#418764
Topic: Continuity of a Function

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Solution

The given function is $f(x) = 5x - 3$

At $x = 0$, $f(0) = 5(0) - 3 = -3$

$\lim_{x \to 3} f(x) = \lim_{x \to 3} f(5x - 3) = 5(0) - 3 = -3$

$\therefore \lim_{x \to 3} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = -3$, $f(-3) = 5(-3) - 3 = -18$

$\lim_{x \to -3} f(x) = \lim_{x \to -3} (5x - 3) = 5(-3) - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)$

Therefore, f is continuous at $x = -3$

At $x = 5$, $f(x) = f(5) = 5(5) - 3 = 25 - 3 = 22$

$\lim_{x \to 5} f(x) = lim_{x \to 5}(5x - 3) = 5(5) - 3 = 22$

$\therefore \lim_{x \to 5} f(x) = f(5)$

Therefore, f is continuous at $x = 5$.

Hence f is continuous at all the given points (In fact f is continuous for all R, Since it is a polynomial)

#418768
Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$\sin(x^2 + 5)$

Solution

Let $f(x) = \sin(x^2 + 5)$

Thus using chain rule,

$f'(x) = \dfrac{d}{dx}(\sin(x^2 + 5)) \cdot \dfrac{d}{dx}(x^2 + 5)$

$= \cos(x^2 + 5) \cdot 2x = 2x\cos(x^2 + 5)$

#418771
Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$\cos(\sin x)$

Solution

Using chain rule,

$\dfrac{d}{dx}[\cos(\sin x)] = -\sin(\sin x). \dfrac{d}{dx}(\sin x)$

$= -\sin(\sin x). \cos x$

$= -\cos x \cdot \sin(\sin x)$

#418774
Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$\sin(ax + b)$

Solution

Let $y = \sin(ax + b)$

Thus using chain rule,

$\dfrac{dy}{dx} = \dfrac{d(\sin(ax + b))}{d(ax + b)} \cdot \dfrac{d(ax + b)}{dx} = a\cos(ax + b)$

#418777

Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$\sec(\tan(\sqrt{x}))$

Solution

Let $f(x) = \sec(\tan(\sqrt{x}))$

Thus using chain rule, we get

$f'(x) = \dfrac{d[\sec(\tan(\sqrt{x}))]}{d(\tan(\sqrt{x}))} \cdot \dfrac{d(\tan(\sqrt{x}))}{d(\sqrt{x})} \cdot \dfrac{d(\sqrt{x})}{dx}$

$= \sec(\tan(\sqrt{x})\tan(\tan(\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}}$

#418783

Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$\dfrac{\sin(ax + b)}{\cos(cx + d)}$

Solution

We have, $f(x) = \dfrac{\sin(ax + b)}{\cos(cx + d)}$

Thus using quotient rule and chain rule simualtaneously,

$f'(x) = \dfrac{a\cos(ax + b).\cos(cx + d) - \sin(ax + b)(-c\sin(cx + d))}{[\cos(cx + d)]^2}$

$= \dfrac{a\cos(ax + b)}{\cos(cx + d)} + c\sin(ax + b).\dfrac{\sin(cx + d)}{\cos(cx + d)} \times \dfrac{1}{\cos(cx + d)}$

$= a\cos(ax + b)\sec(cx + d) + c\sin(ax + b)\tan(cx + d)\sec(cx + d)$

#418927

Topic: Continuity of a Function

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Solution

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n and its value at n is n^n.

Now , $\lim_{x \to n} f(n) = \lim_{x \to n}(x^n) = n^n = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

#419003

Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$2\sqrt{\cot(x^2)}$

Solution

$\frac{d}{dx}\left[2\sqrt{\cot(x^2)}\right]$

$= 2.\frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx}\left[\cot(x^2)\right]$

$= \sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times -cosec^2(x^2) \times \frac{d}{dx}(x^2)$

$= -\sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times -\frac{1}{\sin^2(x^2)} \times (2x)$

$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$

$= \frac{-2\sqrt{2}x}{\sqrt{2\sin x^2 \cos x^2}\sin x^2}$

$= \frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin(2x^2)}}$

#419009

Topic: Chain and Reciprocal Rule

Differentiate the function with respect to x

$\cos(\sqrt{x})$

Solution

$\frac{d}{dx}[\cos(\sqrt{x})] = -\sin(\sqrt{x}).\frac{d}{dx}(\sqrt{x})$

$= -\sin(\sqrt{x}) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = -\sin\sqrt{x} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

#419072

Topic: Algebra of Derivative of Functions

Find $\frac{dy}{dx}$ of $2x + 3y = \sin x$

Solution

$2x + 3y = sinx$

Differentiating both sides w.r.t. x, we obtain

$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x)$

$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$

$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$

$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$

$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$

#419116

Topic: Differentiation of Implicit Functions

Find $\frac{dy}{dx}$ of $2x + 3y = \sin y$

Solution

Given, $2x + 3y = \sin y$

Differentiating both sides respect to x, we get,

$\Rightarrow \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(3y) = \dfrac{d}{dx}(\sin y)$

$\Rightarrow 2 + 3\dfrac{dy}{dx} = \cos y\dfrac{dy}{dx}$

$\Rightarrow 2 = (\cos y - 3)\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{\cos y - 3}$

#419125

Topic: Differentiation of Implicit Functions

Find $\dfrac{dy}{dx}$ of $ax + by^2 = \cos y$

Solution

$ax + by^2 = \cos y$

Differentiating both sides with respect to x, we obtain

$\dfrac{d}{dx}(ax) + \dfrac{d}{dx}(by^2) = \dfrac{d}{dx}(\cos y)$

$\Rightarrow a + b \times 2y\dfrac{dy}{dx} = -\sin y\dfrac{dy}{dx}$

$\Rightarrow (2by + \sin y)\dfrac{dy}{dx} = -a$

$\therefore \dfrac{dy}{dx} = \dfrac{-a}{2by + \sin y}$

#419136

Topic: Differentiation of Implicit Functions

Find $\dfrac{dy}{dx}$ of $xy + y^2 = \tan x + y$

Solution

$xy + y^2 = \tan x + y$

Differentiating both sides w.r.t x we get,

$\Rightarrow y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = \sec^2 x + \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\sec^2 x - y}{x + 2y - 1}$

#419167

Topic: Differentiation of Implicit Functions

Find $\dfrac{dy}{dx}$ of $x^2 + xy + y^2 = 100$

Solution

$x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t x we get,

$\dfrac{d}{dx}(x^2 + xy + y^2) = \dfrac{d}{dx}(100) = 0$

$\Rightarrow 2x + y.1 + x.\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$

$\Rightarrow 2x + y + (x + 2y)\dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$

#419205

Topic: Differentiation of Implicit Functions

Find $\dfrac{dy}{dx}$ of $x^3 + x^2y + xy^2 + y^3 = 81$

Solution

$x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t x we get,

$\frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) = \frac{d}{dx}(81)$

$\Rightarrow 3x^2 + \left[y.\,2x + x^2.\,\frac{dy}{dx}\right] + \left[y^2.\,1 + x.\,2y.\,\frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx} + (3x^2 + 2xy + y^2) = 0$

$\therefore \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2}$

#419224

Topic: Differentiation of Implicit Functions

Find $\frac{dy}{dx}$ of $\sin^2 y + \cos xy = k$

Solution

We have, $\sin^2 y + \cos xy = k$

Differentiating both sides with respect to x, we obtain

$\Rightarrow \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = \frac{d(\pi)}{dx} = 0$ .....(1)

Using chain rule, we obtain

$\frac{d}{dx}(\sin^2 y) = 2\sin y\frac{d}{dx}(\sin y) = 2\sin y\cos y\frac{dy}{dx}$...... (2)

and

$\frac{d}{dx}(\cos xy) = -\sin xy\frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$

$= -\sin xy\left[y.\,1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx}$.....(3)

From (1), (2) and (3), we obtain

$2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$

$\Rightarrow (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$

$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$

$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$

#419233

Topic: Differentiation of Implicit Functions

Find $\frac{dy}{dx}$ of $\sin^2 x + \cos^2 y = 1$

Solution

We have $\sin^2 x + \cos^2 y = 1$

Differentiating both sides w.r.t. x, we obtain

$\frac{d}{dx}(\sin^2 x + \cos^2 y) = \frac{d}{dx}(1)$

$\Rightarrow 2\sin x.\,\frac{d}{dx}(\sin x) + 2\cos y.\,\frac{d}{dx}(\cos y) = 0$

$\Rightarrow 2\sin x\cos x + 2\cos y(-\sin y).\,\frac{dy}{dx} = 0$

$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$

#419276

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}$

#419286

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Solution

$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$= \tan^{-1}\tan(3\theta) = 3\theta = 3\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$

#419290

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \cos^{-1}\left(\frac{1 - x^2}{1+x^2}\right), 0 < x < 1$

Solution

$y = \cos^{-1}\left(\frac{1 - x^2}{1+x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow y = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}\cos(2\theta) = 2\theta = 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$

#419307

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \sin^{-1}\left(\frac{1 - x^2}{1+x^2}\right), 0 < x < 1$

Solution

$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta \Rightarrow y = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}\cos(2\theta) = \frac{\pi}{2} - 2\theta$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x \therefore \frac{dy}{dx} = -\frac{2}{1+x^2}$

#419357

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right),\ -1 < x < 1$

Solution

$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta \Rightarrow y = \frac{\pi}{2} - \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow y = \frac{\pi}{2} - \sin^{-1}\sin(2\theta) = \frac{\pi}{2} - 2\theta$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x \therefore \frac{dy}{dx} = -\frac{2}{1+x^2}$

#419893

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \sin^{-1}(2x\sqrt{1-x^2}),\ -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Solution

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

Then we have,

$y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$

$\therefore \frac{dy}{dx} = 2\cdot\frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$

#419895

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$ of $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right),\ 0 < x < \frac{1}{\sqrt{2}}$

Solution

$y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right) = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$,

Then we have,

$y = \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\cos^{-1}x$

$\therefore \frac{dy}{dx} = 2 \cdot \frac{-1}{\sqrt{1 - x^2}} = -\frac{2}{\sqrt{1 - x^2}}$

#419900

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

e^{x^3}

Solution

Let $y = e^{x^3}$

By using the chain rule, we obtain

$\frac{dy}{dx} = \frac{d}{dx}(e^{x^3})$

$= e^{x^3}\frac{d}{dx}(x^3)$

$= e^{x^3}.3x^2$

$= 3x^2e^{x^3}$

#420860

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

$y = \log(\cos e^x)$

Solution

Let $y = \log(\cos e^x)$

Thus using chain rule,

$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\cos e^x)\right]$

$= \frac{1}{\cos e^x}.\frac{d}{dx}(\cos e^x)$

$= \frac{1}{\cos e^x}.(-\sin e^x).\frac{d}{dx}(e^x)$

$= \frac{-\sin e^x}{\cos e^x}.e^x$

$= -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \neq N$

#420863

Topic: Algebra of Derivative of Functions

Differentiate the given function w.r.t. x.

$y = e^x + e^{x^2} + \ldots + e^{x^5}$

Solution

$\dfrac{d}{dx}(e^x + e^{x^2} + \ldots + e^{x^5})$

$= \dfrac{d}{dx}(e^x) + \dfrac{d}{dx}(e^{x^2}) + \dfrac{d}{dx}(e^{x^3}) + \dfrac{d}{dx}(e^{x^4}) + \dfrac{d}{dx}(e^{x^5})$

$= e^x + \left[e^{x^2} \cdot \dfrac{d}{dx}(x^2)\right] + \left[e^{x^3} \cdot \dfrac{d}{dx}(x^3)\right] + \left[e^{x^4} \cdot \dfrac{d}{dx}(x^4)\right] + \left[e^{x^5} \cdot \dfrac{d}{dx}(x^5)\right]$

$= e^x + (e^{x^2} \times 2x) + (e^{x^3} \times 3x^2) + (e^{x^4} \times 4x^3) + (e^{x^5} \times 5x^4)$

$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$

#420871

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

$y = \sqrt{e^{\sqrt{x}}}, x > 0$

Solution

Let $y = \sqrt{e^{\sqrt{x}}}, x > 0$

$\Rightarrow y^2 = e^{\sqrt{x}}$

Differentiating both sides w.r.t x

$\Rightarrow 2y\dfrac{dy}{dx} = e^{\sqrt{x}}\dfrac{d}{dx}(\sqrt{x})$ [By applying the chain rule]

$\Rightarrow 2y\dfrac{dy}{dx} = e^{\sqrt{x}}\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{4y\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$

#420950

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

$\log(\log x), x > 1$

Solution

Let $y = \log(\log x)$

Thus using chain rule,

$\dfrac{dy}{dx} = \dfrac{d}{dx}[\log(\log x)]$

$= \dfrac{1}{\log x} \cdot \dfrac{d}{dx}(\log x)$

$= \dfrac{1}{\log x} \cdot \dfrac{1}{x}$

$= \dfrac{1}{x\log x}, x > 1$

#420960

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

$\cos(\log x + e^x), x > 0$

Solution

Let $y = \cos(\log x + e^x), x > 0$

Thus using chain rule,

$$\frac{dy}{dx} = -\sin(\log x + e^x).\frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x).\left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + e^x).\left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x), x > 0$$

#421000

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$(\log x)^{\cos x}$

Solution

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$\log y = \cos x.\log(\log x)$ ..... [$\because \log a^x = x\log a$]

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = -\sin x\log(\log x) + \cos x \times \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\sin x\log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x\log(\log x)\right]$$

#421021

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$x^x - 2^{\sin x}$

Solution

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u = x \log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$

$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x \log 2$

Differentiating both sides with respect to x, we obtain

$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 . \frac{d}{dx}(\sin x)$

$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$

$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$

$\therefore \frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$

#421037

Topic: Continuity of a Function

Examine the continuity of f, where f is defined by

$f(x) = \begin{cases} \sin x - \cos x, \text{ if } x \neq 0 \\ -1, \text{ if } x = 0 \end{cases}$

Solution

The given function is $f(x) = \begin{cases} \sin x - \cos x, \text{ if } x \neq 0 \\ -1, \text{ if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$\lim_{x \to c} f(x) = \lim_{x \to c}(\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x \neq 0$

Case II

If $c = 0$, then $f(0) = -1$

$\lim_{x \to 0} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x \to 0} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

#421053

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$$\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

Solution

Let $y=\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$

Also, let $u=\left(x+\frac{1}{x}\right)^{x}$ and $v=x^{\left(1+\frac{1}{x}\right)}$

$\therefore y=u+v$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(1)

Then, $u=\left(x+\frac{1}{x}\right)^{x}$

$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^{x}$

$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$

$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$

$\Rightarrow \frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$

$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$

$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$

$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]$ ...(2)

$v=x^{\left(1+\frac{1}{x}\right)}$

$\Rightarrow \log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$

$\Rightarrow \log v=\left(1+\frac{1}{x}\right)\log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$

$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$

$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$

$$\Rightarrow \frac{dv}{dx} = v\left[-\frac{\log x + x + 1}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[-\frac{x + 1 - \log x}{x^2}\right] \quad ...(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\Rightarrow \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x\left[\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[-\frac{x + 1 - \log x}{x^2}\right]$$

#421070

Topic: Continuity of a Function

Find the values of k so that the function f is continuous at the indicated point:

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & x \neq \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Solution

Given definition of f is

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & if\ x \neq \dfrac{\pi}{2} \\ 3, & if\ x = \dfrac{\pi}{2} \end{cases} \quad \text{.... (1)}$$

Since, f is continuous at $x = \frac{\pi}{2}$

So, $LHL = RHL = f\left(\frac{\pi}{2}\right)$ .... (2)

Now, $LHL = \lim_{x \to \frac{\pi}{2}^-} f(x)$

$= \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right)$

$= \lim_{h \to 0} \dfrac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$

$= \lim_{h \to 0} \dfrac{k\sin h}{2h}$

$= \dfrac{k}{2} \lim_{h \to 0} \dfrac{\sin h}{h}$

$= \dfrac{k}{2} \quad \left(\because \lim_{x \to 0} \dfrac{\sin x}{x} = 1\right)$

Also, by (1)

$f\left(\frac{\pi}{2}\right) = 3$

Substituting these values in (2), we get

$\frac{k}{2} = 3$

$\Rightarrow k = 6$

#421076

Topic: Continuity of a Function

Find the values of k so that the function f is continuous at the indicated point:

$$f(x) = \begin{cases} kx^2, & if\ x \leq 2 \\ 3, & if\ x > 2 \end{cases} \quad \text{at } x = 2$$

Solution

The given function is $f(x) = \begin{cases} kx^2, \text{ if } x \le 2 \\ 3, \text{ if } x > 2 \end{cases}$

The given function f is continuous at $x = 2$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2}(kx^2) = \lim_{x \to 2}(3) = 4k$

$\Rightarrow k \cdot 2^2 = 3 = 4k$

$\Rightarrow 4k = 3 = 4k$

$\Rightarrow 4k = 3$

$\Rightarrow k = \frac{3}{4}$

Therefore, the required value of k is $\frac{3}{4}$.

