to be optimized (i.e maximized or minimized) subject to a set of linear constraints. The function to be optimized is known as the **objective function** .

Linear programming finds many uses in the business and industry, where a decision maker may want to utilize limited available resources in the best possible manner. The limited resources may include material, money, manpower, space and time. Linear Programming provides various methods of solving such problems. In this unit, we present the basic concepts of linear programming problems, their formulation and methods of solution.

0.1.3 Formulation of linear programming problems

Mathematically, the general linear programming problem (LPP) may be stated as:

$$
\begin{aligned}
\text{Maximize or Minimize} \quad & Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ \ b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ \ b_2 \\
& \qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ \ b_m \\
& x_1\,, x_2\,, \ldots, x_n \geq 0
\end{aligned}
\tag{1}
$$

where

(i) the function Z is the objective function.

(ii) $x_1, x_2, \ldots, x_n$ are the decision variables.

(iii) the expression $(\leq, =, \geq)$ means that each constraint may take any one of the three signs.

(iv) c_j $(j = 1, \ldots, n)$ represents the per unit cost or profit to the j^{th} variable.

(v) b_i $(i = 1, \ldots, m)$ is the requirement or availability of the i^{th} constraint.

(vi) $x_1\,, x_2\,, \ldots, x_n \geq 0$ is the set of non-negative restriction on the LPP. In real life problems negative decision variables have no valid meaning.

In this module we shall only discuss cases in which the constraints are strictly inequalities (either have a $\leq$ *or* $\geq$).

In formulating the LPP as a mathematical model we shall follow the following four steps.

1. Identify the **decision variables** and assign symbols to them (eg x, y, z,... or x_1, x_2, x_2, ...). These decision variables are those quantities whose values we wish to determine.
2. Identify the set if constraints and express them in terms of inequalities involving the decision variables.
3. Identify the objective function and express it is terms of the decision variables.
4. Add the *non-negativity condition*.

We will use the following product mix problem to illustrate the formulation of an LPP.

Example 0.1.1 Prototype Example *A paint manufacturer produces two types of paint, one type of standard quality (S) and the other of top quality (T). To make these paints, he needs two ingredients, the pigment and the resin. Standard quality paint requires 2 units of pigment and 3 units of resin for each unit made, and is sold at a profit of R1 per unit. Top quality paint requires 4 units of pigment and 2 units of resin for each unit made, and is sold at a profit of R1.50 per unit. He has stocks of 12 units of pigment, and 10 units of resin. Formulate the above problem as a linear programming problem to maximize his profit?*

We make the following table from the given data.

	Product		Available
Ingredients	S-Type	T-Type	Stock
Pigment	2	4	12
Resin	3	2	10
Profit (R/Unit)	1.0	1.5	

We follow the four steps outlined above for solving LP problems.

1. In our prototype Example 0.1.1, the number of units of S-type and T-type paint are the decision variables.

2. The first constraint is the number of units of pigment available, while the second constraint is the number of units of resin available. It is required that the total pigment and resin used does not exceed 12 and 10, respectively.

Pigment:	for S is 2	Resin:	S = 3
	for T is 4		T = 2

Therefore the required mathematical expressions for the constraints are

$$\begin{aligned} 2S + 4T &\leq 12 \\ 3S + 2T &\leq 10 \end{aligned}$$

3. If we let P be the profit, then the objective in our example is to maximize profits

$$P = S + 1.5T,$$

i.e. the number of units of S times R1 *plus* the number of units of T times R1.5 .

4. In addition to the given constraints, there are nonnegativity constraints which ensure that the solution is meaningful. This is a requirement that whatever the decision, the decision variables should not be negative.

$$S \geq 0\,, T \geq 0$$

We can now write the complete mathematical model of the problem described in Example 0.1.1 as

$$\begin{array}{ll} \text{Maximise:} & P = S + 1.5T \\ \text{Subject to:} & 2S + 4T \leq 12 \\ & 3S + 2T \leq 10 \\ & S \geq 0\,, T \geq 0 \end{array} \tag{2}$$

The above problem is an example of a maximization LPP. Maximization LPPs are usually identified by the $\leq$ in all the constraints. Minimization problems can be identified by a $\geq$ in all the constraints.

In the next example we formulate a minimization LPP.

Example 0.1.2
(Diet problem) A house wife wishes to mix two types of food F_1 and F_2 in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food F_1 costs E60/Kg and Food F_2 costs E80/kg. Food F_1 contains 3 units/kg of vitamin A and 5 units/kg of vitamin B while Food F_2 contains 4 units/kg of vitamin A and 2 units/kg of vitamin B. Formulate this problem as a linear programming problem to minimize the cost of the mixtures.

We make the following table from the given data.

Vitamin content	Food (in Kg)		Requirement (in units)
	F_1	F_2	
Vitamin A (units/kg)	3	4	8
Vitamin B (units/kg)	5	2	11
Cost (E/Kg)	60	80	

In formulating the LPP we use the following steps:

1. The number of kilograms of the foods F_1 and F_2 contained in the mixture are the decision variables. Let the mixture contain x_1 Kg of Food F_1 and x_2 Kg of food F_2.

2. In this example, the constraints are the minimum requirements of the vitamins. The minimum requirement of vitamin A is 8 units. Therefore

$$3x_1 + 4x_2 \geq 8$$

Similarly, the minimum requirement of vitamin B is 11 units. Therefore,

$$5x_1 + 2x_2 \geq 11$$

3. The cost of purchasing 1 Kg of food F_1 is E60.

   The cost of purchasing 1 Kg of food F_2 is E80.

   The total cost of purchasing x_1 Kg of food F_1 and x_2 Kg of food F_2 is

$$C = 60x_1 + 80x_2$$

which is the objective function.

4. The non-negativity conditions are

$$x_1 \geq 0, \quad x_2 \geq 0$$

Therefore the mathematical formulation of the LPP is

$$\begin{array}{ll} \text{Minimize:} & C = 60x_1 + 80x_2 \\ \text{Subject to:} & 3x_1 + 4x_2 \geq 8 \\ & 5x_1 + 2x_2 \geq 11 \\ & x_1 \geq 0\,, x_2 \geq 0 \end{array}$$

0.1.4 The graphical method of solution

The **graphical method** of solving a linear programming problem is used when there are only two decision variables. If the problem has three or more variables, the graphical method is not suitable. In that case we use the **simplex method** which is discussed in the next section.

We begin by giving some important definitions and concepts that are used in the methods of solving linear programming problems.

1. **Solution** A set of values of decision variables satisfying all the constraints of a linear programming problem is called a *solution* to that problem.

2. **Feasible solution** Any solution which also satisfies the non-negativity restrictions of the problem is called a *feasible solution.*

3. **Optimal feasible solution** Any feasible solution which maximizes or minimizes the objective function is called an *optimal feasible solution.*

4. **Feasible region** The common region determined by all the constraints and non-negativity restriction of a LPP is called a *feasible region.*

5. **Corner point** A *corner point* of a feasible region is a point in the feasible region that is the intersection of two boundary lines.

The following theorem is the fundamental theorem of linear programming .

Theorem 0.1.1 *If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the corner points of the feasible region.*

To solve a linear programming problem with two decision variables using the graphical method we use the procedure outlined below;

	Graphical method of solving a LPP
Step 1.	Formulate the linear programming problem.
Step 2.	Graph the feasible region and find the corner points. The coordinates of the corner points can be obtained by either inspection or by solving the two equations of the lines intersecting at that point.
Step 3.	Make a table listing the value of the objective function at each corner point.
Step 4.	Determine the optimal solution from the table in step 3. If the problem is of maximization (minimization) type, the solution corresponding to the largest (smallest) value of the objective function is the optimal solution of the LPP.

We will now use this procedure to solve some LPP where the model has already been determined. We use example (0.1.1) for illustration purposes The graph of the LPP is shown in Figure 1.

Step 2
The boundary of the feasible region consists of the lines obtained from changing the inequalities to equalities; i.e. The lines

$$2S + 4T = 12 \qquad \text{and} \qquad 3S + 2T = 10$$

Step 3
The corner points (or extreme points) and their corresponding objective functional values are:

Extreme Points	Profit ($P = S + 1.5T$)
$(0, 0)$	0
$(\frac{10}{3}, 0)$	$\frac{10}{3}$
$(2, 2)$	5
$(0, 3)$	4.5

Step 4
We therefore deduce that the optimal solution is $S = 2\,, T = 2$ corresponding to a profit $P = 5$. Thus profits are maximized when 2 units of standard quality and 2 units of top quality type paint are produced.