#421086

Topic: Continuity of a Function

Find the values of k so that the function f is continuous at the indicated point:

$f(x) = \begin{cases} kx^2, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

at $x = \pi$

Solution

For $f(x)$ to be continuous at $x = \pi$

$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$

$\Rightarrow k\pi^2 = \cos\pi$

$\Rightarrow k\pi^2 = -1$

$\Rightarrow k = -\frac{1}{\pi^2}$

#421091

Topic: Continuity of a Function

Find the values of k so that the function f is continuous at the indicated point:

$f(x) = \begin{cases} kx + 1, \ if x \le 5 \\ 3x - 5, \ if x > 5 \end{cases}$ at $x = 5$

Solution

The given function is $f(x) = \begin{cases} kx + 1, & \text{if } x \le 5 \\ 3x - 5, & \text{if } x > 5 \end{cases}$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of f at $x = 5$ equals the limit of f at $x = 5$

It is evident that f is defined at $x = 5$ and $f(5) = kx + 1 = 5k + 1$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$

$\Rightarrow \lim_{x \to 5}(kx + 1) = \lim_{x \to 5}(3x - 5) = 5k + 1$

$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$

$\Rightarrow 5k + 1 = 10$

$\Rightarrow 5k = 9 \Rightarrow k = \frac{9}{5}$

Therefore, the required value of k is $\frac{9}{5}$

#421104

Topic: Continuity of a Function

Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & if\, x \le 2 \\ ax + b, & if\, 2 < x < 10 \\ 21, & if\, x \ge 10 \end{cases}$$ is a continuous function.

Solution

Given definition of f is

$$f(x) = \begin{cases} 5, & if\, x \le 2 \\ ax + b, & if\, 2 < x < 10 \\ 21, & if\, x \ge 10 \end{cases} \quad .....(1)$$

Also, given $f(x)$ is a continuous function.

So, $f(x)$ is continuous at all points .

So, $f(x)$ is continuous at $x = 2$

$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$ ....(2)

Now, $RHL = \lim_{x \to 2^+} f(x)$

$= \lim_{h \to 0} f(2 + h)$

$= \lim_{h \to 0} a(2 + h) + b$

$\Rightarrow RHL = 2a + b$

Also, $f(2) = 5$

Substituting these values in (2), we get

$2a + b = 5$ ....(3)

Also, $f(x)$ is continuous at $x = 10$

$\therefore \lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$ ......(4)

Now, $LHL = \lim_{x \to 10^-} f(x)$

$= \lim_{h \to 0} f(10 - h)$

$= \lim_{h \to 0} a(10 - h) + b$

$= 10a + b$

Also, by (1), $f(10) = 21$

Substituting these values in eq (4), we get

$10a + b = 21$ .....(5)

Solving eq (3) from eqn (5), we get

$8a = 16$

$\Rightarrow a = 2$

Put this value in (3), we get

$2(2) + b = 5$

$\Rightarrow b = 1$

#421214

Topic: Rolle's Theorem and Intermediate Value Theorem

Verify Rolles Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$.

Solution

$f(x) = x^2 + 2x - 8$

$\Rightarrow f(x) = (x+4)(x-2)$

$f'(x) = 2x + 2$

We know every polynomial function is continuous and differentiable in R.

In particular f is continuous and differentiable in $[-4, 2]$

Also $f(-4) = f(2) = 0$

Thus there will exist $c \in (-4, 2)$ such that $f'(c) = 0$

$\Rightarrow 2c + 2 = 0 \Rightarrow c = -1 \in (-4, 2)$

Hence, Rolle's theorem verified.

#421216

Topic: Rolle's Theorem and Intermediate Value Theorem

Examine if Rolles theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolles theorem from these example ?

(i) $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Solution

Rolle's theorem holds for a function $f : [a, b] \to R$, if following three conditions holds-

(1) f is continuous on $[a, b]$

(2) f is differentiable on (a, b)

(3) $f(a) = f(b)$

Then, there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i)

Given $f(x) = [x]$ for $x \in [5, 9]$

Since, the greatest integer function is not continuous at integral values.

So, $f(x)$ is not continuous at $x = 5, 6, 7, 8, 9$

So, $f(x)$ is not continuous in $[5, 9]$.

Since, condition (1) does not holds ,so need to check the other conditions.

Hence, Rolle's theorem is not applicable on given function

(ii)

Given function $f(x) = [x]$ for $x \in [-2, 2]$

Since, the greatest integer function is not continuous at integral points.

So, $f(x)$ is not continuous at $x = -2, -1, 0, 1, 2$

$f(x)$ is not continuous in $[-2, 2]$.

(iii)

$f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

Also $f(1) = (1)^2 - 1 = 0$

and $f(2) = (2)^2 - 1 = 3$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolles Theorem.

Hence, Rolles Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

#421219

Topic: Lagrange's Mean Value Theorem

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Solution

The given function is $f(x) = x^2 - 4x - 3$

f being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ and derivative is $f'(x) = 2x - 4$.

Now $f(1) = 1^2 - 4 \cdot 1 - 3 = -6$, $f(4) = 4^2 - 4 \cdot 4 - 3 = -3$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{-3 - (-6)}{3} = \dfrac{3}{3} = 1$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \dfrac{5}{2} \in (1, 4)$

Hence, Mean Value Theorem is verified for the given function.

#421220

Topic: Lagrange's Mean Value Theorem

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval [a, b], where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Solution

The given function is $f(x) = x^3 - 5x^2 - 3x$

f being a polynomial function, so it is continuous in $[1, 3]$ and is differentiable in $(1, 3)$ whose derivative is $3x^2 - 10x - 3$.

$f(1) = 1^3 - 5 \cdot 1^2 - 3 \cdot 1 = -7$, $f(3) = 3^3 - 5 \cdot 3^2 - 3 \cdot 3 = -27$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{-27 - (-7)}{3 - 1} = -10$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$\Rightarrow 3c^2 - 10c - 3 = -10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow (c - 1)(3c - 7) = 0$

$\Rightarrow c = 1, \dfrac{7}{3}$, where $c = \dfrac{7}{3} \in (1, 3)$

Hence, Mean Value Theorem is verified for the given function and $c = \dfrac{7}{3} \in (1, 3)$ is the point for which $f'(c) = 0$

#421864

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x:

$(3x^2 - 9x + 5)^9$

Solution

Let $y = (3x^2 - 9x + 5)^9$

Using chain rule, we obtain

$\dfrac{dy}{dx} = \dfrac{d}{dx}(3x^2 - 9x + 5)^9$

$= 9(3x^2 - 9x + 5)^8 . \dfrac{d}{dx}(3x^2 - 9x + 5)$

$= 9(3x^2 - 9x + 5)^8 . (6x - 9)$

$= 9(3x^2 - 9x + 5)^8 . 3(2x - 3)$

$= 27(3x^2 - 9x + 5)^8(2x - 3)$

#421877

Topic: Algebra of Derivative of Functions

Differentiate the given function w.r.t. x:

$\sin^3 x + \cos^6 x$

Solution

Let $y = \sin^3 x + \cos^6 x$

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^3 x) + \dfrac{d}{dx}(\cos^6 x)$

$= 3\sin^2 x.\dfrac{d}{dx}(\sin x) + 6\cos^5 x.\dfrac{d}{dx}(\cos x)$

$= 3\sin^2 x.\cos x + 6\cos^5 x.(-\sin x)$

$= 3\sin x\cos x(\sin x - 2\cos^4 x)$

#421884

Topic: Logarithmic Differentiation

Differentiate the given function w.r.t. x:

$(5x)^{3\cos 2x}$

Solution

We have $y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{y}\dfrac{dy}{dx} = 3\left[\log 5x\dfrac{d}{dx}(\cos 2x) + \cos 2x\dfrac{d}{dx}(\log 5x)\right]$

$\Rightarrow \dfrac{d}{dx} = 3y\left[\log 5x(-\sin 2x).\dfrac{d}{dx}(2x) + \cos 2x.\dfrac{1}{5x}\dfrac{d}{dx}(5x)\right]$

$\Rightarrow \dfrac{dy}{dx} = 3x\left[-2\sin 2x \log 5x + \dfrac{\cos 2x}{x}\right]$

$\Rightarrow \dfrac{dy}{dx} = 3y\left[\dfrac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$

$\therefore \dfrac{dy}{dx} = (5x)^{3\cos 2x}\left[\dfrac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$

#421890

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\dfrac{dy}{dx}$

$x = 2at^2, y = at^4$

Solution

The given equations are $x = 2at^2, y = at^4$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}(2at^2) = 2a.\dfrac{d}{dt}(t)^2 = 2a.2t = 4at$

and $\dfrac{dy}{dt} = \dfrac{d}{dt}(at^4) = a.\dfrac{d}{dt}(t^4) = a.4.t^3 = 4at^3$

$\therefore \dfrac{dy}{dt} = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{4at^3}{4at} = t^2$

#421892

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\cos\theta, y = b\cos\theta$

Solution

The given equations are $x = a\cos\theta, y = b\cos\theta$

Then, $\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$

and $\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$

#422183

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \sin t, y = \cos 2t$

Solution

We hae $x = \sin t, y = \cos 2t$

$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

and $\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t. \frac{d}{dt}(2t) = -2\sin 2t$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2.\,2\sin t\cos t}{\cos t} = -4\sin t$

#422202

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 4t, y = \frac{4}{t}$

Solution

We have $x = 4t, y = \frac{4}{t}$

$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}(4t) = 4$

and $\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4. \frac{d}{dt}\left(\frac{1}{t}\right) = 4.\left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{-4}{t^2}}{4} = \frac{-1}{t^2}$

#422225

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

Solution

We have $x = \cos\theta - \cos2\theta,\ y = \sin\theta - \sin2\theta$

$\Rightarrow \dfrac{dx}{d\theta} = 2\sin2\theta - \sin\theta$

and $\dfrac{dy}{d\theta} = \cos\theta - 2\cos2\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \dfrac{\cos\theta - 2\cos2\theta}{2\sin2\theta - \sin\theta}$

#422410

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x:

$\cos(a\cos x + b\sin x)$, for some constant a and b.

Solution

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$\dfrac{dy}{dx} = \dfrac{d}{dx}\cos(a\cos x + b\sin x)$

$\Rightarrow \dfrac{dy}{dx} = -\sin(a\cos x + b\sin x).\ \dfrac{d}{dx}(a\cos x + b\sin x)$

$= -\sin(a\cos x + b\sin x).\ [a(-\sin) + b\cos x]$

$= (a\sin x - b\cos x).\ \sin(a\cos x + b\sin x)$

#422537

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\dfrac{dy}{dx}$.

$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

Solution

$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = a(1 - \cos\theta)$

& $\dfrac{dx}{d\theta} = a[0 + (-\sin\theta)] = -a\sin\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \dfrac{-a\sin\theta}{a(1-\cos\theta)} = \dfrac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \dfrac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\dfrac{\theta}{2}$

#422558

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\dfrac{dy}{dx}$.

$x = \dfrac{\sin^3 t}{\sqrt{\cos2t}},\ y = \dfrac{\cos^3 t}{\sqrt{\cos2t}}$

Solution

The given equations are $x = \dfrac{\sin^3 t}{\sqrt{\cos2t}},\ y = \dfrac{\cos^3 t}{\sqrt{\cos2t}}$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}\left[\dfrac{\sin^3 t}{\sqrt{\cos2t}}\right]$

$$= \frac{\sqrt{\cos 2t}.\ \frac{d}{dt}(\sin^3 t) - \sin^3 t.\ \frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.\ 3\sin^2 t\frac{d}{dt}(\sin t) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}}\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}.\ \sin^2 t\cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}.\ \frac{d}{dt}(\cos^3 t) - \cos^3 t.\ \frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.\ 3\cos^2 t.\ \frac{d}{dt}(\cos t) - \cos^3 t.\ \frac{1}{2\sqrt{\cos 2t}}.\ \frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}.\ \cos^2 t(-\sin t) - \cos^3 t.\ \frac{1}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos 2t\cos^2 t\sin t + \cos^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\ \cos^2 t.\ \sin t + \cos^3 t\sin 2t}{3\cos 2t.\ \sin^2 t.\ \cos t + \sin^3 t\sin 2t}$$

$$= \frac{-3\cos 2t.\ \cos^2 t.\ \sin t + \cos^3 t(2\sin t\cos t)}{3\cos 2t.\ \sin^2 t.\ \cos t + \sin^3 t(2\sin t\cos t)}$$

$$= \frac{\sin t\cos t[-3\cos 2t.\ \cos t + 2\cos^3 t]}{\sin t\cos t[3\cos 2t.\ \sin t + 2\sin^3 t]}$$

$$= \frac{[-3(2\cos^2 t - 1)\cos t + 2\cos^3 t]}{[3(1 - 2\sin^2 t)\sin t + 2\sin^3 t]}$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} = -\cot 3t$$

#422745

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$

Solution

The given equations are $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$

Then, $\frac{dx}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$

$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$

$= a\left[-\sin t + \cot\frac{t}{2} \cdot \sec^2\frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right)\right]$

$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}\right]$

$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$

$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$

& $\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$

#422761

Topic: Logarithmic Differentiation

Differentiate the given function w.r.t. x:

$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

Solution

Let $y = x^x + x^a + a^x + a^a$

Also let, $x^x = u, x^a = v, a^x = w,$ and $a^a = s$

$\therefore y = u + v + w + s$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx}$ .....(1)

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x\log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log x)$

$\Rightarrow \frac{du}{dx} = u\left[\log x.1 + x\frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + logx)$ .....(2)

$v = x^a$

$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$

$\Rightarrow \frac{dv}{dx} = ax^{a-1}$ .....(3)

$w = a^x$

$\Rightarrow \log w = \log a^x$

$\Rightarrow \log w = x\log a$

Differentiating both sides with respect to x, we obtain

$\frac{1}{w}\cdot\frac{dw}{dx} = \log a.\frac{d}{dx}(x)$

$\Rightarrow \frac{dw}{dx} = w\log a$

$\Rightarrow \frac{dw}{dx} = a^x\log a$ .....(4)

$s = a^a$

Since a is constant, a^a is also a constant.