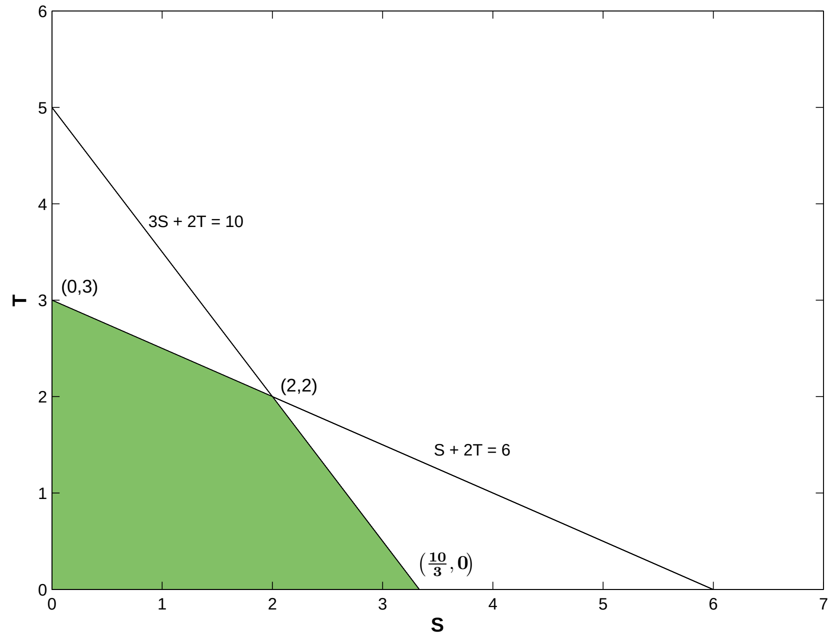

Figure 1: Graphical solution of the model of prototype example

Example 0.1.3

A furniture company produces inexpensive tables and chairs. The production process for each is similar in that both require a certain number of hours of carpentry work and a certain number of labour hours in the painting department.

Each table takes 4 hours of carpentry and 2 hours in the painting department. Each chair requires 3 hours of carpentry and 1 hour in the painting department. During the current production period, 240 hours of carpentry time are available and 100 hours in painting is available. Each table sold yields a profit of E7; each chair produced is sold for a E5 profit.

Find the best combination of tables and chairs to manufacture in order to reach the maximum profit.

Solution:

We begin by summarizing the information needed to solve the problem in the form of a table. This helps us understand the problem being faced.

	Hours required to make 1 Unit		
Department	Tables	Chairs	Available Hours
Carpentry	4	3	240
Painting	2	1	100
Profit	7	5	

The objective is to maximize profit.

The constraints are

1. *The hours of carpentry time used cannot exceed 240 hours per week.*
2. *The hours of painting time used cannot exceed 100 hours per week.*
3. *The number of tables and chairs must be non-negative.*

The decision variables that represent the actual decision to be made are defined as

$$\begin{aligned} x_1 &= \textit{number of tables to be produced} \\ x_2 &= \textit{number of chairs to be produced} \end{aligned}$$

Now we can state the linear programming (LP) problem in terms of x_1 *and* x_2 *and Profit* (P).

$$\begin{array}{lll} \textit{maximize} & P = 7x_1 + 5x_2 & \textit{(Objective function)} \\ \textit{subject to} & 4x_1 + 3x_2 \leq 240 & \textit{(hours of carpentry constraint)} \\ & 2x_1 + x_2 \leq 100 & \textit{(hours of painting constraint)} \\ & x_1 \geq 0, \quad x_2 \geq 0 & \textit{(Non-negativity constraint)} \end{array}$$

To find the optimal solution to this LP using the graphical method we first identify the region of feasible solutions and the corner points of the of the feasible region. The graph for this example is plotted in figure (2)

In this example the corner points are (0,0), (50,0), (30,40) and (0,80). Testing these corner points on $P = 7x_1 + 5x_2$ *gives*

Corner Point	*Profit*
$(0,0)$	0
$(50,0)$	350
$(30,40)$	410
$(0,80)$	400

Because the point (30,40) produces the highest profit we conclude that producing 30 tables and 40 chairs will yield a maximum profit of E410.

Example 0.1.4

A small brewery produces Ale and Beer. Suppose that production is limited by scarce resources of corn, hops and barley malt. To make Ale 5kg of Corn, 4kg of hops and 35kg of malt are required. To make Beer 15kg of corn, 4 kg of hops and 20kg of malt are required. Suppose that only 480 kg of corn, 160kg of hops and 1190 kg of malt are available. If the brewery makes a profit of E13 for each kg of Ale and E23 for each kg of Beer, how much Ale and Beer should the brewer produce in order to maximize profit?

Solution:

The given information is summarized in the table below.

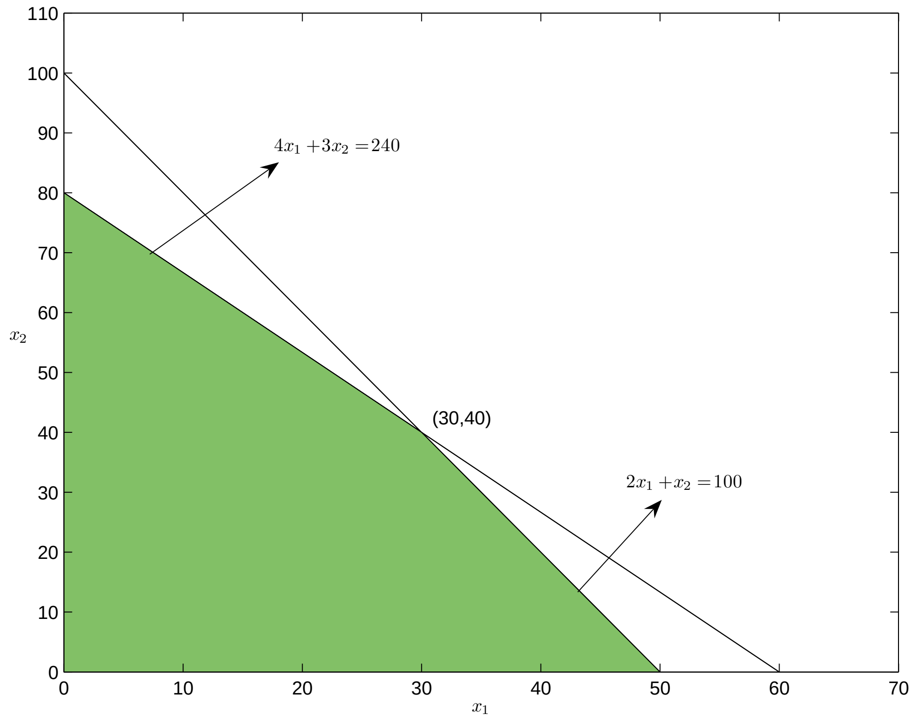

Figure 2: Graphical solution of the carpentry/painting model

Ingredients	*Beverages*		*Available quantity*
	Ale	*Beer*	
Corn(Kg)	*5*	*15*	*480*
Hops (Kg)	*4*	*4*	*160*
Malt (Kg)	*35*	*20*	*1190*
Profit	*13*	*23*	

The decision variables are

1. x_1 *the amount of Ale to be produced.*
2. x_2 *the amount of Beer to be produced.*

The profit function is given by $P = 13x_1 + 23x_2$. *Thus the LP problem can be formulated as follows:*

$$\begin{aligned} \textit{Maximize} \quad & P = 13x_1 + 23x_2 \\ \textit{Subject to} \quad & 5x_1 + 15x_2 \leq 480 \\ & 4x_1 + 4x_2 \leq 160 \\ & 35x_1 + 20x_2 \leq 1190 \\ & x_1 \geq 0 \ , \quad x_2 \geq 0 \end{aligned}$$