$\therefore \frac{ds}{dx} = 0$ .....(5)

From (1), (2), (3), (4) and (5) we obtain

$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x\log a + 0$

$= x^x(1 + \log x) + ax^{a-1} + a^x\log a$

#422775

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\sec\theta, y = b\tan\theta$

Solution

$x = a\sec\theta, y = b\tan\theta$

$\Rightarrow \frac{dx}{d\theta} = a.\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

& $\frac{dy}{d\theta} = b.\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a} \times \frac{1}{\sin\theta} = \frac{b}{a}cosec\theta$

#422784

Topic: Differentiation of Functions in Parametric Form

If x and y are connected parametrically by the given equation, then without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

Solution

The given equations are $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

Then, $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$

$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$

& $\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left(\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta.\frac{d}{d\theta}(\theta)\right)\right]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

#422827

Topic: Differentiation of Functions in Parametric Form

Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

We have, $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$

$\Rightarrow \frac{dx}{dt} = 10(1 - \cos t)$

& $\frac{dy}{dt} = 12.[0 - (-\sin t)] = 12\sin t$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{12\sin t}{10(1-\cos t)} = \frac{12.2\sin\frac{t}{2}.\cos\frac{t}{2}}{10.\,2\sin^2\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$

#422844

Topic: Algebra of Derivative of Functions

Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $-1 \le t \le 1$

Solution

$y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

$\therefore \frac{dy}{dx} = \frac{d}{dx}[\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}]$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\sin^{-1}\sqrt{1-x^2})$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\cdot\frac{d}{dx}(\sqrt{1-x^2})$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}(1-x^2)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}(-2x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

#459518

Topic: Continuity of a Function

Discuss the continuity of the following function:

$f(x) = \sin x + \cos x$

Solution

Here clearly both $\sin x$ and $\cos x$ are defined in their domain.

Let's assume that $g(x) = \sin x$ and $f(x) = \cos x$

Let's first prove that $g(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rightarrow c$, then it means that $h \Rightarrow 0$

$\lim_{x \to c} g(x) = \lim_{x \to c} sin(x)$

Put $x = h + c$

And as mentioned above, when $x \longrightarrow c$ then it means that $h \longrightarrow 0$

Which gives us $\lim_{h \to 0} sin(c + h)$

Expanding $\sin(x + h) = \sin(h)\cos(c) + \cos(h)\sin(c)$

Which gives us $\lim_{h \to 0} \sin(h)\cos(c) + \cos(h)\sin(c)$

$= \sin(c)\cos(0) + \cos(c)\sin(0)$

$= \sin(c)$

So here we get,

And this proves that $\sin(x)$ is continuous all across its domain

Let's prove that $f(x) = \cos(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rrightarrow c$ then it means that $h \Rightarrow 0$

$f(c) = \cos(c)$

$\lim_{x \to c} f(x) = \lim_{x \to c} \cos(x)$

Put $x = h + c$

And as mentioned above, when $x \Rightarrow c$ then it means that $h \Rightarrow 0$

Which gives us $\lim_{h \to 0} \cos(c + h)$

Expanding $\cos(h + c) = \cos(h)\cos(c) - \sin(h)\sin(c)$

Which gives us $\lim_{h \to 0} \cos(h)\cos(c) - \sin(h)\sin(c)$

$= \cos(c)\cos(0) - \sin(c)\sin(0)$

$= \cos(c)$

This gives us

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin(x) = \sin(c) = g(c)$

And this proves that $\cos(x)$ is continuous all across its domain

So by theorem. If function f and function g are contnous then $f+g$ is also continous.

There for $\sin(x) + \cos(x)$ is continious.

#459519

Topic: Continuity of a Function

Discuss the continuity of the following function :

$f(x) = \sin x - \cos x$

Solution

Here clearly both $\sin x$ and $\cos x$ are defined in their domain.

Let's assume that $g(x) = \sin x$ and $f(x) = \cos x$

Let's first prove that $g(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rightarrow c$ then it means that $h \Rightarrow 0$

$\lim_{x \Rightarrow c} g(x) = \lim_{x \Rightarrow c} \sin(x)$

Put $x = h + c$

And as mentioned above, when $x \Rightarrow c$ then it means that $h \Rightarrow 0$

Which gives us $\lim_{h \Rightarrow 0} \sin(c + h)$

Expanding $\sin(x + h) = \sin(h)\cos(c) + \cos(h)\sin(c)$

Which gives us $\lim_{h \Rightarrow 0} \sin(h)\cos(c) + \cos(h)\sin(c)$

$= \sin(c)\cos(0) + \cos(c)\sin(0)$

$= \sin(c)$

So here we get

$\lim_{x \Rightarrow c} g(x) = \lim_{x \Rightarrow c} \sin(x) = \sin(c) = g(c)$

And this proves that $\sin(x)$ is continuous all across its domain

Let's prove that $f(x) = \cos(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rightarrow c$ then it means that $h \Rightarrow 0$

$f(c) = cos(c)$

$\lim_{x \Rightarrow c} f(x) = \lim_{x \Rightarrow c} cos(x)$

Put $x = h + c$

And as mentioned above, when $x \Rightarrow c$ then it means that $h \Rightarrow 0$

Which gives us $\lim_{h \Rightarrow 0} cos(c + h)$

Expanding $\cos(h + c) = \cos(h)\cos(c) - \sin(h)\sin(c)$

Which gives us $\lim_{h \Rightarrow 0} \cos(h)\cos(c) - \sin(h)\sin(c)$

$= \cos(c)\cos(0) - \sin(c)\sin(0)$

$= \cos(c)$

This gives us

$\lim_{x \Rightarrow c} g(x) = \lim_{x \Rightarrow c} \sin(x) = \sin(c) = g(c)$

And this proves that $\cos(x)$ is continuous all across its domain

So by theorem, if function f and function g are continous, then $f - g$ is also continous.

Therefore $\sin(x) - \cos(x)$ is continous.

#459520

Topic: Continuity of a Function

Discuss the continuity of the following function :

$f(x) = \sin x . \cos x$

Solution

Here clearly both $\sin x$ and $\cos x$ are defined in their domain.

Let's assume that $g(x) = \sin x$ and $f(x) = \cos x$

Let's first prove that $g(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rightarrow c$ then it means that $h \Rightarrow 0$

$\lim_{x \Rightarrow c} g(x) = \lim_{x \Rightarrow c} sin(x)$

Put $x = h + c$

And as mentioned above, when $x \Rightarrow c$ then it means that $h \Rightarrow 0$

Which gives us $\lim_{h \Rightarrow 0} \sin(c + h)$

Expanding $\sin(x + h) = \sin(h)\cos(c) + \cos(h)\sin(c)$

Which gives us $\lim_{h \Rightarrow 0} \sin(h)\cos(c) + \cos(h)\sin(c)$

$= \sin(c)\cos(0) + \cos(c)\sin(0)$

$= \sin(c)$

So here we get,

$\lim_{x \to c} g(x) = \lim_{x \Rightarrow c} \sin(x) = \sin(c) = g(c)$

And this proves that $\sin(x)$ is continuous all across its domain

Let's prove that $f(x) = \cos(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rightarrow c$ then it means that $h \Rightarrow 0$

$f(c) = \cos(c)$

$\lim_{x \Rightarrow c} f(x) = \lim_{x \Rightarrow c} \cos(x)$

Put $x = h + c$

And as mentioned above, when $x \Rightarrow c$ then it means that $h \Rightarrow 0$

Which gives us $\lim_{h \Rightarrow 0} \cos(c + h)$

Expanding $\cos(h + c) = \cos(h)\cos(c) - \sin(h)\sin(c)$

Which gives us $\lim_{h \Rightarrow 0} \cos(h)\cos(c) - \sin(h)\sin(c)$

$= \cos(c)\cos(0) - \sin(c)\sin(0)$

$= \cos(c)$

This gives us

$\lim_{x \Rightarrow c} g(x) = \lim_{x \Rightarrow c} \sin(x) = \sin(c) = g(c)$

And this proves that $\cos(x)$ is continuous all across its domain

So by theorem, if function f and function g are continous, then $f.\ g$ is also continous.

There for $\sin(x).\ \cos(x)$ is continous.

#459521

Topic: Continuity of a Function

Discuss the continuity of the cosine, cosecant and secant functions.

Solution

Function y = f(x) is continuous at point x = a if the following three conditions are satisfied :

i.) f(a) is defined ,

ii.) $\lim_{x \to a} f(x)$ exists (i.e., is finite) ,

and

iii.) $\lim_{x \to a} f(x) = f(a)$

We can also see the continuity by graph of function

i) Cosine

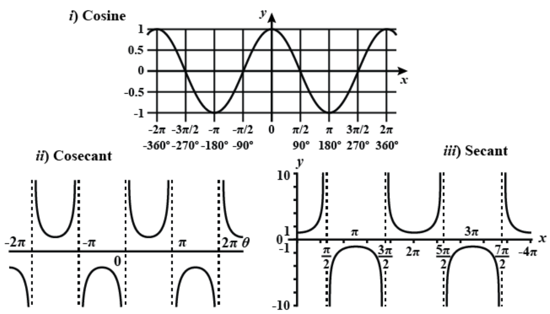
Cosine is continuous function

*ii)*cosecant

Cosecant is discontinuous at $x = n\pi$ where n is integer

*iii)*Secant

Secant is discontinuous at $x = \frac{n\pi}{2}$ where n is integer



#459523

Topic: Continuity of a Function

Show that the function defined by $f(x) = \cos\left(x^2\right)$ is continuous in its domain.

Solution

Let's assume that $h(x) = x^2$ and $g(x) = \cos x$

So $f(x) = (goh) = \cos(x^2)$

Let's prove that $g(x) = \cos(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x \Rightarrow c$ then it means that $h \Rightarrow 0$

$f(c) = \cos(c)$

$\lim_{x \Rightarrow c} f(x) = \lim_{x \Rightarrow c} \cos(x)$

Put $x = h + c$

And as mentioned above, when $x \Rightarrow c$ then it means that $h \Rightarrow 0$

Which gives us $\lim_{h \Rightarrow 0} \cos(c + h)$

Expanding $\cos(h + c) = \cos(h)\cos(c) - \sin(h)\sin(c)$

Which gives us $\lim_{h \Rightarrow 0} \cos(h)\cos(c) - \sin(h)\sin(c)$

$= \cos(c)\cos(0) - \sin(c)\sin(0)$

$= \cos(c)$

And this proves that $\cos(x)$ is continuous all across its domain

Now let's see if x^2 is continues in its domain

$\lim_{x \Rightarrow h} h(x) = \lim_{x \Rightarrow h} x^2 = h^2 = h(c)$ and this proves that $h(x)$ is continuos in its domain.

So by theorem: If function f and function g are continuous then fog is also continuous.

There for $\cos(x^2)$ is continuous across its domain.

#459525

Topic: Continuity of a Function

Show that the function defined by $f(x) = |\cos x|$ is continuous.

Solution

Let's assume that $h(x) = \cos(x)$ and $g(x) = |x|$

So $f(x) = (goh) = |\cos(x)|$

Let's prove that $h(x) = \cos(x)$ is continuous in it's domain.

Let c be a real number, put $x = c + h$

So if $x = c$ then it means that $h \to 0$

$h(c) = \cos(c)$

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos(x)$

Put $x = h + c$

And as mentioned above, when $x \to c$ then it means that $h \to 0$

Which gives us $\lim_{h \to 0} cos(c + h)$

Expanding $\cos(h + c) = \cos(h)\cos(c) - \sin(h)\sin(c)$

Which gives us $\lim_{h \to 0} \cos(h)\cos(c) - \sin(h)\sin(c)$

$= \cos(c)\cos(0) - \sin(c)\sin(0)$

$= \cos(c)$

This gives us

$\lim_{x \to c} h(x) = \lim_{x \to c} \sin(x) = \sin(c) = h(c)$

And this proves that $\cos(x)$ is continuous all across its domain

Here $g(x)$ is given by $|x|$

And $|x| = -x$ for $x < 0$ and x for $x > 0$

For $c < 0$

$\lim_{x \to c} g(x) = -c = g(c)$ and

For $c > 0$

$\lim_{x \to c} g(x) = c = g(c)$ which shows that $|x|$ is continuous in its domain

So by theorem: If function h and function g are continuous then goh is also continuous.

There for $|\cos(x)|$ is continuous across its domain.

#459545

Topic: Differentiation of Implicit Functions

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that $\frac{dy}{dx} = \frac{1}{(1+x)^2}$

Solution

Given, $x\sqrt{1+y}+y\sqrt{1+x}=0$

If we differentiate this function with respect to x, then we get

$$\sqrt{1+y}+\frac{x}{2\sqrt{1+y}}\frac{dy}{dx}+\frac{y}{2\sqrt{1+x}}+\frac{dy}{dx}\sqrt{1+x}=0$$

Taking out $\frac{dy}{dx}$ one side, we get $\frac{dy}{dx}=\frac{1}{(1+x)^2}$

#459549

Topic: Continuity and Differentiability

If $f(x)=|x|^3$. show that $f''(x)$ exists for all real x and find it.

Solution

Given, $f(x)=|x|^3$

Removing mod, we get

$$f(x)=\begin{cases}x^3 & if \quad x\geq 0\\ -x^3 & if \quad x<0\end{cases}$$

For case: when $x>0$

$f'(x)=3x^2$

$\Rightarrow f''(x)=6x$

Now lets see for $x<0$

$f'(x)=-3x^2$

$\Rightarrow f''(x)=-6x$

This shows that $f''(x)$ exist and is given by

$$f''(x)=\begin{cases}6x & if \quad x\geq 0\\ -6x & if \quad x<0\end{cases}$$

#459552

Topic: Continuity of a Function

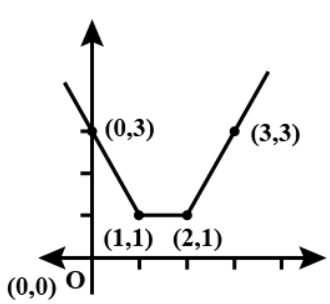
Does there exist a function which is continuous everywhere but not differentiable at exactly tow points? Justify your answer.

Solution

Yes there exist such a function

$f(x)=|x-1|+|x-2|$

As you can see the graph of the function, this function is continuous at every point. But it is differentiable at exactly two points, viz $(1,1)$ and $(2,1)$ because of a sharp turn.



#459554

Topic: Higher Order Derivatives

If $y = e^{a\cos^{-1}x}, \ -1 \le x \le 1$, show that $\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Solution

$y = e^{acos^{-1}x}\frac{dy}{dx} = e^{a\cos^{-1}x} \times a\frac{-1}{\sqrt{1-x^2}}\frac{d^2y}{dx^2} = e^{a\cos^{-1}x} \times \frac{a^2}{1-x^2} - e^{acos^{-1}x} \times a \times \frac{1}{2} \times \frac{2x}{(1-x^2)^{\frac{3}{2}}}$

Now,

$(1-x^2)^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2ya^2e^{a\cos^{-1}x} - e^{a\cos^{-1}x} \times a \times \frac{x}{(1-x^2)^{\frac{1}{2}}} + x \times \left(e^{a\cos^{-1}x} \times a\frac{-1}{\sqrt{1-x^2}}\right) - a^2 \times e^{a\cos^{-1}x} = 0$

#418766

Topic: Continuity of a Function

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Solution

We have $f(x) = 2x^2 - 1$

At $x = 3$, $f(x) = f(3) = 2(3)^2 - 1 = 17$

and $\lim_{x \to 3} f(x) = \lim_{x \to 3}(2x^2 - 1) = 2(3)^2 - 1 = 17$

Clearly $\lim_{x \to 3} f(x) = f(3)$

Thus, f is continuous at $x = 3$.

#418770

Topic: Continuity of a Function

Examine the following functions for continuity.

(i) $f(x) = x - 5$ (ii) $f(x) = \frac{1}{x-5}, x \neq 5$

(iii) $f(x) = \frac{x^2 - 25}{x + 5}, \ x \neq -5$ (iv) $f(x) = |x - 5|$

Solution

(i)

Given function is polynomial of degree one,

and we know every polynomial function is continuous for all $x \in R$

Hence f is continuous.

(ii)

The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we obtain

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x-5} = \frac{1}{k-5}$$

Also, $f(k) = \frac{1}{k-5}$ (As k $\neq$ 5)

$\therefore \ \lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iii)

The given function is $f(x) = \frac{x^2 - 25}{x+5}, x \neq 5$

For any real number $x \neq -5$, we obtain

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{(x+5)(x-5)}{x+5} = x - 5$$

$\therefore \ \lim_{x \to c} f(x) = f(c)$

Hence,

f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iv)

The given function is $f(x) = |x - 5| = \begin{cases} 5 - x, \text{if} x < 5 \\ x - 5, \text{if} x \geq 5 \end{cases}$

The function is defined at all points of the real line.

Let c be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I : $c < 5$

Then, $f(c) = 5 - c$

$\lim_{x \to c} f(x) = \lim_{x \to c}(5 - x) = 5 - c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all real numbers less than 5.

Case II : $c = 5$

Then $f(c) = f(5) = (5 - 5) = 0$

Then, $\lim_{x \to 5} f(x) = \lim_{x \to 5}(5 - x) = (5 - 5) = 0$

$\lim_{x \to 5} f(x) = \lim_{x \to 5}(x - 5) = 0$

$\therefore \lim_{x \to 5} f(x) = \lim_{x \to 5} f(x) = f(c)$

Therefore, f is continuous at $x = 5$

Case III : $c > 5$

Then, $f(c) = f(5) = c - 5$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x - 5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

#418772

Topic: Discontinuity of a Function

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution

The given function $g(x)$ is defined at all integral points.

Let n be an integer.

Then $g(n) = n - n = n - n = 0$

$L.H.L.$ at $x = n = \lim_{x \to n^-} g(x) = \lim_{x \to n^-}(x - \{x\}) = n.(n - 1) = 1$

$R.H.L.$ at $x = n = \lim_{x \to n^+} g(x) = \lim_{x \to n^+}(x - \{x\}) = n - n = 0$

Since $L.H.L. \neq R.H.L.$

Therefore g is not continuous at $x = n.$ i.e., $g(x)$ is discontinuous at all integral points.

#418941

Topic: Continuity of a Function

Is the function f defined by

$$f(x) = \begin{cases} x, & if\, x \leq 1 \\ 5, & if\, x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Solution

The given function is

$$f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$$

At $x = 0$

It is evident that f is defined at 0 and its value at 0 is 0.

Then $\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1} f(x) = \lim_{x \to 1} (5) = 5$

$\therefore \lim_{x \to 1} f(x) \ne \lim_{x \to 1} f(x)$

Therefore, f is not continuous at $x = 1$.

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

Then, $\lim_{x \to 2} f(x) = \lim_{x \to 2} (5) = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$

#418989

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} 2x + 3, & if\, x \le 2 \\ 2x - 3, & if\, x > 2 \end{cases}$$

Solution

The given function is $f(x) = \begin{cases} 2x+3, \text{ if } x \le 2 \\ 2x-3, \text{ if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i)

$c < 2$

Then, $f(c) = 2c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x + 3) = 2c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$

Case (ii)

$c > 2$

Then, $f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x - 3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x such that, $x > 2$

Case (iii)

$c = 2$

Then, the left hand limit of f at $x = 2$ is,

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x + 3) = 2 \times 2 + 3 = 7$

The right hand limit of f at $x = 2$ is,

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of f at $x = 2$ do not coincide.

Therefore, f is not continuous at $x = 2$

Hence, $x = 2$ is the only point of discontinuity of f.

#418993

Topic: Algebra of Derivative of Functions

Differentiate the function with respect to x

$\cos x^3 . \sin^2(x^5)$

Solution

Let $f(x) = \cos x^3 . \sin^2(x^5)$

$\therefore f'(x) = \dfrac{d}{dx}[\cos x^3 . \sin^2(x^5)] = \sin^2(x^5) \times \dfrac{d}{dx}(\cos x^3) + \cos x^3 \times \dfrac{d}{dx}[\sin^2(x^5)]$

$= \sin^2(x^5) \times (-\sin x^3) \times \dfrac{d}{dx}(x^3) + \cos x^3 \times 2\sin(x^5) . \dfrac{d}{dx}[\sin x^5]$

$= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2\sin x^5 \cos x^3 . \cos x^5 \times \dfrac{d}{dx}(x^5)$

$= -3x^2 \sin x^3 . \sin^2(x^5) + 2\sin x^5 \cos x^5 \cos x^3 \times 5x^4$

$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)$

#418996

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$$

Solution

The given function is $f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$

The given function is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c < -3$, then $f(c) = -c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-x + 3) = -c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < -3$

Case II :

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3}(-x + 3) = -(-3) + 3 = 6$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3}(-2x) = -2x(-3) = 6$

$\therefore \lim_{x \to 3} f(x) = f(-3)$

Therefore, f is continuous at $x = -3$

Case III :

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(-2x) = -2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in $(-3, 3)$.

Case IV :

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$\lim_{x \to 3} f(x) = \lim_{x \to 3}(-2x) = -2 \times 3 = -6$

The right hand limit of f at $x = 3$ is,

$\lim_{x \to 3} f(x) = \lim_{x \to 3}(6x + 2) = 6 \times 3 + 2 = 20$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V :

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(6x + 2) = 6c + 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points x, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f.

#419265

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, \text{ if } \quad x \ne 0 \\ 0, \text{ if } \quad x = 0 \end{cases}$$

Solution

The given function is $f(x) = \begin{cases} \dfrac{|x|}{x}, \text{if} x \neq 0 \\ 0, \text{if} x = 0 \end{cases}$

It is known that, for $x < 0$, $|x| = -x$ and for $x > 0$, $|x| = x$

Therefore given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{|x|}{x} = \dfrac{-x}{x} = -1, \text{if} x < 0 \\ 0, \text{if} x = 0 \\ \dfrac{|x|}{x} = \dfrac{x}{x} = 1, \text{if} x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points $x < 0$

Case II :

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (-1) = -1$

The right hand limit of f at $x = 0$ is,

$\lim_{x \to 0} f(x) = lim_{x \to 0}(1) = 1$

It is observed that the left and right hand limit of $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Case III :

If $c > 0$, then $f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

#419272

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|} \quad \text{if} \quad x < 0 \\ -1 \quad \text{if} \quad x \geq 0 \end{cases}$$

Solution

The given function is $f(x) = \begin{cases} \frac{x}{|x|} \text{ if } x < 0 \\ -1 \text{ if } x \geq 0 \end{cases}$

It is known that $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1 \text{ if } x < 0 \\ -1 \text{ if } x \geq 0 \end{cases}$

$\Rightarrow f(x) = -1$ for all $x \in R$

Let c be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

#419282

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

Solution

The given function is $f(x) = \begin{cases} x+1, \text{ if } x \geq 1 \\ x^2+1, \text{ if } x < 1 \end{cases}$

The given function is defined at all the points of the real line.

Let c be a point on the real line.

Case I: $c < 1$, then $f(c) = c^2 + 1$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x^2 + 1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II : $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$

The left hand limit of f at $x = 1$ is,

$lim_{x \to 1} f(x) = lim_{x \to 1} (x^2 + 1) = 1^2 = 2$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1} f(x) = \lim_{x \to 1}(x + 1) = 1 + 1 = 2$

$\therefore \lim_{x \to 1} f(x) = f(1)$

Therefore, f is continuous at $x = 1$

Case III : $c > 1$, then $f(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x + 1) = c + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, the given function f has no point of discontinuity.

#419284

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

Solution

The given function f is $f(x) = \begin{cases} x^3 - 3, \text{if} x \le 2 \\ x^2 + 1, \text{if} x > 2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If c < 2, then $f(c) = c^3 - 3$ and $\displaystyle \underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim}(x^3 - 3) = c^3 - 3$

$\therefore\displaystyle \underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < 2

Case II

If c = 2, then $f(c) = f(2) = 2^3 - 3 = 5$

$\displaystyle \underset{x \rightarrow 2^-}{\lim} = \underset{x \rightarrow 2^-}{\lim} (x^3 - 3) = 2^3 - 3 = 5$

$\displaystyle \underset{x \rightarrow 2^+}{\lim} f(x) = \underset{x \rightarrow 2^+}{\lim} (x^2 + 1) = 2^2 + 1 = 5$

$\therefore\displaystyle \underset{x \rightarrow 2}{\lim} f(x) = f(2)$

Therefore, f is continuous at x = 2

Case III :

if c > 2, then $f(c) = c^2 + 1$

$\displaystyle \underset{x \rightarrow c}{\lim} = \underset{x \rightarrow c}{\lim} (x^2 +1) = c^2 + 1$

$\therefore\displaystyle \underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 2

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

#419291

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where f is defined by

$f(x) = \begin{cases} x^{10} - 1, \quad \text{if} \quad x \le 1 \\ x^2, \quad \text {if} \quad x > 1 \end{cases}$

Solution

The given function is $f(x) = \begin{cases} x^{10} - 1, \quad \text{if} \quad x \le 1 \\ x^2, \quad \text{if} \quad x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If c < 1, then $f(c) = c^{10} - 1$ and $\displaystyle \underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim}(x^{10} - 1) = c^{10} - 1$

$\therefore \displaystyle\underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < 1

Case II :

If c = 1, then the left hand limit of x at x = 1 is,

$\displaystyle\underset{x \rightarrow 1}{\lim} f(x) = \underset{x \rightarrow 1}{\lim}(x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$

The right hand limit of f at x = 1 is,

$\displaystyle\underset{x \rightarrow 1}{\lim} f(x) = \underset{x \rightarrow 1}{\lim}(x^2) = 1^2 = 1$

It is observed that the left and right hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III :

If c > 1, then $f(c) = c^2$

$\displaystyle\underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} (x^2) = c^2$

$\therefore\displaystyle\underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

#419325

Topic: Continuity of a Function

Is the function defined by

$f(x) = \begin{cases} x + 5,\ if\ x \le 1 \\ x - 5,\ if\ x > 1 \end{cases}$

a continuous function ?

Solution

The given function is $\begin{cases} x + 5, \text{if} \quad x \le 1 \\ x - 5, \text{if} \quad x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c - 5$ and $\displaystyle \underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} (x - 5) = c - 5$

$\therefore\displaystyle\underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore f is continuous at all points x, such that $x < 1$

Case II

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left hand limit of f at $x = 1$ is,

$\displaystyle\underset{x \rightarrow 1}{\lim} f(x) = \underset{x \rightarrow 1}{\lim} (x + 5)= 1 + 5 = 6$

The right hand limit of f at $x = 1$ is,

$\displaystyle\underset{x \rightarrow 1}{\lim} f(x) = \underset{x \rightarrow 1}{lim} (x - 5) = 1 - 5 = -4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III

If $c > 1$, then $f(c) = c - 5$ and $\displaystyle f(x) = \underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} (x - 5) = c - 5$

$\therefore\displaystyle\underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

#419335

Topic: Discontinuity of a Function

Discuss the continuity of the function f, where f is defined by

$f(x) = \begin{cases} 3,\ if\ 0 \le x \le 1 \\ 4,\ if\ 1< x < 3 \\ 5,\ if\ 3 \le x \le 10 \end{cases}$

Solution

The given function is $f(x) = \begin{cases} 3,\ if\ 0 \le x \le 1 \\ 4,\ if\ 1< x < 3 \\ 5,\ if\ 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I :

If $0 \le c < 1$, then $f(c) = 3$ and $\underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} (3) = 3$

$\therefore \underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous in the interval [0, 1].

Case II :

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x =- 1$ is

$\underset{x \rightarrow 1}{\lim} f(x) = \underset{x \rightarrow 1}{\lim} (3) = 3$

The right hand limit of f at $x = 1$ is,

$\underset{x \rightarrow 1}{\lim} f(x) = \underset{x \rightarrow 1}{\lim} (4) = 4$

It is observed that the left and right hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III :

If $1 < c < 3$, then $f(c) = 4$ and $\underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} (4) = 4$

$\therefore \underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV :

If $c = 3$, then $f(c) = 5$

The left hand limit of f at x = 3 is,

$\underset{x \rightarrow 3}{\lim} f(x) = \underset{x \rightarrow 3}{\lim} (4) = 4$

The right hand limit of f at x = 3 is,

$\underset{x \rightarrow 3}{\lim} f(x) = \underset{x \rightarrow 3}{\lim}((5) = 5$

It is observed that left and right hand limit of fat x = 3 do not coincide.

Therefore, f is not continuous at x = 3.

Case V:

If $3 < c \le 10$, then $f(c) = 5$ and $\underset{x \rightarrow c}{lim} f(x) = \underset{x \rightarrow c}{\lim} (5) = 5$

$\underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at x = 1 and x = 3

#419359

Topic: Continuity of a Function

Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax + 1,\ if\ x \le 3 \\ bx + 3,\ if\ x > 3 \end{cases}$

is continuous at $x = 3$.

Solution

The given function is $f(x) = \begin{cases} ax + 1, & \text{if} \quad x \le 3 \\ bx + 3, & \text{If} \quad x > 3 \end{cases}$

If f is continuous at $x = 3$, then

$\displaystyle \underset{x \rightarrow 3^-}{\lim} f(x) = \underset{x \rightarrow 3^+}{\lim} f(x) = f(3)$ ....(1)

Now,

$\displaystyle \underset{x \rightarrow 3^-}{\lim} f(x) = \underset{x \rightarrow 3^-}{\lim}(ax + 1) = 3a + 1$

$\displaystyle \underset{x \rightarrow 3^+}{\lim} f(x) = \underset{x \rightarrow 3^+}{\lim}(bx + 3) = 3b + 3$

and $f(3) = 3a +1$

Therefore from (1), we obtain

$3a + 1 = 3b + 3 = 3a + 1$

$\Rightarrow 3a + 1 = 3b + 3$

$\Rightarrow 3a = 3b + 2\Rightarrow a - b = \cfrac{2}{3}$

Therefore the required relationship is given by, $a - b = \cfrac{2}{3}$

#419370

Topic: Continuity of a Function

For what value of λ is the function defined by

$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if} \quad x \le 0 \\ 4x + 1, & \text{if} \quad x > 0 \end{cases}$

continuous at $x = 0$? What about continuity at $x = 1$?

Solution

$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if} \quad x \le 0 \\ 4x + 1, & \text{if} \quad x > 0 \end{cases}$

If f is continuous at $x = 0$, then

$\displaystyle \underset{x \rightarrow 0^-}{\lim} f(x) = \underset{x \rightarrow 0^+}{\lim} f(x) = f(0)$

$\Rightarrow\displaystyle \underset{x \rightarrow 0}{\lim}\lambda(x^2 - 2x) = \underset{x \rightarrow 0}{\lim}(4x + 1) = \lambda(0^2 - 2 x 0)$

$\Rightarrow \lambda(0^2 - 2 \cdot 0) = 4 \cdot 0 + 1 = 0$

$\Rightarrow 0 = 1 = 0$, which is not possible

Therefore, there is no value of λ for which f is continuous at $x = 0$

At $x = 1$,

$f(1) = 4x + 1 = 4 \cdot 1 + 1 = 5$

$\displaystyle \underset{x \rightarrow 1}{\lim} (4x + 1) = 4 \cdot 1 + 1 = 5$

$\therefore\displaystyle \underset{x \rightarrow 1}{\lim} f(x) = f(1)$

Therefore, for any values of λ, f is continuous at $x = 1$

#419896

Topic: Algebra of Derivative of Functions

Differentiate the given function w.r.t. x.

$\displaystyle \frac{e^x}{\sin x}$

Solution

Let $\displaystyle y = \dfrac{e^x}{\sin x}$

Thus by using the quotient rule, we obtain

$\displaystyle \dfrac{dy}{dx} = \dfrac{\sin x \dfrac{d}{dx} (e^x) - e^x \dfrac{d}{dx} (\sin x)}{\sin^2 x}$

$\displaystyle = \dfrac{\sin x . (e^x) - e^x . (\cos x)}{\sin^2 x}$

$\displaystyle = \dfrac{e^x(\sin x - \cos x)}{\sin^2 x} , x \neq n \pi , n \in Z$

#419898

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

$\displaystyle e^{\sin^{-1} x}$

Solution

Let $\displaystyle y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$\displaystyle \dfrac{dy}{dx} = \dfrac{d}{dx} (\displaystyle e^{\sin^{-1} x})$

$\Rightarrow \displaystyle \dfrac{dy}{dx} = \displaystyle e^{sin^{-1} x} . \dfrac{d}{dx} (\sin^{-1}x)$

$\displaystyle = e^{\sin^{-1} x} . \dfrac{1}{\sqrt{1 - x^2}}$

$\displaystyle = \dfrac{e^{\sin^{-1} x}}{{\sqrt{1 - x^2}}}$

$\displaystyle\therefore \dfrac{dy}{dx} = \dfrac{e^{\sin^{-1} x}}{{\sqrt{1 - x^2}}} , x \in (-1, 1)$

#420856
Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x.