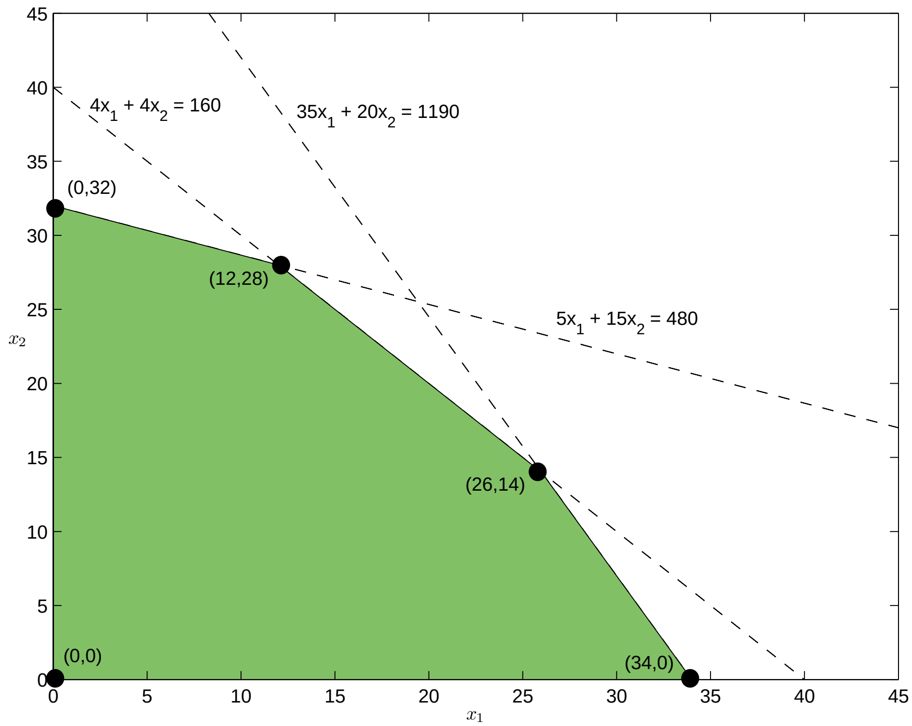

Figure 3: Graphical solution of the brewery model

The graph for this example is plotted in figure (3)

The corner points in this example are (0,0), (0,32), (12,28), (26,14) and (34,0). Testing these corner points on $P = 13x_1 + 23x_2$ *gives*

Corner Point	*Profit*
$(0, 0)$	0
$(0, 32)$	736
$(12, 28)$	800
$(26, 14)$	660
$(34, 0)$	442

Because the point (12,28) produces the highest profit we conclude that producing 12 Kg of Ale and 28 Kg of Beer will yield a maximum profit of E800.

Example 0.1.5 *(Medicine) A patient in a hospital is required to have at least 84 units of* drug A *and 120 units of* drug B *each day. Each gram of substance M contains 10 units of* drug A *and 8 units of* drug B*, and each gram of substance N contains 2 units of* drug A *and 4 units of* drug B*. Now suppose that both M and N contain an undesirable* drug C*, 3 units per gram in M and 1 unit per gram in N. How many grams of substances M and N should be mixed to meet the minimum daily requirements at the same time minimize the intake of* drug C*? How many units of the undesirable* drug C *will be in this mixture?*

Solution: *We start by summarizing the given data in the following table;*

	AMOUNT OF DRUG PER GRAM		MINIMUM DAILY
	Substance M	*Substance N*	*REQUIREMENT*
Drug A	*10 Units*	*2 units*	*84 units*
Drug B	*8 units*	*4 units*	*120 units*
Drug C	*3 units*	*1 unit*	

To form the mathematical model, we start by identifying the decision variables.

Let: x_1 = *Number of grams of substance M used.*
x_2 = *Number of grams of substance N used.*

The objective is to minimize the intake of drug C. In terms of the decision variables, the objective function is

$$C = 3x_1 + x_2$$

which gives the amount of the undesirable drug C in x_1 grams of M and x_2 grams of N.

The following conditions must be satisfied to meet daily requirements:

$$\begin{pmatrix} \text{Number of units of} \\ \text{drug A} \\ \text{in } x_1 \text{ grams of substance M} \end{pmatrix} + \begin{pmatrix} \text{Number of units of} \\ \text{drug A} \\ \text{in } x_2 \text{ grams of substance N} \end{pmatrix} \geq 84$$

$$\begin{pmatrix} \text{Number of units of} \\ \text{drug B} \\ \text{in } x_1 \text{ grams of substance M} \end{pmatrix} + \begin{pmatrix} \text{Number of units of} \\ \text{drug B} \\ \text{in } x_2 \text{ grams of substance N} \end{pmatrix} \geq 120$$

(Number of grams of substance M used) $\geq$ *0*

(Number of grams of substance N used) $\geq$ *0*

Writing the above constraint inequalities in terms of the decision variables x_1 and x_2 and including the objective function we obtain the following linear programming model.

$$\begin{array}{ll} \textit{Minimize} & C = 3x_1 + x_2 \\ \textit{Subject to} & 10x_1 + 2x_2 \geq 84 \\ & 8x_1 + 4x_2 \geq 120 \\ & x_1 \geq 0\ , \quad x_2 \geq 0 \end{array}$$

Figure 4 shows the graph of the feasible region obtained by plotting the system of inequalities. The evaluation of the objective function at each corner point is show in the table below.

CORNER POINT (x_1, x_2)	$C = 3x_1 + x_2$
(0,42)	*42*
(4,22)	*34*
(15,0)	*45*

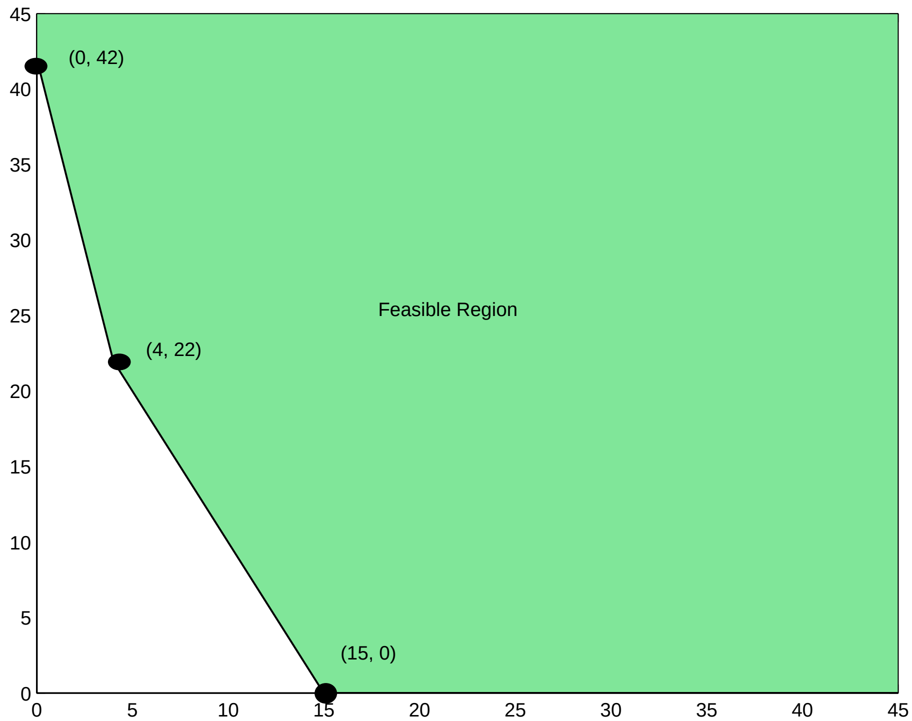

Figure 4: Graphical solution of the medicine minimization example

The graphical method is the easiest way to solve a small LP problem. However this method is useful only when there are two decision variables. When there are more than two decision variables, it is not possible to plot the solution on a two-dimensional graph and we must turn to more complex methods.

The graphical nature of the above method makes its use limited to problems involving only two decision variables. For such problems it is possible to represent the constraints graphically. A graphical solution for a problem with a higher number of decision variables than two cannot be practically obtained because of the complexity of the graphs in higher dimensional spaces. An additional limitation of this method is that if the graph is not good, the answer may be very inaccurate.

A very useful method of solving linear programming problems of any size is the so called Simplex method. The use of computers has made this method a viable tool for solving linear programming problems involving a very large number of decision variables.

0.1.5 Summary

In this section we have formulated linear programming problems and used a graphical method to obtain solutions to such problems. The types of problems we considered were maximization and minimization problems in which an objective function was either maximized or minimized subject to a set of constraints.

0.1.6 Exercise: Maximization problems

Use the graphical method to solve each of the following LP problems.

1. A wheat and barley farmer has 168 hectare of ploughed land, and a capital of E2000. It costs E14 to sow one hectare wheat and E10 to sow one hectare of barley. Suppose that his profit is E80 per hectare of wheat and E55 per hectare of barley. Find the optimal number of hectares of wheat and barley that must be ploughed in order to maximize profit? What is the maximum profit? [80,88], Profit E11 240

2. An company manufactures two electrical products: air conditioners and large fans. The assembly process for each is similar in that both require a certain amount of wiring and drilling. Each air conditioner takes 3 hours of wiring and 2 hours of drilling. Each fan must go through 2 hours of wiring and 1 hour of drilling. During the next production period, 240 hours of wiring time are available and up to 140 hours of drilling time may be used. Each air conditioner sold yields a profit of E25. Each fan assembled may be sold for a profit of E15. Formulate and solve this linear programming mix situation to find the best combination of air conditioners and fans that yields the highest profit. [40 air conditioners, 60 fans, profit E1900]

3. A manufacturer of lightweight mountain tents makes a standard model and an expedition model for national distribution. Each standard tent requires 1 labour hour from the cutting department and 3 labour hours from the assembly department. Each expedition tent requires 2 labour hours from the cutting department and 4 labour hours from the assembly department. The maximum labour hours available per day in the cutting department and the assembly department are 32 and 84 respectively. If the company makes a profit of E50 on each standard tent and E80 on each expedition tent, use the graphical method to determine how many tents of each type should be manufactured each day to maximize the total daily profit? [E1480]

4. A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labour hours from the cutting department and 0.8 labour hours from the assembly department. Each four-person boat requires 1.8 labour hours from the cutting department and 1.2 labour hours from the assembly department. The maximum labour hours available per month in the cutting department and the assembly department are 864 and 672 respectively. The company makes a profit of E25 on each two-person boat and E40 on each four-person boat. Use the graphical method to find the maximum profit. [E21 600]

5. LESCO Engineering produces chairs and tables. Each table takes four hours of labour from the carpentry department and two hours of labour from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of E70 and each chair a profit of E50. How many chairs and tables should be made in order to maximize profit? [40,30], P = E410

6. A company manufactures two products X and Y. Each product has to be processed in three departments: welding, assembly and painting. Each unit of X spends 2 hours in the welding department, 3 hours in assembly and 1 hour in painting. The corresponding times for a unit of Y are 3,2 and 1 respectively. The man-hours available in a month are 1500 for the welding department, 1500 in assembly and 550 in painting. The contribution to profits and fixed

overheads are E100 for product X and E120 for product Y. Formulate the appropriate linear programming problem and solve it graphically to obtain the optimal solution for the maximum contribution. [150, 400], P = 63000

7. Suppose a manufacturer of printed circuits has a stock of 200 resistors, 120 transistors and 150 capacitors and is required to produce two types of circuits.

   Type A requires 20 resistors, 10 transistors and 10 capacitors.

   Type B requires 10 resistors, 20 transistors and 30 capacitors.

   If the profit on type A circuits is E5 and that on type B circuits is E12, how many of each circuit should be produced in order to maximize profit? [6,3], P = 66

8. A small company builds two types of garden chairs.

   Type A requires 2 hours of machine time and 5 hours of craftsman time.

   Type B requires 3 hours of machine time and 5 hours of craftsman time.

   Each day there are 30 hours of machine time available and 60 hours of craftsman time. The profit on each type A chair is E60 and on each type B chair is E84. Formulate the appropriate linear programming problem and solve it graphically to obtain the optimal solution that maximizes profit. [6,6], P = 864

9. Namboard produces two gift packages of fruit. Package A contains 20 peaches, 15 apples and 10 pears. Package B contains 10 peaches, 30 apples and 12 pears. Namboard has 40 000 peaches, 60 000 apples and 27 000 pears available for packaging. The profit on package A is E2.00 and the profit on B is E2.50. Assuming that all fruit packaged can be sold, what number of packages of types A and B should be prepared to maximize the profit? [750 type A, 1625 type B]

10. A factory manufactures two products, each requiring the use of three machines. The first machine can be used at most 70 hours; the second machine at most 40 hours; and the third machine at most 90 hours. The first product requires 2 hours on Machine 1, 1 hour on Machine 2, and 1 hour on Machine 3; the second product requires 1 hour each on machines 1and 2 and 3 hours on Machine 3. If the profit in E40 per unit for the first product and E60 per unit for the second product, how many units of each product should be manufactured to maximize profit? [24,22, P = 2280]

0.1.7 Exercises 3.2: Minimization problems

1. A house wife wishes to mix together two kinds of food, I and II, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below;

	Vitamin A	Vitamin B	Vitamin C
Food I	1	2	3
Food II	2	2	1

One Kg of food I costs E6 and one Kg of food II costs E10. Formulate the above problem as a linear programming problem and find the least cost of the mixture which will produce the diet. [2,4, cost = E52]

2. A chicken farmer can buy a special food mix A at 20c per Kg and special food mix B at 40c per Kg. Each Kg of mix A contains 3000 units of nutrient N_1 and 1000 units of nutrient N_2; each Kg of mix B contains 4000 units of nutrient N_1 and 4000 units of nutrient N_2. If the minimum daily requirements for the chickens collectively are 36000 units of nutrient N_1 and 20000 units of nutrient N_2, how many pounds of each food mix should be used each day to minimize daily food costs while meeting (or exceeding) the minimum daily nutrient requirements? What is the minimum daily cost? [8kg of mix A, 3 kg of mix B; C = E2.80 per day]

3. A farmer can buy two types of plant food, mix A and mix B. Each cubic metre of mix A contains 20 kg of phosphoric acid, 30 kg of nitrogen, and 5 kg of potash. Each cubic metre of mix B contains 10 kg of phosphoric acid, 30 kg of nitrogen and 10 kg of potash. The minimum monthly requirements are 460 kg of phosphoric acid, 960 kg of nitrogen, and 220 kg of potash. If mix A costs E30 per cubic metre and mix B costs E35 per cubic metre, how many cubic metres of each mix should the farmer blend to meet the minimum monthly requirements at a minimal cost? What is the cost? [20 m^3, 12 m^3, E1020]

4. A city council voted to conduct a study on inner city community problems. A nearby university was contacted to provide sociologists and research assistants. Allocation of time and costs per week are given in the table. How many sociologists and how many research assistants should be hired to minimize the cost and meet the weekly labour-hour requirements? What is the weekly cost?

	LABOUR HOURS		MINIMUM LABOUR-HOURS NEEDED PER WEEK
	Sociologist	Research Assistant	
FIELDWORK	10	30	180
RESEARCH CENTRE	30	10	140
COSTS PER WEEK (E)	500	300	

5. A laboratory technician in a medical research centre is asked to formulate a diet from two commercially packaged foods, food A and food B, for a group of animals. Each kg of food A contains 8 units of fat, 16 units of carbohydrates, and 2 units of protein. Each Kg of food B contains 4 units of fat, 32 units of carbohydrate and 8 units of protein. The minimum daily requirements are 176 units of fat, 1024 units of carbohydrate, and 384 units of protein. If

food A costs 5c per Kg and food B costs 5c per Kg, how many kilograms of each food should be used to meet the minimum daily requirements at the least cost? What is the cost of this amount?

6. A can of cat food, guaranteed by the manufacturer to contain at least 10 units of protein, 20 units of mineral matter, and 6 units of fat, consists of a mixture of four different ingredients. Ingredient A contains 10 units of protein, 2 units of mineral matter, and $\frac{1}{2}$ unit of fat per 100g. Ingredient B contains 1 unit of protein, 40 units of mineral matter, and 3 units of fat per 100g. Ingredient C contains 1 unit of protein, 1 unit of mineral matter, and 6 units of fat per 100g. Ingredient D contains 5 units of protein, 10 units of mineral matter, and 3 units of fat per 100g. The cost of each ingredient is 3c, 2c, 1c, and 4c per 100g, respectively. How many grammes of each should be used to minimise the cost of the cat food, while still meeting the guaranteed composition?

0.1.8 The simplex method - maximisation

The Simplex method is based on an understanding of the algebra of the linear programming problem being solved. We begin by stating a general maximising linear programming problem involving n unknown (or decision) variables and m constraints as

$$\begin{array}{ll} \text{Maximise:} & z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{Subject to:} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ & x_1\,, x_2\,, \ldots, x_n \geq 0 \end{array} \tag{3}$$

or equivalently

$$\begin{aligned} \max \quad z &= \sum_{i=1}^{n} c_i x_i \\ \text{subj. to:} \quad \sum_{i=1}^{n} a_{ki} x_i &\leq b_k \qquad (4) \\ k &= 1, 2, \ldots, m \\ x_i &\geq 0\,, i = 1, 2, \ldots, n \qquad (5) \end{aligned}$$

We note in particular that all the constraints involve the $\leq$ sign. We will use this type of maximising linear programming problem to introduce the Simplex method. Other types of inequalities as well as the minimising problem will be discussed later. The number m, of constraints, can be less, equal or even greater than n.