$y=\displaystyle \sin(\tan^{-1} e^{-x})$

Solution

Let $\displaystyle \sin(\tan^{-1} e^{-x})$

Thus by chain rule,

$\displaystyle \frac{dy}{dx} = \frac{d}{dx} \left[\sin (\tan^{-1} e^{-x}) \right]$

$\displaystyle = \cos (\tan^{-1} e^{-x}) . \frac{d}{dx} (\tan^{-1} e^{-x})$

$\displaystyle = \cos (\tan^{-1} e^{-x}) . \frac{1}{1 + (e^{-x})^2} \frac{d}{dx} (e^{-x})$

$\displaystyle = \frac{\cos (\tan^{-1} e^{-x})}{1 + e^{-2x}} . e^{-x} . \frac{d}{dx} (-x)$

$\displaystyle = \frac{e^{-x} \cos (\tan^{-1} e^{-x})}{1 + e^{-2x}} . (-1)$

$\displaystyle = \frac{-e^{-x} \cos (\tan^{-1} e^{-x})}{1 + e^{-2x}}$

#420955
Topic: Algebra of Derivative of Functions

Differentiate the given function w.r.t. x.

$\displaystyle \frac{\cos x}{\log x} , x > 0$

Solution

Let $\displaystyle y = \frac{\cos x}{\log x} , x > 0$

Thus using quotient rule,

$\displaystyle \frac{dy}{dx} = \frac{\frac{d}{dx} (\cos x) \times \log x - \cos x \times \frac{d}{dx} (\log x)}{(\log x)^2}$

$\displaystyle = \frac{ -\sin x \log x - \cos x \times \frac{1}{x} }{(\log x)^2}$

$\displaystyle = \frac{-[x \log x . \sin x + \cos x] }{x (\log x)^2} , x > 0$

#420974
Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$\cos x. \cos 2x . \cos 3x$

Solution

Let $y = \cos x. \cos 2x . \cos 3x$

Taking logarithm on both the sides, we obtain

$\log y = \log (\cos x. \cos 2x . \cos 3x)= \log (\cos x) + \log (\cos 2x) + \log (\cos 3x)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} . \frac{d}{dx} (\cos x) + \frac{1}{\cos 2x} . \frac{d}{dx} (\cos 2x) + \frac{1}{\cos 3x} . \frac{d}{dx} (\cos 3x)$

$\Rightarrow \displaystyle \frac{dy}{dx} = y \left[- \frac{\sin x}{\cos x} - \frac{\sin 2 x}{\cos 2x} . \frac{d}{dx} (2x) - \frac{\sin 3x}{\cos 3x} . \frac{d}{dx} (3x) \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = - \cos x . \cos 2x . \cos 3x \left[\tan x + 2 \tan 2x + 3 \tan 3x \right]$

#420986

Topic: Discontinuity of a Function

Find all points of discontinuity of f, where

$f(x) = \begin{cases} \displaystyle{\frac{\sin x}{x}}, \text{if} x < 0 \\ x + 1, \text{if} x \ge 0 \end{cases}$

Solution

The given function is $f(x) = \begin{cases} \displaystyle{\frac{\sin x}{x}}, \text{if} x < 0 \\ x + 1, \text{if} x \ge 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I

If $c < 0$, then $f(c) = \displaystyle{\frac{\sin c}{c}}$ and $\displaystyle \underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} \displaystyle{\frac{\sin x}{x}} =$

$\displaystyle{\frac{\sin c}{c}}$

Therefore, f is continuous at all points x, such that $x < 0$

Case II

If $c > 0$, then $f(c) = c + 1$ and $\displaystyle \underset{x \rightarrow c}{\lim} f(x) = \underset{x \rightarrow c}{\lim} (x + 1) = c + 1$

$\therefore\displaystyle \underset{x \rightarrow c}{\lim} f(x) = f(c)$

Therefore, f is continuous at all points such that $x > 0$

Case III

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left hand limit of f at $x = 0$ is,

$\displaystyle \underset{x \rightarrow 0}{\lim} f(x) = \underset{x \rightarrow 0}{\lim}\displaystyle{\frac{\sin x}{x}} = 1$

The right hand limit of f at $x = 0$ is,

$\displaystyle \underset{x \rightarrow 0}{\lim} f(x) = \underset{x \rightarrow 0}{\lim} (x + 1) = 1$

$\therefore\displaystyle \underset{x \rightarrow 0^+}{\lim} f(x) = \underset{x \rightarrow 0^-}{\lim} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

#420991

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$\displaystyle \sqrt{ \frac{(x - 1)(x - 2)}{(x - 3)(x - 4) (x - 5)} }$

Solution

Let $\displaystyle y = \sqrt{ \frac{(x - 1)(x - 2)}{(x - 3)(x - 4) (x - 5)} }$

Taking logarithm on both the sides, we obtain

$\displaystyle \log y = \log \sqrt{ \frac{(x - 1)(x - 2)}{(x - 3)(x - 4) (x - 5)} }$

$\Rightarrow \displaystyle \log y = \frac{1}{2} log \left[\frac{(x - 1)(x - 2)}{(x - 3)(x - 4) (x - 5)} \right]$

$\Rightarrow \displaystyle \log y = \frac{1}{2}[\log [(x - 1)(x - 2)] - \log[(x - 3)(x - 4) (x - 5)]]$

$\Rightarrow \displaystyle \log y = \frac{1}{2}[\log (x - 1) + \log (x - 2) - \log(x - 3) - \log (x - 4) - \log (x - 5)]$

Differentiating both sides with respect to x, we obtain

$\Rightarrow \displaystyle \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x - 1} . + \frac{1}{x - 2} - \frac{1}{x - 3} - \frac{1}{x - 4}- \frac{1}{x - 5} \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = \frac{y}{2} \left(\frac{1}{x - 1} + \frac{1}{x - 2} - \frac{1}{x - 3} - \frac{1}{x - 4} - \frac{1}{x - 5} \right)$

$\therefore \displaystyle \frac{dy}{dx} = \frac{1}{2} \sqrt{ \frac{(x - 1)(x - 2)}{(x - 3)(x - 4) (x - 5)} } \left[\frac{1}{x - 1} + \frac{1}{x - 2} - \frac{1}{x - 3} - \frac{1}{x - 4} - \frac{1}{x - 5} \right]$

#421036

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$(x + 3)^2 . (x + 4)^3 . (x + 5)^4$

Solution

Let $y = (x + 3)^2 . (x + 4)^3 . (x + 5)^4$

Taking logarithm on both the sides, we obtain

$\log y = \log (x + 3)^2 + \log (x + 4)^3 + \log (x + 5)^4$

$\Rightarrow \log y = 2 \log (x + 3) + 3 \log (x + 4) + 4 \log (x + 5)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{y} = 2 . \frac{1}{ x + 3} + 3 . \frac{1}{ x + 4} + 4 . \frac{1}{ x + 5}$

$\Rightarrow \displaystyle \frac{dy}{dx} = y \left[\frac{2}{ x + 3} + \frac{3}{ x + 4} + \frac{4}{ x + 5} \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = (x + 3)^2 . (x + 4)^3 . (x + 5)^4 . \left[\frac{2}{ x + 3} + \frac{3}{ x + 4} + \frac{4}{ x + 5} \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = (x + 3)^2 . (x + 4)^3 . (x + 5)^4 . \left[\frac{2(x + 4)(x + 5) + 3(x+ 3)(x + 5) + 4(x + 3)(x + 4)}{(x + 3)(x + 4)(x + 5)} \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = (x + 3)^2 . (x + 4)^3 . (x + 5)^4 . \left[2 (x^2 + 9x + 20) + 3 (x^2 + 8x + 15) + 4 (x^2 + 7x + 12) \right]$

$\therefore \displaystyle \frac{dy}{dx} = (x + 3)^2 . (x + 4)^3 . (x + 5)^4 . (9x^2 + 70x + 133)$

#421115

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$(\log x)^x + x^{\log x}$

Solution

Let $y = (\log x)^x + x^{\log x}$

Also, let $u = (\log x)^x$ and $v = x^{\log x}$

$\therefore y = u + v$

$\Rightarrow \displaystyle \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

$u = (\log x)^x$

$\Rightarrow \displaystyle \log u = \log [(\log x)^x]$

$\Rightarrow \displaystyle \log u = x \log (\log x)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{u}\frac{du}{dx}=\log(\log x)+x\cdot \frac{1}{\log x}\cdot \frac{1}{x}$

$\Rightarrow \displaystyle \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{x}{\log x} . \frac{1}{x} \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (\log x)^x \left[\frac{\log(\log x) . \log x + 1}{\log x} \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (\log x)^{x - 1} \left[1 + \log x . \log(\log x) \right]$ ...(2)

$v = x^{\log x}$

$\Rightarrow \displaystyle \log v = \log(x^{\log x})$

$\Rightarrow \displaystyle \log v = \log x . \log x = (\log x)^2$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left[(\log x)^2 \right]$

$\Rightarrow \displaystyle \frac{1}{v} . \frac{dv}{dx} = 2 (\log x) . \frac{d}{dx} (\log x)$

$\Rightarrow \displaystyle \frac{dv}{dx} = 2 v (\log x) . \frac{1}{x}$

$\Rightarrow \displaystyle \frac{dv}{dx} = 2 x^{\log x} \frac{\log x}{x}$

$\Rightarrow \displaystyle \frac{dv}{dx} = 2 x^{\log x - 1} . \log x$ ...(3)

Therefore, from (1), (2), and (3), we obtain

$\Rightarrow \displaystyle \frac{dy}{dx} = (\log x)^{x - 1} \left[1 + \log x . \log(\log x) \right] + 2 x^{\log x - 1} . \log x$

#421157

Topic: Higher Order Derivatives

Find the second order derivatives of $x^2 + 3x + 2$

Solution

Let $y = x^2 + 3x + 2$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2)} = 2x + 3 + 0 = 2x + 3$

$\therefore\displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx} (2x) + \frac{d}{dx} (3)} = 2 + 0 = 2$

#421158
Topic: Higher Order Derivatives

Find the second order derivatives of x^{20}

Solution

Let $y = x^{20}$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx}(x^{20})} = 20x^{19}$

$\therefore\displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 20\frac{d}{dx}(x^{19})} = 20.19 x^{18} = 380 x^{18}$

#421167
Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$(\sin x)^x + \sin^{-1} \sqrt{x}$

Solution

Let $y = (\sin x)^x + \sin^{-1} \sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1} \sqrt{x}$

$\therefore y = u + v$

$\Rightarrow \displaystyle \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(1)

$u = (\sin x)^x$

$\Rightarrow \displaystyle \log u = \log (\sin x)^x$

$\Rightarrow \displaystyle \log u = x \log (\sin x)$

Differentiating both sides with respect to x, we obtain

$\Rightarrow \displaystyle \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x) \times \log (\sin x) + x \times \frac{d}{dx} [\log (\sin x)]$

$\Rightarrow \displaystyle \frac{du}{dx} = u \left[1 . \log (\sin x) + x . \frac{1}{\sin x} . \frac{d}{dx} (\sin x) \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (\sin x)^x \left[\log (\sin x) + \frac{x}{\sin x} . \cos x \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x)$ ...(2)

$v = \sin^{-1} \sqrt{x}$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{dv}{dx} = \frac{1}{\sqrt{1 - (\sqrt{x})^2}} . \frac{d}{dx} (\sqrt{x})$

$\Rightarrow \displaystyle \frac{dv}{dx} = \frac{1}{\sqrt{1 - x}} . \frac{1}{2 \sqrt{x}}$

$\Rightarrow \displaystyle \frac{dv}{dx} = \frac{1}{2 \sqrt{x - x^2}}$ ...(3)

Therefore, from (1), (2) and (3), we obtain

$\displaystyle \frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2 \sqrt{x - x^2}}$

#421189
Topic: Higher Order Derivatives

Find the second order derivatives of $x.\cos x$

Solution

Let $y = x.\cos x$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx}(x.\cos x) = \cos x \frac{d}{dx} (x) + x \frac{d}{dx}(\cos x)} = \cos x.1 + x(-\sin x) = \cos x - x \sin x$

$\therefore \displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x \sin x] = \frac{d}{dx} (\cos x) - \frac{d}{dx} (x \sin x)}$

$= -\sin x - \left[\sin x\displaystyle{\frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)}\right]$

$=-\sin x - (\sin x + x\cos x)$

$= -(x\cos x + 2\sin x)$

#421193

Topic: Higher Order Derivatives

Find the second order derivatives of $\log x$

Solution

Let, $y = \log x$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx} (\log x) = \frac{1}{x}}$

$\therefore \displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}}$

#421198

Topic: Higher Order Derivatives

Find the second order derivatives of $x^3.\log x$

Solution

Let $y =x^3\log x$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \log x \frac{d}{dx} (x^3) + x^3.\frac{d}{dx}(\log x)}$

$= \log x.3x^2 + x^3.\displaystyle{\frac{1}{x}} = \log x.3x^2 + x^2$

$\quad = x^2(1 + 3 \log x)$

$\therefore \displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}[x^2(1 + 3\log x)]}$

$= (1 + 3\log x).\displaystyle{\frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(1 + 3\log x)}$

$\quad =(1 + 3\log x).2x + x^2\displaystyle{\frac{3}{x}}$

$\quad = 2x + 6x \log x+ 3x= 5x + 6x\log x= x(5 + 6 \log x)$

#421201

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$\displaystyle x^{\sin x} + (\sin x)^{\cos x}$

Solution

Let $\displaystyle y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = \displaystyle x^{\sin x}$ and $v = \displaystyle (\sin x)^{\cos x}$

$\therefore y = u + v$

$\Rightarrow \displaystyle \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(1)

$u = \displaystyle x^{\sin x}$

$\Rightarrow \displaystyle \log u = \log (x^{\sin x})$

$\Rightarrow \displaystyle \log u = \sin x \log x$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (\sin x) . \log x + \sin x . \frac{d}{dx} (\log x)$

$\Rightarrow \displaystyle \frac{du}{dx} = u \left[\cos x \log x + \sin x . \frac{1}{x} \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right]$ ...(2)

$v = \displaystyle (\sin x)^{\cos x}$

$\displaystyle \log v = \log (\sin x)^{\cos x}$

$\displaystyle \log v = \cos x \log (\sin x)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} (\cos x) \times \log (\sin x) + \cos x \times \frac{d}{dx} [\log (\sin x)]$

$\Rightarrow \displaystyle \frac{dv}{dx} = v \left[-\sin x \log (\sin x) + \cos x . \frac{1}{\sin x} . \frac{d}{dx} (\sin x) \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = (\sin x)^{\cos x} \left[-\sin x \log \sin x + \frac{\cos x}{\sin x} \cos x \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = (\sin x)^{\cos x} \left[-\sin x \log \sin x + \cot x \cos x \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log \sin x \right]$ ...(3)

From (1), (2) and (3), we obtain

$\displaystyle \frac{dy}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right] + (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log \sin x \right]$

#421202

Topic: Higher Order Derivatives

Find the second order derivatives of $e^x \sin 5x$

Solution

Let $y = e^x \sin 5x$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx} (e^x \sin 5x) = \sin 5x.\frac{d}{dx} (e^x) + e^x \frac{d}{dx}(sin 5x)}$