The Simplex method is similar to the graphical method in that it uses the extreme points of the feasible region to search for the solution. The main difference is that with the Simplex method, once the initial vertex has been chosen, movement from one vertex to another is in such a way that the value of the objective function improves with each move. Although there are $n + m$ variables in m equations, the solution of the problem concerns the n variables in the original constraints. If $m < n$ then some of the decision variables will have zero values.

Before we can employ the Simplex method we need to rewrite the problem in a standard form in which the constraints are equations rather than inequalities.

Standard Form

We consider the k-th constraint of the general linear programming problem (3)

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \leq b_k \tag{6}$$

We convert the k-th inequality constraint to an equality constraint by introducing a new variable, $x_{n+k} \geq 0$, called a **slack variable**. The name of the variable derives from the fact that if the left hand side of the constraint is to balance with the right hand side of the constraint, then something has to be added to the left side.

If we do this for each of the m constraints we can write the standard form of the system (4) as

$$\begin{array}{ll} \text{Maximise} & z = \sum_{i=1}^{n} c_i x_i \\ \text{Subject to:} & \sum_{i=1}^{n} a_{ki} x_i + x_{n+k} = b_k \\ & k = 1, 2, \ldots, m \\ & x_i \geq 0\,, x_{n+k} \geq 0\,, \quad i = 1, 2, \ldots, n \end{array} \tag{7}$$

We can write the standard form of the linear programming problem as a set of matrix equations

$$\begin{aligned} z &= C\mathbf{x} \\ A\mathbf{x} &= \mathbf{b} \end{aligned} \tag{8}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & & \ldots & \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 \end{bmatrix} \tag{9}$$

and

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T , \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} , \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{10}$$

We note the following about the standard linear programming problem:

1. The objective function is unchanged. The slack variables can be included in the objective function with zero coefficients.

2. The m constraints of the new system are represented by m equations and there are now $n+m$ unknown variables (the solution variables plus the slack variables);

3. All the variables including the slack variables are nonnegative;

4. The right side values are nonnegative.

Definition 0.1.1 *A set of variables x_i, together which satisfy the equality constraints $Ax = b$ are said to be* **basic variables**. *These basic variables form a* **basic solution** *or a* **basis**. *If all the basic variables are nonnegative then they form a* **basic feasible solution**. *We note that a basic feasible solution may not necessarily optimise the objective function.*

In relation to the graphical approach we point out that every basic feasible solution is an extreme point of the feasible region, and conversely, every extreme point is a basic feasible solution.

As we discuss the Simplex procedure we will use our prototype example of the paint mix problem presented by the linear programme (2), whose solution has been previously found using the graphical method.

The linear programming problem (2) is restated below with a slight change in the names of the decision variables (x_1 instead of S and x_2 instead of T) as

$$\begin{array}{ll} \text{Maximise:} & P = x_1 + 1.5x_2 \\ \text{Subject to:} & 2x_1 + 4x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 10 \\ & x_1 \geq 0\,, x_2 \geq 0 \end{array} \tag{11}$$

Example 0.1.6
Writing the linear programme (11) in standard form we obtain

$$\begin{array}{ll} \textit{Maximise:} & P = x_1 + 1.5x_2 \\ \textit{Subject to:} & 2x_1 + 4x_2 + x_3 = 12 \\ & 3x_1 + 2x_2 + x_4 = 10 \\ & x_1 \geq 0\,, x_2 \geq 0 \end{array} \tag{12}$$

where x_3 and x_4 are the slack variables.

Once we have written a linear programming in standard we are ready to solve it using the Simplex method.

The Simplex Procedure

The Simplex procedure involves the following steps:

Step 1: **Find the initial basic feasible solution**
The simplest choice of an initial feasible basic is to assume that none of the decision variables are basic. Hence we initially assume that the basic solution consists only of the slack variables.
i.e. Set $x_i = 0\,, i = 1, 2, \ldots, n$ and $x_{n+k} \neq 0\,, k = 1, 2, \ldots, m$. This choice of the initial solution means that we initially assume that objective functional value is zero. In terms of the graphical method we start at the origin so as to move along the best possible route to the optimal solution.

Example 0.1.7 *If in the constraints of the standard form (12) we set $x_1 = 0,\ x_2 = 0$ we have the initial basic feasible solution $x_3 = 12,\ x_4 = 10,$ and $P = 0$.*

Step 2: **Set up the initial Simplex tableau**
In this step we arrange the various matrices and vectors involved in the matrix form of the linear programming problem in the simplex tableau. This tableau contains all the information about the current basic variables and their corresponding values, the optimality status of the solution. The method then continues to use the principle of the Gauss-Jordan procedure to compute the next improved solution.

A typical initial simplex tableau has the form shown in Table 1

In the tableau:

1. The top row shows both the $n + m$ decision and slack variables $x_1, x_2, \ldots, x_{n+1}, \ldots, x_{n+m}$ as labels for the corresponding columns;

2. The coefficients of the constraints are shown in the middle rows;

3. The last row is the z-equation, showing the objective coefficients;

Basic	x_1	x_2	$\dots$	x_n	x_{n+1}	x_{n+2}	$\dots$	x_{n+m}	
x_{n+1}	a_{11}	a_{12}	$\dots$	a_{1n}	1	0	$\dots$	0	b_1
x_{n+2}	a_{21}	a_{22}	$\dots$	a_{2n}	0	1	$\dots$	0	b_2
$\vdots$	$\vdots$			$\vdots$	0			$\vdots$	$\vdots$
x_{n+m}	a_{m1}	a_{m2}	$\dots$	a_{mn}	0	0	$\dots$	1	b_m
z	$-c_1$	$-c_2$	$\dots$	$-c_n$	0	0	$\dots$	0	0

Table 1: The general simplex tableau.

4. The extreme left column shows the basic variables;
5. Each basic variable
   - appears in exactly one equation in which it has a coefficient 1. The column it labels has all zeros except in the row in which it is shown as a basic variable
   - has a value shown on the extreme right column.

Initially the negative coefficients in the z-equation are a result of writing the objective equation as

$$z - c_1x_1 - c_2x_2 - \dots - c_nx_n = 0 \tag{13}$$

so that z itself is treated like a variable. When the decision variables are initially set to zero, the initial value of z is also zero. The value of z will vary as the decision variables assume nonzero values. In particular for the maximising problem z will increase as any of the nonbasic variables with a negative entry in the z-row is increased.

Example 0.1.8
The initial Simplex tableau of our example is

Tableau 1:

Basic	x_1	x_2	x_3	x_4	*R.H.S.*
x_3	*2*	$\boxed{4}$	*1*	*0*	*12*
x_4	*3*	*2*	*0*	*1*	*10*
P	-1	-1.5	*0*	*0*	*0*

The initial basic variables are $x_3 = 12$ and $x_4 = 10$ which you can read from the extreme left and right columns of the tableau.

Step 3: **Test for optimality**
At any stage of the procedure you can check whether the current basic solution is optimal. This information is contained in the objective row of the tableau. If all the entries in the objective row are **nonnegative**, then the current basic solution is optimal. In particular all the columns associated with the basic solution will have zero coefficients in the objective row while the columns associated with the nonbasic variables will have positive coefficients.

For our example, in the last row of Tableau 1 we have the negative coefficients -1 and -1.5 corresponding to x_1 and x_2. Thus the present solution is not optimal.

Step 4: **Choose the variable to enter/leave the basic set**
First you decide which of the nonbasic variables will bring about the best improvement on the objective value if entered into the basic set. i.e. if it is increased from zero. The nonbasic variables corresponding to the negative coefficients in the objective row are candidates for entry into the basic set. The **entering variable** is the one associated with the column with the most negative coefficient in the z-row. This column is known as the **pivot column**.

Since this variable will become basic, one of the basic variables will have to become nonbasic (or will leave the basic set) and be reduced to zero. The **leaving variable** determined by the quotients of the right hand column and the pivot column. You first compute quotients of the right hand side and **positive** coefficients of the pivot column. Thus you compare only positive quotients. Once you have computed all positive quotients, you choose the row which has the least quotient. This is called the **pivot row**. The leaving variable is the one which corresponds to the pivot row on the left hand side of the tableau (among the basic variables).

The element at the intersection of the pivot row and pivot column is called the **pivot coefficient**. It is normally highlighted by circling it (we will highlight it by boldface type). It is necessary to identify this element in order to move on to the next step.

In our example the variables x_1 and x_2 are the candidates for entry into the basis. The most negative coefficient in the last row is -1.5 in the column labeled x_2. This is the pivot column and thus the entering variable is x_2. This variable is raised from zero a nonzero value which will be determined in the next step.

Now we need to decide which variable should leave the basis. If we divide the coefficients of the last column by corresponding coefficients in the x_2-column we obtain the quotients $\frac{12}{4} = 3$ and $\frac{10}{2} = 5$. The smallest quotient is 3, corresponding to x_3 in the extreme left column. Thus the x_3-row is the pivot row, the variable x_3 has to leave the basis as x_2 enters, and assumes the value 3. The pivot coefficient is 4. (in box)

We are now ready to proceed to the next step.

Step 5: **Update the Simplex tableau**
The next step is to update the tableau by reducing it using the Gauss reduction principle. By updating the tableau you are essentially determining the effect of the introduction of the new basic variable and discarding the one that has left the basis.

By Gauss reduction, you reduce the pivot coefficient to one and all other coefficients in that column to zero. Let us illustrate this using our example once again.

In updating the tableau we first divide the x_3-row by 4 to reduce the pivot coefficient to 1. The coefficients 2 and -1.5 in the pivot column should be reduced to 0 by either adding or subtracting a suitable multiple of the pivot row. i.e. R_2 becomes $R_2 - 2R_1, R_3$ becomes $R_3 + 1.5R_1$. Performing these Gaussian operations leads to the tableau

Tableau 2

	x_1	x_2	x_3	x_4	
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$	0	3
x_4	$\boxed{2}$	0	$-\frac{1}{2}$	1	4
P	$-\frac{1}{4}$	0	$\frac{3}{8}$	0	4.5

Thus the objective functional value has improved from 0 to 4.5 as x_2 is raised from zero to 3. Note

that $4.5 = 1.5 \times 3$, *the contribution made by* x_2 *in the objective.* Step 6: **Repeat Steps 3 - 5**
Test for optimality and pivot again until the optimal solution is obtained or some other conclusion is made of the problem.

Once again we test whether the current solution is optimal. Looking at the last row of the Tableau 2 above we see that there is still a negative coefficient, so the solution is not optimal. We repeat the last three steps of the Simplex procedure. Since $-\frac{1}{4}$ *in the objective row is the only negative coefficient, the corresponding column is the pivot column and* x_1 *enters the basis. The leaving variable is obtained by comparing the quotients* $\frac{3}{\frac{1}{2}} = 6$ *and* $\frac{4}{2} = 2$. *Hence* x_4 *should leave the basis and give way to* x_1. *The pivot coefficient is 2, at the intersection of the pivot row and pivot column.*

Updating the tableau by Gauss reduction leads to the tableau

Tableau 3

	x_1	x_2	x_3	x_4	
x_2	*0*	*1*	$\frac{3}{8}$	$-\frac{1}{4}$	*2*
x_1	*1*	*0*	$-\frac{1}{4}$	$\frac{1}{2}$	*2*
P	*0*	*0*	$\frac{5}{16}$	$\frac{1}{8}$	*5*

Since there are no more negative coefficients in the objective row of Tableau 3 we conclude that the current solution is optimal.

The optimal solution

Once the optimality test is met (i.e. all coefficients in the objective row are nonnegative), we can extract the solution from the final tableau. The optimal solution consists of the basic decision variables appearing in the extreme left column. The corresponding values appear in the extreme right column. Any decision variable which is not in the basic set has a zero value.

Referring this to our particular example, the extreme columns of the final tableau give the solution

$$x_2 = 2, \quad x_1 = 2 \quad P = 5$$

which agrees with what we obtained earlier using the graphical approach.

We note what was mentioned earlier about the coefficients of the basic variables in the last row and the existence of a unit matrix in the tableau.

If we relate this procedure to the geometrical solution we observe the following movements: From the origin the search for the solution moved to the vertex $(0, 2)$ then to $(2, 2)$.

We will now solve the following linear programming problem to illustrate the implementation of the complete Simplex algorithm.

Example 0.1.9
Consider the following linear programming problem in standard form

$$\begin{array}{ll} \textit{Maximise} & z = 120x_1 + 100x_2 \\ \textit{Subj. to} & 2x_1 + 2x_2 + x_3 = 8 \\ & 5x_1 + 3x_2 + x_4 = 15 \\ & x_1,\, x_2,\, x_3,\, x_4 \geq 0 \end{array}$$

The initial tableau is shown in Tableau 1. The initial basic feasible solution is obtained by setting $x_1 = x_2 = 0$, so that $x_3 = 8$ and $x_4 = 15$.

Tableau 1

	x_1	x_2	x_3	x_4	
x_3	2	2	1	0	8
x_4	$\boxed{5}$	3	0	1	15
z	-120	-100	0	0	0

The initial solution is not optimal since there are negative coefficients in the z-row.
The entering variable is x_1 corresponding to the most negative coefficient, -120.
The required quotients to determine the leaving variable are $\frac{8}{2} = 4$, and $\frac{15}{3} = 3$ of which 3 is smaller.
Hence x_4 is the leaving variable. The pivot element is therefore 5.

In pivoting, x_1 now replaces x_4 in the extreme left column. The new x_1-row entries are obtained by dividing by 5. The variable x_3 remains in the basic solution but the coefficients in the corresponding row are obtained by carrying out a row operations on x_3-row and z-row so that the x_1-column entries are zero except for the pivot coefficient.

The new tableau is shown in Tableau 2.

Tableau 2

	x_1	x_2	x_3	x_4	
x_3	*0*	$\boxed{\frac{4}{5}}$	*1*	$-\frac{2}{5}$	2
x_1	*1*	$\frac{3}{5}$	*0*	$\frac{1}{5}$	3
z	0	-28	0	24	360

The new basic feasible solution, $x_1 = 3$, $x_2 = 0$, is not optimal. Hence we continue pivoting. The next variable to enter the basis is x_2. The leaving variable is x_3 (quotients are 5/2 and 5). Pivoting on $\frac{4}{5}$ leads to the new tableau shown in Tableau 3.

Tableau 3

	x_1	x_2	x_3	x_4	
x_2	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$
z	0	0	35	10	430

Since all the entries in the z-row of the last tableau, Tableau 3 are positive, we deduce that optimality has been reached.

The solution is obtained from reading the first and last columns of the tableau. On the first column, the basic variables are x_1 and x_2. The corresponding values in the last column are $\frac{3}{2}$ and $\frac{5}{2}$. The maximum value of z is 430, corresponding to z in the last column.

Hence the optimal solution is

$$x_1 = \frac{5}{2}, \quad x_2 = \frac{3}{2}, \quad z = 430$$

Example 0.1.10

$$
\begin{array}{ll}
\textit{Maximize} & P = 70x_1 + 50x_2 + 35x_3 \\
\textit{subject to} & 4x_1 + 3x_2 + x_3 \le 240 \\
& 2x_1 + x_2 + x_3 \le 100 \\
& -4x_1 + x_2 \le 0 \\
& x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.
\end{array}
$$

Solution:

We add the slack variables x_4, x_5 and x_6 to convert the problem to standard form.

$$
\begin{array}{ll}
\textit{Maximize} & P = 70x_1 + 50x_2 + 35x_3 \\
\textit{subject to} & 4x_1 + 3x_2 + x_3 + x_4 = 240 \\
& 2x_1 + x_2 + x_3 + x_5 = 100 \\
& -4x_1 + x_2 + x_6 = 0 \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \ge 0
\end{array}
$$

The initial Tableau is shown below

Tableau 1

Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*
x_4	4	3	1	1	0	0	240
x_5	$\boxed{2}$	1	1	0	1	0	100
x_6	-4	1	0	0	0	1	0
P	-70	-50	-35	0	0	0	0

The initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$ so that $x_4 = 240$, $x_5 = 100$ and $x_6 = 0$. This solution is not optimal since there are negative coefficients in the last row containing P. The entering variable is x_1 corresponding to the most negative coefficient, -70. The quotients are $\frac{100}{2} = 50$ and $\frac{240}{4} = 60$ of which 50 is the smallest (note that we don't consider the quotient $\frac{0}{-4} = 0$). Thus x_5 is the leaving variable and the pivot element is 2.