$= e^x (\sin 5x + 5\cos 5x)$

$\therefore \cfrac{d^2y}{dx^2}=\displaystyle{\frac{d}{dx}[e^x(\sin 5x + 5 \cos 5x)]}$

$= (\sin 5x + 5\cos 5x)\displaystyle{\frac{d}{dx}(e^x) + e^x.\frac{d}{dx} (\sin 5x + 5\cos 5x)}$

$= (\sin 5x + 5\cos 5x)e^x + e^x\left[\displaystyle{\cos 5x\frac{d}{dx}(5x) + 5(-\sin 5x).\frac{d}{dx}(5x)}\right]$

$= e^x (\sin 5x + 5\cos 5x) + e^x (5 \cos 5x - 25 \sin 5x)$

$= e^x (10\cos 5x - 24 \sin 5x) = 2e^x(5 \cos 5x - 12 \sin 5x)$

#421207

Topic: Higher Order Derivatives

Find the second order derivatives of $e^{6x} \cos 3x$

Solution

Let $y = e^{6x} \cos 3x$

$\displaystyle{\frac{dy}{dx} = \frac{d}{dx}(e^{6x}.\cos 3x) = \cos 3x.\frac{d}{dx}(e^{6x}) + e^{6x}.\frac{d}{dx}(\cos 3x)}$

$= \cos 3x.e^{6x}\displaystyle{\frac{d}{dx}(6x) + e^{6x}.(-\sin 3x).\frac{d}{dx}(3x)}$

$= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x$ ....(1)

$\therefore \displaystyle{\frac{d^2x}{dy^2} = \frac{d}{dx}(6e^{6x} \cos 3x - 3e^{6x} \sin3x) = 6\frac{d}{dx}(e^{6x}\cos 3x) -3 \frac{d}{dx}(e^{6x} \sin 3x)}$

$= 6[6e^{6x} \cos 3x - 3e^{6x} \sin 3x] - 3[\sin 3x \displaystyle{\frac{d}{dx}(e^{6x}) + e^{6x} \frac{d}{dx}(\sin 3x)}$

$= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3[\sin 3x.e^{6x}.6 + e^{6x} \cos 3x.3]$

$=27e^{6x} \cos 3x - 36e^{6x} \sin 3x= 9e^{6x}(3 \cos 3x - 4 \sin 3x)$

#421208

Topic: Higher Order Derivatives

Find the second order derivatives of $\tan^{-1} x$

Solution

Let $y = \tan^{-1} x$

$\Rightarrow \displaystyle{\frac{d}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}}$

$\therefore \displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1 + x^2}\right) = \frac{d}{dx}(1 + x^2)^{-1} = (-1).(1 + x^2).\frac{d}{dx}(1 + x^2)}$

$=\displaystyle{\frac{1}{(1 + x^2)^2} \times 2x = \frac{-2x}{(1 + x^2)^2}}$

#421209

Topic: Higher Order Derivatives

Find the second order derivatives of $\log (\log x)$

Solution

Let $y = \log (\log x)$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx} [\log (\log x)] = \frac{1}{\log x} . \frac{d}{dx} (\log x) = \frac{1}{x \log x} = (x \log x)^{-1}}$

$\therefore \displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx} [(x \log x)^{-1}] = (-1). (x \log x)^{-2}.\frac{d}{dx}(x \log x)}$

$=\displaystyle{\frac{-1}{(x \log x)^2}.\left[\log x.\frac{d}{dx}(x) + x\frac{d}{dx}(\log x) \right]}$

$=\displaystyle{\frac{-1}{(x \log x)^2}\left[\log x.1 + x\frac{1}{x}\right] = \frac{-(1 + \log x)}{(x \log x)^2}}$

#421211

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$\displaystyle x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Solution

Let $\displaystyle y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Also, let $\displaystyle u = x^{x \cos x}$ and $\displaystyle v = \frac{x^2 + 1}{x^2 - 1}$

$\therefore y = u + v$

$\Rightarrow \displaystyle \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(1)

$\displaystyle u = x^{x \cos x}$

$\Rightarrow \displaystyle \log u = \log (x^{x \cos x})$

$\Rightarrow \displaystyle \log u = x \cos x \log x$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x) . \cos x . \log x + x \frac{d}{dx} (\cos x) . \log x + x \cos x . \frac{d}{dx} (\log x)$

$\Rightarrow \displaystyle \frac{du}{dx} = u \left[1 . \cos x . \log x + x . (- \sin x) \log x + x \cos x . \frac{1}{x} \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = x^{x \cos x} \left(\cos x . \log x - x \sin x \log x + \cos x \right)$

$\Rightarrow \displaystyle \frac{du}{dx} = x^{x \cos x} \left[\cos x (1 + \log x) - x \sin x \log x \right]$ ...(2)

$\displaystyle v = \frac{x^2 + 1}{x^2 - 1}$

$\Rightarrow \displaystyle \log v = \log {(x^2 + 1)} - \log {(x^2 - 1)}$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{v} \frac{dv}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}$

$\Rightarrow \displaystyle \frac{dv}{dx} = v \left[\frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 + 1) (x^2 - 1)} \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = \frac{x^2 + 1}{x^2 - 1} \times \left[\frac{- 4x }{(x^2 + 1) (x^2 - 1)} \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = \frac{- 4x }{(x^2 - 1)^2}$ ...(3)

From (1), (2) and (3), we obtain

$\Rightarrow \displaystyle \frac{dy}{dx} = x^{x \cos x} \left[\cos x (1 + \log x) - x \sin x \log x \right] - \frac{4x }{(x^2 - 1)^2}$

#421212

Topic: Higher Order Derivatives

Find the second order derivatives of $\sin (\log x)$

Solution

Let $y = \sin (\log x)$

$\Rightarrow \displaystyle{\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)] = \cos(\log x)\frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}}$

$\therefore \displaystyle{\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos (\log x)}{x}\right]}$

$= \displaystyle{\frac{x.\frac{d}{dx}[\cos(\log x)] - \cos(\log x)\frac{d}{dx}(x)}{x^2}}$

$=\displaystyle{\frac{x\left[-\sin (\log x).\frac{d}{dx} (\log x)\right] - \cos(\log x).1}{x^2}}$

$=\displaystyle{\frac{-x\sin(\log x).\frac{1}{x} - \cos (\log x)}{x^2}}$

$=\displaystyle{\frac{-[\sin (\log x) + \cos (\log x)]}{x^2}}$

#421213

Topic: Higher Order Derivatives

If $y = \cos^{-1} x$, find $\displaystyle{\frac{d^2y}{dx^2}}$ in terms of y alone.

Solution

$y = \cos^{-1} x$

$\Rightarrow \displaystyle{\frac{dy}{dx}= -\frac{1}{\sqrt{1 - x^2}} = -(1 - x^2)^{\frac{-1}{2}}}$

$\displaystyle{\frac{d^2y}{dx^2}} = \frac{d}{dx} \left[-(1 - x^2)^{\frac{-1}{2}}\right]$

$=\displaystyle{-\left(-\frac{1}{2}\right)(1 - x^2)^{\frac{-3}{2}}.\frac{d}{dx}(1 - x^2)}$

$= \displaystyle{\frac{1}{2\sqrt{(1 - x^2)^3}}}\times (2x)$

$\Rightarrow \displaystyle{\frac{d^2y}{dx^2} = \frac{-x}{2\sqrt{(1 - x^2)^3}}}$ .....(i)

Now $y = \cos^{-1} x \Rightarrow x = \cos y$

Putting $x = \cos y$ in equation (i), we obtain

$\displaystyle{\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(1 - \cos^2 y)^3}}}$

$=\displaystyle{\frac{-\cos y}{\sin^3 y}}$

$=\displaystyle{\frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}}$

$\Rightarrow \displaystyle{\frac{d^2y}{dx^2} = -\cot y.co\sec^2 y}$

#421222

Topic: Lagrange's Mean Value Theorem

Examine the applicability of Mean Value Theorem for all three functions

(i) $f(x) = [x]$ for $x\in[5, 9]$

(ii) $f(x) = [x]$ for $x\in[-2, 2]$

(iii) $f(x) = x ^2 - 1$ for $x\in[1, 2]$

Solution

Mean Value Theorem holds for a function $f :[a, b] \rightarrow R$, if following two conditions holds

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b)

Then, there exists some $c\in (a, b)$ such that

$f'(c) = \cfrac{f(b)-f(a)}{b-a}$

Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i)

Given function $f(x) = [x]$ for $x\in [5, 9]$

Since, the greatest integer function is not continuous at integral points.

So, $f(x)$ is not continuous at $x = 5,6, 7 ,8 ,9$

Hence, Mean Value theorem is not applicable to given function.

(ii)

Given function $f(x) = [x]$ for $x\in [-2, 2]$

Since, the greatest integer function is not continuous at integral points.

So, $f(x)$ is not continuous at $x = \{-2,-1, 0 ,1 ,2\}$

Hence, Mean Value theorem is not applicable to given function.

(iii)

$f(x) = x^2 - 1$ for $x\in [1, 2]$

Since, $f(x)$ is a polynomial function.

Polynomial functions are continuous and differentiable everywhere.

So, $f(x)$ is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

Hence, f satisfies the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for given function $f(x)$

It can be proved as follows.

$f(1) = 1^2 - 1 = 0, f(2)= 2^2 - 1 = 3$

$\therefore \displaystyle{\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1}} = 3$

Also, $f'(x) = 2x$

$\therefore f'(c) = 3$

$\Rightarrow 2c = 3$

$\Rightarrow c = \cfrac{3}{2}$

$\displaystyle c = 1.5$, where $1.5 \in [1,2]$

#421226

Topic: Logarithmic Differentiation

Differentiate the function w.r.t. x.

$\displaystyle (x \cos x)^x + (x \sin x)^{\tfrac{1}{x}}$

Solution

Let $\displaystyle y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Also, let $\displaystyle u = (x \cos x)^x$ and $\displaystyle v = (x \sin x)^{\frac{1}{x}}$

$\therefore y = u + v$

$\displaystyle \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(1)

$\displaystyle u = (x \cos x)^x$

$\Rightarrow \displaystyle \log u = \log (x \cos x)^x$

$\Rightarrow \displaystyle \log u = x \log (x \cos x)$

$\Rightarrow \displaystyle \log u = x [\log x + \log \cos x]$

$\Rightarrow \displaystyle \log u = x \log x + x \log \cos x$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x \log x) + \frac{d}{dx} (x \log \cos x)$

$\Rightarrow \displaystyle \frac{du}{dx} = u \left[\left(\log x . \frac{d}{dx} (x) + x . \frac{d}{dx} (\log x) \right) + \left(\log \cos x . \frac{d}{dx} (x) + x . \frac{d}{dx} (\log \cos x) \right) \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (x \cos x)^x \left[\left(\log x . 1 + x . \frac{1}{x} \right) + \left(\log \cos x . 1 + x . \frac{1}{ \cos x} . \frac{d}{dx} (\cos x) \right) \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (x \cos x)^x \left[\left(\log x + 1 \right) + \left(\log \cos x + \frac{x}{ \cos x} . (- \sin x) \right) \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (x \cos x)^x \left[\left(1 + \log x \right) + \left(\log \cos x - x \tan x \right) \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + \left(\log x + \log \cos x \right) \right]$

$\Rightarrow \displaystyle \frac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right]$ ... (2)

$\displaystyle v = (x \sin x)^{\frac{1}{x}}$

$\Rightarrow \displaystyle \log v = \log (x \sin x)^{\frac{1}{x}}$

$\Rightarrow \displaystyle \log v = \frac{1}{x} \log (x \sin x)$

$\Rightarrow \displaystyle \log v = \frac{1}{x} (\log x + \log \sin x)$

$\Rightarrow \displaystyle \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{x} \log x \right) + \frac{d}{dx} \left[\frac{1}{x} \log (\sin x) \right]$

$\Rightarrow \displaystyle \frac{1}{v} \frac{dv}{dx} = \left[\log x . \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} . \frac{d}{dx} (\log x) \right] + \left[\log (\sin x) . \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} . \frac{d}{dx} (\log (\sin x)) \right]$

$\Rightarrow \displaystyle \frac{1}{v} \frac{dv}{dx} = \left[\log x . \left(- \frac{1}{x^2} \right) + \frac{1}{x} . \frac{1}{x} \right] + \left[\log (\sin x) . \left(- \frac{1}{x^2} \right) + \frac{1}{x} . \frac{1}{\sin x} . \frac{d}{dx} (\sin x) \right]$

$\Rightarrow \displaystyle \frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2} (1 - \log x) + \left[- \frac{\log (\sin x)}{x^2} + \frac{1}{x \sin x} . \cos x \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x}{x^2} + \frac{- \log (\sin x) + x \cot x}{x^2} \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x - \log (\sin x) + x \cot x}{x^2} \right]$

$\Rightarrow \displaystyle \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log (x \sin x) + x \cot x}{x^2} \right]$ ...(3)

From (1), (2) and (3), we obtain

$\displaystyle \frac{dy}{dx} = (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right] + (x \sin x)^{\frac{1}{x}} \left[\frac{x \cot x + 1 - \log (x \sin x)}{x^2} \right]$

#421230

Topic: Logarithmic Differentiation

Find $\displaystyle \frac{dy}{dx}$ of $x^y + y^x = 1$

Solution

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$\displaystyle \frac{du}{dx} + \frac{dv}{dx} = 0$ ...(1)

$u = x^y$

$\Rightarrow \log u = \log (x^y)$

$\Rightarrow \log u = y \log x$

Differentiating both sides with respect to x, we obtain

$\Rightarrow \displaystyle \frac{1}{u} \frac{du}{dx} = \log x \frac{dy}{dx} + y . \frac{d}{dx} (\log x)$

$\Rightarrow \displaystyle \frac{du}{dx} = x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right)$ ...(2)

$v = y^x$

$\Rightarrow \log v = \log (y^x)$

$\Rightarrow \log v = x \log y$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{v} \frac{dv}{dx} = \log y . \frac{d}{dx} (x) + x . \frac{d}{dx} (\log y)$

$\Rightarrow \displaystyle \frac{dv}{dx} = v \left(\log y . 1 + x . \frac{1}{y} . \frac{dy}{dx} \right)$

$\Rightarrow \displaystyle \frac{dv}{dx} = y^x \left(\log y + \frac{x}{y} . \frac{dy}{dx} \right)$ ...(3)

From (1), (2) and (3) we obtain

$\displaystyle x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\log y + \frac{x}{y} . \frac{dy}{dx} \right) = 0$

$\Rightarrow \displaystyle (x^y \log x + xy^{x - 1}) \frac{dy}{dx} = - (yx^{y - 1} + y^x \log y)$

$\therefore \displaystyle \frac{dy}{dx} = - \frac{yx^{y - 1} + y^x \log y}{x^y \log x + xy^{x - 1}}$

#421233
Topic: Logarithmic Differentiation

Find $\displaystyle \frac{dy}{dx}$ of $y^x = x^y$

Solution

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$x \log y = y \log x$

Differentiating both sides with respect to x, we obtain

$\displaystyle \log y . \frac{d}{dx} (x) + x . \frac{d}{dx} (\log y) = \log x . \frac{d}{dx} (y) + y . \frac{d}{dx} (\log x)$

$\Rightarrow \displaystyle \log y . 1 + x . \frac{1}{y} . \frac{dy}{dx} = \log x . \frac{dy}{dx} + y . \frac{1}{x}$

$\Rightarrow \displaystyle \log y + \frac{x}{y} \frac{dy}{dx} = \log x . \frac{dy}{dx} + \frac{y}{x}$

$\Rightarrow \displaystyle \left(\frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y$

$\Rightarrow \displaystyle \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x}$

$\therefore \displaystyle \frac{dy}{dx} = \frac{y}{x} \left(\frac{y - x \log y}{x - y \log x} \right)$

#421237
Topic: Logarithmic Differentiation

Find $\displaystyle \frac{dy}{dx}$ of $(\cos x)^y = (\cos y)^x$

Solution

Given, $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides, we obtain

$y \log \cos x = x \log \cos y$

Differentiating both sides, we obtain

$\displaystyle \log \cos x . \frac{dy}{dx} + y . \frac{d}{dx} (\log \cos x) = \log \cos y . \frac{d}{dx} (x) + x . \frac{d}{dx} (\log \cos y)$

$\Rightarrow \displaystyle \log \cos x \frac{dy}{dx} + y . \frac{1}{\cos x} . \frac{d}{dx} (\cos x) = \log \cos y . 1 + x . \frac{1}{\cos y} . \frac{d}{dx} (\cos y)$

$\Rightarrow \displaystyle \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} . (- \sin x) = \log \cos y + \frac{x}{\cos y} (-\sin y) . \frac{dy}{dx}$

$\Rightarrow \displaystyle \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$

$\Rightarrow \displaystyle (\log \cos x + x \tan y) \frac{dy}{dx} = y \tan x + \log \cos y$

$\displaystyle \therefore \frac{dy}{dx} = \frac{ y \tan x + \log \cos y}{ x \tan y + \log \cos x}$

#421239

Topic: Logarithmic Differentiation

Find $\displaystyle \frac{dy}{dx}$ of $\displaystyle xy = e^{x - y}$

Solution

Given, $\displaystyle xy = e^{x - y}$

Taking logarithm on both the sides, we obtain

$\displaystyle \log (xy) = \log (e^{x - y})$

$\Rightarrow \displaystyle \log x + \log y = (x - y)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \displaystyle \left(1 + \frac{1}{y} \right) \frac{dy}{dx} = 1 - \frac{1}{x}$

$\Rightarrow \displaystyle \left(\frac{y + 1}{y} \right) \frac{dy}{dx} = \frac{x - 1}{x}$

$\therefore \displaystyle \frac{dy}{dx} = \frac{y (x - 1)}{x(y + 1)}$

#421873

Topic: Logarithmic Differentiation

Find the derivative of the function given by $f(x) = (1 + x) (1 + x^2) (1 + x^4) (1 + x^8)$ and hence find $f'(1)$.