Dividing row 2 by 2 gives

	Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*
$R_1:$	x_4	4	3	1	1	0	0	240
$R_2:$	x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	50
$R_3:$	x_6	-4	1	0	0	0	1	0
$R_4:$	P	-70	-50	-35	0	0	0	0

To obtain Tableau 2 we perform the following row operations

$$-4R_2 + R_1, \quad 4R_2 + R_3, \quad 70R_2 + R_4$$

This gives

Tableau 2

Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*
x_4	0	$\boxed{1}$	-1	1	-2	0	40
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	50
x_6	0	3	2	0	2	1	200
P	0	-15	0	0	35	0	3500

Since -15 is the only negative coefficient in the P-row, it follows that the entering variable is x_2. The quotients required to determine the leaving variable are $\frac{200}{3} = 66\frac{2}{3}$, $\frac{50}{\frac{1}{2}} = 100$, $\frac{40}{1} = 40$ of which 40 is the smallest. Thus x_4 is the leaving variable and 1 is the pivot coefficient.

To obtain Tableau 4 we perform the following row operations

$$-\frac{1}{2}R_1 + R_2, \qquad -3R_1 + R_3, \qquad 15R_1 + R_4$$

This gives

Tableau 3

Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*
x_2	0	1	-1	1	-2	0	40
x_1	1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	30
x_6	0	0	$\boxed{5}$	-3	8	1	80
P	0	0	-15	15	5	0	4100

Looking at the coefficient of the P-row in Tableau 3 we note that the only negative coefficient is -15. Thus, x_3 is the entering variable. The quotients are $\frac{80}{5} = 16$, $\frac{30}{1} = 30$. Thus, x_6 is the leaving variable and 5 is the pivot coefficient.

Dividing row 3 by 5 gives

Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*
x_2	0	1	-1	1	-2	0	40
x_1	1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	30
x_3	0	0	1	$-\frac{3}{5}$	$\frac{8}{5}$	$\frac{1}{5}$	16
P	0	0	-15	15	5	0	4100

To obtain Tableau 4 we perform the following row operations

$$R_3 + R_1, \qquad -R_3 + R_2, \qquad 15R_3 + R_4$$

Tableau 4

Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*
x_2	0	1	0	$\frac{2}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	56
x_1	1	0	0	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{1}{5}$	14
x_3	0	0	1	$-\frac{3}{5}$	$\frac{8}{5}$	$\frac{1}{5}$	16
P	0	0	0	6	39	3	4340

Since all the entries in the P-row of Tableau 4 are positive, we deduce that the optimal solution has been reached. The solution is

$$x_1 = 14, \quad x_2 = 56, \quad x_3 = 16 \quad \textit{and} \quad \textit{maximum } P = 4340$$

Special cases in the simplex procedure

Certain situations may arise which do not comply to the assumptions so far made in implementing the Simplex procedure. Some of these situations and how they are handled are discussed below.

Tie breaking

It may happen that during pivoting there is a tie in the entering and leaving variables; i.e. the most negative coefficient of the z-equation appears under more than one variable; or the smallest quotient corresponds to more than one variable. Normally the tie is broken (called tie breaking) by making an arbitrarily selection of the entering or leaving variables among those that qualify.

Unbounded solution

Unboundedness describes linear programs that do not have finite solutions.Under very rare occasions in the Simplex method it may turn out that every coefficient in the pivot column is either zero or negative (called the unbounded solution situation). Hence there would be no way of computing a leaving variable. In this case, it may be necessary to check if there has been no computational errors or else z would be unbounded.

0.1.9 Summary

In this section we solved maximization linear programming problems using the Simplex method with slack variables. The Simplex method is a very practical way of solving linear programming involving two or more decision variables.

0.1.10 Exercises 3.3: The Simplex method

Solve the following LP problems using the Simplex Method

1.

$$\begin{aligned} \text{maximize} \quad P = 70x_1 &+ 50x_2 \\ \text{subject to} \quad 4x_1 + 3x_2 &\leq 240 \\ 2x_1 + x_2 &\leq 100 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$P = 4100, x_1 = 30, x_2 = 40$$

2.

$$\begin{aligned} \text{maximize} \quad P = 10x_1 &+ 5x_2 \\ \text{subject to} \quad 4x_1 + x_2 &\leq 28 \\ 2x_1 + 3x_2 &\leq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$P = 80, x_1 = 6, x_2 = 4$$

3.

$$\begin{aligned} \text{maximize} \quad P = 70x_1 &+ 50x_2 + 35x_3 \\ \text{subject to} \quad 4x_1 + 3x_2 + x_3 &\leq 240 \\ 2x_1 + x_2 + x_3 &\leq 100 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$P = 4550, x_1 = 0, x_2 = 70, x_3 = 30$$

4.

$$\begin{aligned} \text{maximize} \quad P = 2x_1 &+ x_2 \\ \text{subject to} \quad 5x_1 + x_2 &\leq 9 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$P = 6, x_1 = 1, x_2 = 4$$

5.

$$\begin{aligned} \text{maximize} \quad P = 30x_1 &+ 40x_2 \\ \text{subject to} \quad 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 7 \\ x_1 + 2x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$P = 260, x_1 = 2, x_2 = 5$$

6.

$$
\begin{array}{rrcl}
\text{maximize} & P = -x_1 & + & 2x_2 \\
\text{subject to} & -x_1 + x_2 & \leq & 2 \\
 & -x_1 + 3x_2 & \leq & 12 \\
 & x_1 - 4x_2 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

$P = 7, x_1 = 3, x_2 = 5$

0.1.11 The minimisation problem : Dual problem

We have so far discussed the Simplex method as applied to solving a maximizing linear programming problem. One way to solve a minimisation problem is to solve an equivalent maximising problem called the **dual problem**. The theory of duality simply states that every linear programming problem can be written in two forms: the **primal** form and the **dual** form. The original problem is called the **primal problem**. The objective of a dual problem is opposite that of the given primal problem. Thus a primal minimisation problem has a dual maximisation problem. The same holds for a maximisation problem. That is, a primal maximisation problem has a dual minimisation problem.

Sometimes it is easier to solve the dual problem than it is to solve the primal problem. The relationship between the two types of problems is given in the following statement.

There is an important result called the Von Neumann duality principle which relates the optimal value of the dual problem to that of the primal problem. The statement of the result is that **the optimal solution of a primal linear programming problem, if it exists, has the same value at the optimal solution of the dual problem.** Thus the optimal value determined for the dual problem is the same optimal value for the primal problem.

Solving a Minimization problem

A minimization problem is in standard form if the objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

is to be minimized subject to the constraints.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. To solve this problem we use the following steps.

1. Form the **augmented matrix** for the given system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ & & & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ c_1 & c_2 & \cdots & c_n & \vdots & 1 \end{bmatrix}$$

2. Form the **transpose** of this matrix.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & \vdots & c_1 \\ a_{12} & a_{22} & \cdots & a_{m2} & \vdots & c_2 \\ & & & & \vdots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} & \vdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ b_1 & b_2 & \cdots & b_m & \vdots & 1 \end{bmatrix}$$

3. Form the dual maximization problem corresponding to this standard matrix. That is, find the maximum of the objective function given by

$$w = b_1y_1 + b_2y_2 + \ldots + b_my_n$$

subject to the constraints.

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m &\leq c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m &\leq c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m &\leq c_n \end{aligned}$$

where $y_1 \geq 0$, $y_2 \geq 0$,... and $y_m \geq 0$ and $y_m \geq 0$

4. Apply the **Simplex Method** to the dual maximization problem. The maximization value of w will be the minimum value of z. Moreover, the values of x_1, x_2, ... and x_n will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

We will illustrate the concept of duality by way of a minimization linear programming problem.

Constructing the dual problem

Example 0.1.11

Consider the minimization problem

$$\begin{aligned} \textit{Minimize}: \quad & C = 16\,x_1 + 45\,x_2 \\ \textit{Subject to}: \quad & 2x_1 + 5x_2 \geq 50 \\ & x_1 + 3x_2 \geq 27 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{14}$$

The following steps are involved in constructing the dual problem from a given primal problem.

1. Construct a special augmented matrix from the constraints coefficients of the primal problem without introducing slack/surplus variables and append the objective coefficients.

$$\begin{aligned} 2x_1 + 5x_2 &\geq 50 \\ x_1 + 3x_2 &\geq 27 \\ 16x_1 + 45x_2 &= C \end{aligned} \qquad A = \left[\begin{array}{cc|c} 2 & 5 & 50 \\ 1 & 3 & 27 \\ \hline 16 & 45 & 1 \end{array}\right]$$

2. Obtain the transpose of the augmented matrix

$$A^T = \left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right]$$

3. Write out the dual problem from the transpose matrix. This new problem will always be a maximization problem with $\leq$ problem constraints. To avoid confusion, we shall use different variables in this new problem:

$$\left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right] \qquad \begin{array}{c} 2y_1 + y_2 \leq 16 \\ 5y_1 + 3y_2 \leq 45 \\ 50y_1 + 27y_2 = P \end{array}$$

The dual of the minimization problem is the following maximization problem:

$$\begin{array}{lr} \text{Maximize} & P = 50y_1 + 27y_2 \\ \text{Subject to} & 2y_1 + y_2 \leq 16 \\ & 5y_1 + 3y_2 \leq 45 \\ & y_1 \geq 0, \quad y_2 \geq 0 \end{array}$$

4. Solve the dual problem in the usual way.

Note the following changes when constructing the dual problem, in addition to the change of notation:

1. The objective becomes the opposite of that of the primal problem.
2. $\leq$ signs become $\geq$ and vice versa.
3. There are as many decision variables in the dual problem as there are constraints in the primal problem.
4. There are as many constraints in the dual problem as there are decision variables in the primal problem.
5. The objective coefficients of primal problem become the right side (resource) values of the dual problem.
6. The right side (resource) values of the primal problem become the objective coefficients of the dual problem.

Example 0.1.12
Form the dual problem of

$$\begin{array}{lr} \textit{Minimize} & C = 40x_1 + 12x_2 + 40x_3 \\ \textit{Subject to} & 2x_1 + x_2 + 5x_3 \geq 20 \\ & 4x_1 + x_2 + x_3 \geq 30 \\ x_1, x_2, x_3 \geq 0 & \end{array}$$

Step 1. Form the matrix A

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 5 & 20 \\ 4 & 1 & 1 & 30 \\ \hline 40 & 12 & 40 & 1 \end{array}\right]$$

Step 2. Find the transpose of A, A^T.

$$A^T = \left[\begin{array}{cc|c} 2 & 4 & 40 \\ 1 & 1 & 12 \\ 5 & 1 & 40 \\ \hline 20 & 30 & 1 \end{array}\right]$$

Step 3. State the dual problem.

$$\begin{array}{lr} \textit{Maximize} & P = 20y_1 + 30y_2 \\ \textit{Subject to} & 2y_1 + 4y_2 \leq 40 \\ & y_1 + y_2 \leq 12 \\ & 5y_1 + y_2 \leq 40 \\ & y_1,\ y_2 \geq 0 \end{array}$$

In the next example we solve a minimization problem by solving its dual.

Example 0.1.13
Find the minimum value of

$$C = 3x_1 + 2x_2 \qquad \textit{Objective function}$$

subject to the constraints

$$\left.\begin{array}{ccccc} 2x_1 & + & x_2 & \geq & 6 \\ x_1 & + & x_2 & \geq & 4 \end{array}\right\} \qquad \textit{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution:

The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 2 & 1 & \vdots & 6 \\ 1 & 1 & \vdots & 4 \\ \cdots & \cdots & \vdots & \cdots \\ 3 & 2 & \vdots & 1 \end{bmatrix}$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\begin{bmatrix} 2 & 1 & \vdots & 3 \\ 1 & 1 & \vdots & 2 \\ \cdots & \cdots & \vdots & \cdots \\ 6 & 4 & \vdots & 1 \end{bmatrix}$$

This implies that the dual maximization problem is as follows.

Dual maximization problem: *Find the maximum value of*

$$P = 6y_1 + 4y_2$$

Subject to the constraints

$$\left.\begin{array}{rcrcl} 2y_1 & + & y_2 & \leq & 3 \\ y_1 & + & y_2 & \leq & 2 \end{array}\right\}$$

where $y_1 \geq 0$ and $y_2 \geq 0$.

After writing the dual problem in standard form we obtain the initial tableau

Tableau 1

Basic	y_1	y_2	y_3	y_4	
y_3	$\boxed{2}$	*1*	*1*	*0*	*3*
y_4	*1*	*1*	*0*	*1*	*2*
P	-6	-4	*0*	*0*	*0*

We see from the tableau that the pivot column is the y_1-column. The quotients are $\frac{3}{2}$ and $\frac{2}{1} = 1$. Hence the y_3-row is the pivot row. Thus y_1 is the entering variable which replaces y_3, the leaving variable. The pivot element at the intersection of the pivot row and pivot column is 2. To update the tableau we

Performing the Gauss reductions we obtain Tableau 2 given below.

Tableau 2

Basic	y_1	y_2	y_3	y_4	
y_1	*1*	$\frac{1}{2}$	$\frac{1}{2}$	*0*	$\frac{3}{2}$
y_4	*0*	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	*1*	$\frac{1}{2}$
P	0	-1	*3*	*0*	*9*

We deduce that the current solution is not optimal. (Why?) Updating once more we obtain Tableau 3 given below

Tableau 3

Basic	y_1	y_2	y_3	y_4	
y_1	*1*	*0*	*1*	*-1*	*1*
y_2	*0*	*1*	*-1*	*2*	*1*
P	0	0	*2*	*2*	*10*

The current solution is optimal since all the coefficients in the last row are nonnegative.

Reading the solution of the primal problem

We are now going to extract the solution of the primal problem from the final simplex tableau of the dual problem. The optimal objective value is

$$P = C = 10$$

Since the above final tableau is for the dual problem, we recall that in transposing the primal problem the objective coefficients of the original variables became the right-hand values of the constraints. This means that each original variable now corresponds to a slack variable. Thus we do not read the solution in the same way as for the primal simplex tableau.The optimal values of the original variables correspond to the slack variables in the final tableau of the dual problem.

For our particular example, the decision variables x_1 and x_2 of the primal problem correspond to the slack variables of the dual problem and their values are the corresponding coefficients in the last row of the final simplex tableau. Thus the solution is contained in the y_3 and y_4 columns and is

$$x_1 = 2, \qquad x_2 = 2$$

and the objective value is in the usual column i.e

$$y_1 = 1, \qquad y_2 = 1$$

.

Note that if we substitute the basic variables of the dual problem in the dual objective function we have

$$P = 6y_1 + 4y_2 = (6)(1) + (4)(1) = 10.$$

We get the same objective value if we substitute $x_1 = 2, \quad x_2 = 2$ into the primal objective function

$$C = 3x_1 + 2x_2 = (3)(2) + (2)(2) = 10$$

This verified the Von Neumann Optimality Principle.

0.1.12 Summary

In this section we solved minimization linear programming problems by forming the dual and solving it by the Simplex method with slack variables.

0.1.13 Exercises 3.4: The dual problem

Solve the following minimization problems by maximizing the Dual.

1.

$$\begin{aligned} \text{minimize} \quad C = 40x_1 + 12x_2 &+ 40x_3 \\ \text{subject to} \quad 2x_1 + x_2 + 5x_3 &\geq 20 \\ 4x_1 + x_2 + x_3 &\geq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$C = 320, \; x_1 = 5, x_2 = 10$$

2.

$$\begin{aligned} \text{minimize} \quad C = 21x_1 &+ 50x_2 \\ \text{subject to} \quad 2x_1 + 5x_2 &\geq 12 \\ 3x_1 + 7x_2 &\geq 17 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$C = 121, \; x_1 = 1, x_2 = 2$$

3.

$$\begin{aligned} \text{minimize} \quad C = 2x_1 + 10x_2 &+ 8x_3 \\ \text{subject to} \quad x_1 + x_2 + x_3 &\geq 6 \\ x_2 + 2x_3 &\geq 8 \\ -x_1 + 2x_2 + 2x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$C = 36, \; x_1 = 2, x_2 = 0, x_3 = 4$$

4.

$$\begin{aligned} \text{minimize} \quad C = 16x_1 + 9x_2 &+ 21x_3 \\ \text{subject to} \quad x_1 + x_2 + 3x_3 &\geq 12 \\ 2x_1 + x_2 + x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$C = 136, \; x_1 = 4, x_2 = 8, x_3 = 0$$