Solution

The given equation is $f(x) = (1 + x) (1 + x^2) (1 + x^4) (1 + x^8)$

Taking logarithm on both the sides, we obtain

$\log f(x) = \log (1 + x) + \log (1 + x^2) + \log (1 + x^4) + \log (1 + x^8)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{f(x)} . \frac{d}{dx} [f(x)] = \frac{d}{dx} \log (1 + x) + \frac{d}{dx} \log (1 + x^2) + \frac{d}{dx} \log (1 + x^4) + \frac{d}{dx} \log (1 + x^8)$

$\Rightarrow \displaystyle \frac{1}{f(x)} . f'(x) = \frac{1}{1 + x} . \frac{d}{dx} (1 + x) + \frac{1}{1 + x^2} . \frac{d}{dx} (1 + x^2) + \frac{1}{1 + x^4} . \frac{d}{dx} (1 + x^4) + \frac{1}{1 + x^8} . \frac{d}{dx} (1 + x^8)$

$\Rightarrow \displaystyle f'(x) = f(x) \left[\frac{1}{1 + x} + \frac{1}{1 + x^2} . 2x + \frac{1}{1 + x^4} . 4x^3 + \frac{1}{1 + x^8} . 8x^7 \right]$

$\therefore \displaystyle f'(x) = (1 + x) (1 + x^2) (1 + x^4) (1 + x^8) \left[\frac{1}{1 + x} + \frac{2x}{1 + x^2} + \frac{4x^3}{1 + x^4} + \frac{8x^7}{1 + x^8} \right]$

Hence, $\displaystyle f'(1) = (1 + 1) (1 + 1^2) (1 + 1^4) (1 + 1^8) \left[\frac{1}{1 + 1} + \frac{2 \times 1}{1 + 1^2} + \frac{4 \times 1^3}{1 + 1^4} + \frac{8 \times 1^7}{1 + 1^8} \right]$

$\displaystyle = 2 \times 2 \times 2 \times 2 \left[\frac{1}{2} + \frac{2 }{2} + \frac{4}{2 } + \frac{8}{2} \right]$

$\displaystyle = 16 \times \left(\frac{1 + 2 + 4 + 8}{2} \right)$

$\displaystyle = 16 \times \frac{15}{2} = 120$

#421880

Topic: Logarithmic Differentiation

Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below,

(i) By using product rule

(ii) By expanding the product to obtain a single polynomial

(iii) By logarithmic differentiation

Do they all give the same answer?

Solution

Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

(i) Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

$\therefore y = uv$

$\displaystyle \Rightarrow \frac{dy}{dx} = \frac{du}{dx} . v + u . \frac{dv}{dx}$

By using product rule,

$\displaystyle \Rightarrow \frac{dy}{dx} = \frac{d}{dx} (x^2 - 5x + 8) . (x^3 + 7x + 9) + (x^2 - 5x + 8) . \frac{d}{dx} (x^3 + 7x + 9)$

$\displaystyle \Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8) (3x^2 + 7)$

$\displaystyle \Rightarrow \frac{dy}{dx} = 2x (x^3 + 7x + 9) - 5 (x^3 + 7x + 9) + x^2 (3x^2 + 7) - 5x (3x^2 + 7) + 8 (3x^2 + 7)$

$\displaystyle \Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56$

$\displaystyle \therefore \frac{dy}{dx} = 5x^4 - 20 x^3 + 45 x^2 - 52 x + 11$

(ii) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$\displaystyle = x^2 (x^3 + 7x + 9) - 5x (x^3 + 7x + 9) + 8(x^3 + 7x + 9)$

$\displaystyle = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$\displaystyle = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$\displaystyle \therefore \frac{dy}{dx} = \frac{d}{dx} (x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)$

$\displaystyle = \frac{d}{dx} (x^5) - 5 \frac{d}{dx} (x^4) + 15 \frac{d}{dx} (x^3) - 26 \frac{d}{dx} (x^2) + 11 \frac{d}{dx} (x) + \frac{d}{dx} (72)$

$\displaystyle = 5 x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$

$\displaystyle = 5 x^4 - 20 x^3 + 45 x^2 - 52 x + 11$

(iii) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

Taking logarithm on both the sides, we obtain

$\log y = \log (x^2 - 5x + 8) + \log (x^3 + 7x + 9)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \log (x^2 - 5x + 8) + \frac{d}{dx} \log (x^3 + 7x + 9)$

$\Rightarrow \displaystyle \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2 - 5x + 8} . \frac{d}{dx} (x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9} . \frac{d}{dx} (x^3 + 7x + 9)$

$\Rightarrow \displaystyle \frac{dy}{dx} = y \left[\frac{1}{x^2 - 5x + 8} \times (2x - 5) + \frac{1}{x^3 + 7x + 9} \times (3x^2 + 7) \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9} \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right]$

$\Rightarrow \displaystyle \frac{dy}{dx} = 2x (x^3 + 7x + 9) - 5(x^3 + 7x + 9) + 3x^2 (x^2 - 5x + 8) + 7(x^2 - 5x + 8)$

$\Rightarrow \displaystyle \frac{dy}{dx} = (2x^4 +14 x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 - 15x^3 + 24x^2) + (7x^2 - 35x + 56)$

$\Rightarrow \displaystyle \frac{dy}{dx} = 5 x^4 - 20 x^3 + 45 x^2 - 52 x + 11$

From the above three observations, it can be concluded that all the $\displaystyle \frac{dy}{dx}$ results of are same.

#422118

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x:

$\sin^{-1}(x \sqrt x), 0 \le x \le 1$

Solution

Let y = \sin^{-1}(x \sqrt x)

Using chain rule, we obtain

\displaystyle{\frac{dy}{dx} = \frac{d}{dx} \sin^{-1} (x\sqrt x)}

=\displaystyle{\frac{1}{\sqrt{1 - (x\sqrt x)^2}} \times \frac{d}{dx}(x\sqrt x)}

=\displaystyle{\frac{1}{\sqrt{1 - x^3}}.\frac{d}{dx}(x^{\frac{3}{2}})}

=\displaystyle{\frac{1}{\sqrt{1 - x^3}} \times \frac{3}{2}.x^{\frac{1}{2}}}

=\displaystyle{\frac{3\sqrt x}{2\sqrt{1 - x^3}}}

=\displaystyle{\frac{3}{2}\sqrt{\frac{x}{1 - x^3}}}

#422262

Topic: Algebra of Derivative of Functions

Differentiate the given function w.r.t. x:

\displaystyle{\frac{\cos^{-1} \dfrac{x}{2}}{\sqrt{2x + 7}}}, -2 < x < 2

Solution

Let y =\displaystyle{\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x + 7}}}

Thus using quotient rule, we obtain

\displaystyle{\frac{dy}{dx} = \frac{\sqrt{2x + 7}\dfrac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right) \dfrac{d}{dx}(\sqrt{2x + 7})}{(\sqrt{2x + 7})^2}}

= \displaystyle{\dfrac{\sqrt{2x + 7} \left[\dfrac{-1}{\sqrt{1 - \left(\frac{x}{2}\right)^{2}}}.\dfrac{d}{dx}\left(\dfrac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right).\dfrac{1}{2\sqrt{2x + 7}}\dfrac{d}{dx}(2x + 7)}{2x + 7}}

= \displaystyle{\frac{\sqrt{2x +7} \dfrac{-1}{\sqrt{4 - x^2}} - \left(\cos^{-1}\frac{x}{2}\right)\dfrac{1}{2\sqrt{2x + 7}}\times 2}{2x + 7}}

\displaystyle = -\left[\frac{1}{\sqrt{4 - x^2} \sqrt{2x + 7}} + \frac{\cos^{-1} \frac{x}{2}}{(2x + 7)^{\frac{3}{2}}}\right]

#422391

Topic: Chain and Reciprocal Rule

Differentiate the given function w.r.t. x:

\cot^{-1}\displaystyle{\left[\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} \sqrt{1 - \sin x}}\right]}, 0 < x <\displaystyle{\frac{\pi}{2}}

Solution

Let y = \cot^{-1}\displaystyle{\left[\frac{\sqrt{1 + \sin x} + \sqrt{1 -\sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right]} ...(i)

Then, \displaystyle{\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}}

=\displaystyle{\frac{(\sqrt{1 + \sin x} + \sqrt{1 - \sin x)^2}}{(\sqrt{1 + \sin x} - \sqrt{1 - \sin x})(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})}}

=\displaystyle{\frac{(1 + \sin x) + (1 - \sin x) + 2\sqrt{(1 - \sin x)(1 + \sin x)}}{(1 + \sin x) - (1 - \sin x)}}

=\displaystyle{\frac{2 + 2\sqrt{1 - \sin^2 x}}{2\sin x}}

=\displaystyle{\frac{1 + \cos x}{\sin x}}={\frac{2 \cos^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}}}=\displaystyle{\cot \frac{x}{2}}

Therefore equation (1) becomes,

y = \cot^{-1}\displaystyle{\left(\cot \frac{x}{2}\right)}= \frac{x}{2}

\therefore \displaystyle{\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx}(x)= \frac{1}{2}}

#422485

Topic: Logarithmic Differentiation

Differentiate the given function w.r.t. x:

(\sin x - \cos x)^{(\sin x - \cos x)}, \displaystyle{\frac{\pi}{4} < x < \frac{3\pi}{4}}

Solution

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both the sides, we obtain

$\log y = \log[(\sin x - \cos x)^{(\sin x - \cos x)}]$

$\Rightarrow \log y = (\sin x - \cos x).\log(\sin x -\cos x)$

Differentiating both sides with respect to x, we obtain

$\displaystyle \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx} [(\sin x - \cos x).\log(\sin x - \cos x)$

$\Rightarrow \displaystyle{\frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).\frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x).\frac{d}{dx} \log(\sin x - \cos x)}$

$\Rightarrow \displaystyle{\frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).(\cos x + \sin x) + (\sin x - \cos x).\frac{1}{(\sin x - \cos x)}.\frac{d}{dx}(\sin x - \cos x)}$

$\Rightarrow \displaystyle{\frac{dy}{dx} = (\sin x - \cos x)^{\sin x - \cos x} [(\cos x + \sin x).\log(\sin x - \cos x) + (\cos x + \sin x)]}$

$\therefore \displaystyle{\frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)[1 + \log(\sin x - \cos x)]}$

#422788

Topic: Logarithmic Differentiation

Differentiate the given function w.r.t. x:

$x^{x^{2} - 3} + (x - 3)^{x^{2}}$, for $x > 3$

Solution

Let $y = x^{x^{2} - 3} + (x - 3)^{x^{2}}$

Also, let $u = x^{x^{2} - 3}$ and $v = (x - 3)^{x^{2}}$

$\therefore y = u + v$

Differentiating both sides with respect to x, we obtain

$\displaystyle{\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}}$ .....(1)

$u = x^{x^{2} - 3}$

$\therefore \log u = \log (x^{x^{2} - 3})$

$\Rightarrow \log u = (x^2 - 3) \log x$

Differentiating with respect to x, we obtain

$\displaystyle{\frac{1}{u} \frac{du}{dx} = \log x.\frac{du}{dx} = \log x.\frac{d}{dx}(x^2 - 3) + (x^2 - 3).\frac{d}{dx} (\log x)}$

$\Rightarrow \displaystyle{ \frac{1}{u}\frac{du}{dx} = \log x.2x + (x^2 - 3).\frac{1}{x}}$

$\displaystyle{\frac{du}{dx} = x^{x^{2} - 3}\left[\frac{x^2 - 3}{x} + 2x \log x\right]}$

Also,

$v = (x - 3)^{x^2}$

$\therefore \log v = \log(x - 3)^{x^2}$

$\Rightarrow \log v = x^2 \log (x - 3)$

Differentiating both sides with respect to x, we obtain

$\displaystyle{\frac{1}{v}\frac{dv}{dx} = \log (x - 3).\frac{d}{dx} (x^2) + x^2.\frac{d}{dx}[\log (x - 3)]}$

$\Rightarrow \displaystyle{\frac{1}{v}\frac{dv}{dx} = \log (x - 3).2x + x^2 .\frac{1}{x - 3}.\frac{d}{dx} (x - 3)}$

$\Rightarrow \displaystyle{\frac{dv}{dx} = v\left[2x\log (x - 3) + \frac{x^2}{x - 3} . 1\right]}$

$\Rightarrow \displaystyle{\frac{dv}{dx} = (x - 3)^{x^2}\left[\frac{x^2}{x - 3} + 2x\log (x - 3)\right]}$

Substituting the expressions of $\displaystyle{\frac{du}{dx}}$ and $\displaystyle{\frac{dv}{dx}}$ in equation (1), we obtain

$\displaystyle{\frac{dy}{dx} = x^{x^{2} - 3}\left[\frac{x^2 - 3}{x} + 2x \log x\right] + (x - 3)^{x^2}\left[\frac{x^2}{x - 3} + 2x\log (x - 3)\right]}$

#459528

Topic: Continuity and Differentiability

Prove that the greatest integer function defined by

$f(x)=\left[x \right] ,0<x<3$ is not differentiable at $x=1$ and $x=2$

Solution

By definition, if $f(x)$ is not continuous at a point, it is not differentiable at that point too.

So let us check for continuity of $f(x) = [x]$ at $x =1$ and $x = 2$

$\displaystyle \lim_{x\rightarrow 1^{-}} f(x) = \lim_{x\rightarrow 1^-} [x] = 0$

$\displaystyle \lim_{x\rightarrow 1^+}f(x) = \lim_{x\rightarrow 1^+} [x] = 1$

$\Rightarrow \lim_{x\rightarrow 1^-}f(x) \neq \lim_{x\rightarrow 1^+}f(x)$

Therefore, $f(x)$ is neither continuous nor differentiable at $x=1$

$\displaystyle \lim_{x\rightarrow 2^-}f(x) = \lim_{x\rightarrow 2^-}[x] = 1$

$\displaystyle \lim_{x\rightarrow 2^+}f(x) = \lim_{x\rightarrow 2^+}[x] = 2$

$\Rightarrow \lim_{x\rightarrow 2^-}f(x) \neq \lim_{x\rightarrow 2^+}f(x)$

Therefore, $f(x)$ is neither continuous nor differentiable at $x=2$

#459531

Topic: Continuity and Differentiability

Prove that the function f is given by $f(x)=\left| x-1 \right| ,x\in R$ is not differentiable at $x=1$

Solution

Here $f(x)$ can be written as

$$f(x)=\begin{cases} x-1\quad x\ge 1 \\ 1-x\quad x<1 \end{cases}$$

${f}^{'}(x)$ to the left of 1 is 1

${f}^{'}(x)$ to the right of 1 is -1

Clearly both of them are not equal so $f(x)$ is not differentiable at $x=1$

#459533

Topic: Logarithmic Differentiation

If u, v and w are functions of x, then show that

$\cfrac { d }{ dx } \left(u,v,w \right) =\cfrac { du }{ dx } v.w+u.\cfrac { dv }{ dx } .w+u.v\cfrac { dw }{ dx }$

in two ways- first by repeated application of product rule, second by logarithmic differentiation.

Solution

Using Product rule, in which u and v are taken as one

$$\dfrac{d(uvw)}{dx}= \dfrac{d(uv)}{dx}w+\dfrac{d(w)}{dx}uv\\ w.v.\dfrac{d(u)}{dx}+w.u.\dfrac{d(v)}{dx}+u.v.\dfrac{d(w)}{dx}$$

Now using logarithmic,

$y=uvw$

taking log on both sides, we have

$$\log y = \log u+\log v+\log w\\ \dfrac{1}{y} \dfrac{dy}{dx}=\dfrac{1}{u} \dfrac{du}{dx}+\dfrac{1}{v} \dfrac{dv}{dx}+\dfrac{1}{w} \dfrac{dw}{dx}\\ \therefore \dfrac{dy}{dx}= y\times \left(\dfrac{1}{u} \dfrac{du}{dx}+\dfrac{1}{v} \dfrac{dv}{dx}+\dfrac{1}{w} \dfrac{dw}{dx}\right)\\ \dfrac{dy}{dx}=w.v.\dfrac{d(u)}{dx}+w.u.\dfrac{d(v)}{dx}+u.v.\dfrac{d(w)}{dx}$$

since $y=uvw$

#459535

Topic: Logarithmic Differentiation

If $x=\sqrt { { a }^{ \sin ^{ -1 }{ t } } } ,\quad y=\sqrt { { a }^{ \cos ^{ -1 }{ t } } }$, show that $\cfrac{dy}{dx}=-\cfrac{y}{x}$

Solution

Given, $x=\sqrt {a^{\sin ^{-1}t}}$, $y= \sqrt {a^{\cos^{-1}t}}$

$\therefore \displaystyle \frac { dx }{ dt } =\cfrac { 1 }{ 2\sqrt { { a }^{ \sin ^{ -1 }{ t } } } } .{ a }^{ \sin ^{ -1 }{ t } }\log { a } .\cfrac { 1 }{ \sqrt { 1-{ t }^{ 2 } } }$ ..(i)

and $\displaystyle \frac { dy }{ dt } =\cfrac { 1 }{ 2\sqrt { { a }^{ \cos ^{ i }{ t } } } } .{ a }^{ \cos ^{ -1 }{ t } }\log { a } .\cfrac { -1 }{ \sqrt { 1-{ t }^{ 2 } } }$ ...(ii)

When we divide (ii) from (i), we will get

$\displaystyle \cfrac{dy}{dx}=\cfrac{\frac { 1 }{ 2\sqrt { { a }^{ \sin ^{ -1 }{ t } } } } .{ a }^{ \sin ^{ -1 }{ t } }\log { a } .\frac { 1 }{ \sqrt { 1-{ t }^{ 2 } } }}{\frac { 1 }{ 2\sqrt { { a }^{ \cos ^{ i }{ t } } } } .{ a }^{ \cos ^{ -1 }{ t } }\log { a } .\frac { -1 }{ \sqrt { 1-{ t }^{ 2 } } }}$

$\dfrac { dy }{ dx } = -\dfrac { y }{ x }$

#459536

Topic: Higher Order Derivatives

If $y=5\cos{x}-3\sin {x}$, prove that $\cfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } } +y=0$

Solution

Given, $y=5 \cos x-3 \sin x$ .....(i)

On differentiating w.r.t. x, we get

$\dfrac{dy}{dx}=-5\sin x-3\cos x$ .....(ii)

$\dfrac { { d }^{ 2 }y }{ { d }x^{ 2 } } = -5\cos x+3\sin x$ ....(iii)

On adding equations (i) and (iii), we get

$\dfrac { { d }^{ 2 }y }{ { d }x^{ 2 } } +y = -5\cos x+3\sin x+5\cos x-3\sin x =0$

#459538

Topic: Higher Order Derivatives

If $y=3\cos { \left(\log { x } \right) } +4\sin { \left(\log { x } \right) }$, show that ${ x }^{ 2 }{ y }_{ 2 }+x{ y }_{ 1 }+y=0$

Solution

$y=3 \cos(\log x)+4\sin (\log x)\\ y_1 = \cfrac{dy}{dx}=3(-\sin (\log x))\times \cfrac{1}{x} + 4(\cos (\log x)) \times \cfrac {1}{x}$

$y_2= 3(-\cos(\log x)) \times \cfrac{1}{x^2}+3(\sin (\log x)) \times \cfrac{1}{x^2}+4(-\sin (\log x))\times \cfrac{1}{x^2}-4(\cos(\log x))\times \cfrac{1}{x^2}$ ...By Using Product rule

Now, by putting the value of y_2 & y_1

$x^2 y_2 + x y_1 +y \\ =3(-\cos (\log x))+3(\sin(\log x))-4(\sin(\log x))-4(\cos (\log x))-3(\sin (\log x)) \\+4(\cos (\log x))+ 3\cos(\log x)+4\sin(\log x)$

$=0$

#459540

Topic: Higher Order Derivatives

If $y=A{ e }^{ mx }+B{ e }^{ nx }$, show that $\cfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } } -\left(m+n \right) \cfrac { dy }{ dx } +mny=0$

Solution

Given $y=A{e}^{mx}+B{e}^{nx}$

On differentiating w.r. t. x, we get

$\dfrac { dy }{ dx } =Am{ e }^{ mx }+Bn{ e }^{ nx }$

Again differentiating w.r.t. x,

$\dfrac { { d }^{ 2 }y }{ { d }x^{ 2 } } =A{ m }^{ 2 }{ e }^{ mx }+B{ n }^{ 2 }{ e }^{ nx }$

We need to prove $\dfrac {d^2y}{dx^2}-(m+n)\dfrac {dy}{dx}+mny=0$

Putting them in the equation, we get

$A{ m }^{ 2 }{ e }^{ mx }+B{ n }^{ 2 }{ e }^{ nx } - (m+n)(Am{ e }^{ mx }+Bn{ e }^{ nx })+mn(A{e}^{mx}+B{e}^{nx})$

$= A{ m }^{ 2 }{ e }^{ mx }+B{ n }^{ 2 }{ e }^{ nx } - A{m}^{2}{ e }^{ mx }-Bmn{e}^{nx}-Amn{e}^{mx}-B{n}^{2}{ e }^{ nx }+mnA{e}^{mx}+mnB{e}^{nx}$

$=0$

#459541

Topic: Higher Order Derivatives

If $500{e}^{7x}+600{e}^{-7x}$, show that $\cfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } } =49y$

Solution

Given, $500 e^{7x}+600 e^{-7x}$

On differentiating w.r.t. x, we get

$\dfrac{dy}{dx}=7\times 500{e}^{7x}-7\times 600{e}^{-7x}$

Again differentiating w.r.t. x,

$\dfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } } =49\times 500{ e }^{ 7x }+49\times 600{ e }^{ -7x }$

$=49(500{e}^{7x}+600{e}^{-7x})$

$=49y$

#459542

Topic: Logarithmic Differentiation

If ${e}^{y}(x+1)=1$, show that $\cfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } } ={ \left(\cfrac { dy }{ dx } \right) }^{ 2 }$

Solution

We have, $e^y(x+1)=1$

$\Rightarrow e^{-y}=x+1$

Now take log both sides,

$\Rightarrow -y =\ln(x+1)\Rightarrow y = -\ln(x+1)$

$\Rightarrow \cfrac{dy}{dx}=-\cfrac{1}{x+1}$

And $\cfrac{d^2y}{dx^2}=\cfrac{1}{(x+1)^2}$

Clearly $\cfrac{d^2y}{dx^2}=\left(\cfrac{dy}{dx}\right)^2$

Hence proved.

#459543

Topic: Higher Order Derivatives

If $y={ \left(\tan ^{ -1 }{ x } \right) }^{ 2 }$, show that ${({x}^{2}+1)}^{2}{y}_{2}+2x({x}^{2}+1){y}_{1}=2$

Solution

$y= (\tan^{-1}x)^2$

$\Rightarrow y_1 = 2(\tan^{-1}x)\times \dfrac{1}{1+x^2}$

$\Rightarrow y_2 = \dfrac{2}{(1+x^2)^2}+ 2(\tan^{-1}x)\times \dfrac{-2x}{(1+x^2)^2}$

Now, $(x^2+1)^2y_2 +2x(x^2+1)y_1 = (x^{2}+1)^{2}\left(\dfrac{2}{(1+x^{2})^{2}}-\dfrac{4x(\tan^{-1}x)}{(1+x^{2})^{2}}\right)+2x(x^{2}+1)\left(\dfrac{2(\tan^{-1}x)}{1+x^{2}}\right)$

$= 2-4x(\tan^{-1}x) +2x(2(\tan^{-1}x))$

$= 2-4x(\tan^{-1}x) +4x(\tan^{-1}x)$

$=2$

#459544

Topic: Lagrange's Mean Value Theorem

If $f:\left[-5,5 \right] \rightarrow R$ is differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\ne f(5)$.

Solution

It is given that f(x) is a differentiable function, so it is clear that it is also a continuous function.

Now let's apply mean value theorem.

So as per the theorem, there exist a c \epsilon (-5,5) such that

f^{ ' }\left(c \right) =\dfrac { f\left(5 \right) -f\left(-5 \right) }{ (5)-(-5) }

10f^{ ' }\left(c \right) =f\left(5 \right) -f\left(-5 \right)

Given: {f}^{'}(c)\neq 0

\Rightarrow 10{f}^{'}(c)\neq0

\Rightarrow f(5)-f(-5)\neq0

\Rightarrow f(5)\neq f(-5)

Hence proved.

#459546
Topic: Higher Order Derivatives

If {(x-a)}^{2}+{(y-b)}^{2}={c}^{2}, for some c> 0, prove that \cfrac { { \left[1+{ \left(\cfrac { dy }{ dx } \right) }^{ 2 } \right] }^{ \tfrac { 3 }{ 2 } } }{ \cfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } } } is a constant and independent of a and b.

Solution

(x-a)^2 + (y-b)^2 = c^2

\Rightarrow 2(x-a) + 2(y-b)\dfrac{dy}{dx} = 0

\Rightarrow \cfrac{dy}{dx} = -\cfrac{(x-a)}{(y-b)}

\dfrac{d^2y}{dx^2} = - \cfrac{(y-b) - (x-a)\dfrac{dy}{dx}}{(y-b)^2}

\Rightarrow \cfrac{d^2y}{dx^2} = \cfrac{(y-b) + \dfrac{(x-a)^2}{(y-b)}}{(y-b)^2}

\Rightarrow \cfrac{d^2y}{dx^2} = \cfrac{(y-b)^2+ (x-a)^2}{(y-b)^3}

\Rightarrow (y-b) \cfrac{d^2y}{dx^2} = 1 + \left(\cfrac{x-a}{y-b}\right)^2

\Rightarrow (y-b)\cfrac{d^2y}{dx^2} = 1 + \left(\cfrac{dy}{dx}\right)^2

\therefore \cfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\tfrac32}}{\dfrac{d^2y}{dx^2}} = (y-b)\left[1+\left(\cfrac{dy}{dx}\right)^2\right]^{\tfrac32 - 1}\\=(y-b) \left[1+\cfrac{(x-a)^2}{(y-b)^2}\right]^{\tfrac12}\\=(y-b)\left[\cfrac{(y-b)^2 + (x-a)^2}{(y-b)^2}\right]^{\tfrac12}\\=(y-b)\left[\cfrac{c^2}{(y-b)^2}\right]^{\tfrac12}\\=c

Hence, the given expression is a constant c which is independent of a and b.

#459547
Topic: Chain and Reciprocal Rule

If \cos {y}=x\cos{(a+y)}, with \cos{a}\ne \pm 1, prove that \cfrac{dy}{dx}=\cfrac { \cos ^{ 2 }{ \left(a+y \right) } }{ \sin { a } }

Solution

We have, \cos y =x\cos(a+y)\rightarrow (1)

Differentiate both sides w.r.t. x

-\sin y\dfrac{dy}{dx}=\cos(a+y)-x\sin(a+y)\dfrac{dy}{dx}

\Rightarrow \dfrac{dy}{dx} =\dfrac{\cos(a+y)}{x\sin (a+y)-\sin y}=\dfrac{\cos(a+y)}{\dfrac{\cos y}{\cos (a+y)}\sin (a+y)-\sin y}

\quad =\dfrac{\cos^2(a+y)}{\cos y\sin (a+y)-\sin y\cos(a+y)}=\dfrac{\cos^2(a+y)}{\sin a}

Since \sin(A-B)=\sin A\cos B-\sin B\cos A

#459548
Topic: Higher Order Derivatives

If x=a(\cos{t}+t\sin {t}) and y=a(\sin {t}-t \cos{t}), find \cfrac { { d }^{ 2 }y }{ d{ x }^{ 2 } }

Solution

$x=a(\cos t+t\sin t) \;\;and\;\; y=a(\sin t-t\cos t)$

first we will find $\cfrac{d^2x}{dt^2}$ and $\cfrac{d^2y}{dt^2}$

and $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$

$x=a(\cos t+t \sin t)\\ \dfrac{dx}{dt} = a(-\sin t+t\cos t+\sin t)\\ \dfrac{d^2x}{dt^2} = a(-\cos t+\cos t-t \sin t+\cos t)=a(\cos t-t \sin t)$

$y=a(\sin t-t\cos t)\\ \dfrac{dy}{dt} = a(\cos t+t\sin t-\cos t)\\ \dfrac{d^2y}{dt^2} = a(-\sin t+t \cos t+\sin t+\sin t)=a(\sin t+t\cos t)$

$\therefore \dfrac{d^2y}{dx^2} = \dfrac{(\cos t-t\sin t)}{(\sin t+t\cos t)}$

#459551

Topic: Chain and Reciprocal Rule

Using the fact that $\sin {(A+B)}=\sin {A}\cos{B}+\cos{A}\sin {B}$ and the differentiation, obtain the sum formula for cosines.

Solution

$\sin(A+B)=\sin A \cos B+\cos A \sin B\\ cos(A+B) \left(\dfrac{dA}{dx} + \dfrac{dB}{dx}\right)= \sin A(-\sin B\dfrac{dB}{dx})+\cos B \cos A\dfrac{dA}{dx}+\cos A \cos B\dfrac{dB}{dx}+\sin B(-\sin A)\dfrac{dA}{dx}\\ \cos (A+B) \left(\dfrac{dA}{dx} + \dfrac{dB}{dx}\right)= \dfrac{dB}{dx}(\cos A \cos B-\sin A\sin B)+\dfrac{dA}{dx}(\cos B \cos A-\sin A \sin B)\\ \cos (A+B) \left(\dfrac{dA}{dx} + \dfrac{dB}{dx}\right)= (\cos A \cos B-\sin A \sin B)\left(\dfrac{dA}{dx} + \dfrac{dB}{dx}\right)\\ \therefore \cos (A+B)= (\cos A \cos B-\sin A\sin B)$