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Maths Formula Continuity and Differentiability

Mathematics Notes for Class 12 chapter 5.

Continuity and Differentiability

Derivative

The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x .

Differentiation of a Function

Let $f(x)$ is a function differentiable in an interval $[a, b]$. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function $g: [a, b] \rightarrow \mathbb{R}$, such that, $\forall x \in [a, b]$, $g(x) = f'(x)$.

This new function is said to be differentiation (differential coefficient) of the function $f(x)$ with respect to x and it is denoted by $df(x) / dx$ or $Df(x)$ or $f'(x)$.

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Differentiation 'from First Principle'

Let $f(x)$ is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Standard Differentiations

1. $d / dx (x^n) = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{R}$
2. $d / dx (k) = 0$, where k is constant.
3. $d / dx (e^x) = e^x$
4. $d / dx (a^x) = a^x \log_e a$, $a > 0$, $a \neq 1$

5. $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$
6. $\frac{d}{dx}(\log_a x) = \frac{1}{x}(\log_a e) = \frac{1}{x \log_e a}$
7. $\frac{d}{dx}(\sin x) = \cos x$
8. $\frac{d}{dx}(\cos x) = -\sin x$
9. $\frac{d}{dx}(\tan x) = \sec^2 x, x \neq (2n+1)\frac{\pi}{2}, n \in I$
10. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x, x \neq n\pi, n \in I$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x, x \neq (2n+1)\frac{\pi}{2}, n \in I$
12. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x, x \neq n\pi, n \in I$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
14. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
18. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
19. $\frac{d}{dx}(\sinh x) = \cosh x$
20. $\frac{d}{dx}(\cosh x) = \sinh x$
21. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

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Fundamental Rules for Differentiation

$$(i) \frac{d}{dx} \{cf(x)\} = c \frac{d}{dx} f(x), \text{ where } c \text{ is a constant.}$$

$$(ii) \frac{d}{dx} \{f(x) \pm g(x)\} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) \quad (\text{sum and difference rule})$$

$$(iii) \frac{d}{dx} \{f(x) g(x)\} = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) \quad (\text{product rule})$$

Generalization If $u_1, u_2, u_3, \dots, u_n$ be a function of x , then

$$\begin{aligned} \frac{d}{dx} (u_1 u_2 u_3 \dots u_n) &= \left(\frac{du_1}{dx} \right) [u_2 u_3 \dots u_n] \\ &+ u_1 \left(\frac{du_2}{dx} \right) [u_3 \dots u_n] + u_1 u_2 \left(\frac{du_3}{dx} \right) \\ &[u_4 u_5 \dots u_n] + \dots + [u_1 u_2 \dots u_{n-1}] \left(\frac{du_n}{dx} \right) \end{aligned}$$

$$(iv) \frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{\{g(x)^2\}} \quad (\text{quotient rule})$$

(v) if $d/d(x) f(x) = \phi(x)$, then $d/d(x) f(ax + b) = a \phi(ax + b)$

(vi) Differentiation of a constant function is zero i.e., $d/d(x) (c) = 0$.

Geometrically Meaning of Derivative at a Point

Geometrically derivative of a function at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $\{c, f(c)\}$.

Slope of tangent at $P = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \{df(x)/d(x)\}_{x=c}$ or $f'(c)$.

Different Types of Differentiable Function

1. Differentiation of Composite Function (Chain Rule)

If f and g are differentiable functions in their domain, then $f \circ g$ is also differentiable and

$$(f \circ g)'(x) = f'(g(x)) g'(x)$$

More easily, if $y = f(u)$ and $u = g(x)$, then $dy/dx = dy/du * du/dx$.

If y is a function of u , u is a function of v and v is a function of x . Then,

$$dy/dx = dy/du * du/dv * dv/dx.$$

2. Differentiation Using Substitution

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below .

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a \sin \theta$ or $a \cos \theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a \tan \theta$ or $a \cot \theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $a \operatorname{cosec} \theta$
(iv)	$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a \cos 2\theta$
(v)	$a \sin x + b \cos x$	$a = r \cos \alpha, b = r \sin \alpha$
(vi)	$\sqrt{x-\alpha}$ and $\sqrt{\beta-x}$	$x = \alpha \sin^2 \theta + \beta \cos^2 \theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos \theta)$

3. Differentiation of Implicit Functions

If $f(x, y) = 0$, differentiate with respect to x and collect the terms containing dy / dx at one side and find dy / dx .

Shortcut for Implicit Functions For Implicit function, put $d/dx \{f(x, y)\} = -\partial f / \partial x / \partial f / \partial y$, where $\partial f / \partial x$ is a partial differential of given function with respect to x and $\partial f / \partial y$ means Partial differential of given function with respect to y .

4. Differentiation of Parametric Functions

If $x = f(t)$, $y = g(t)$, where t is parameter, then

$$dy / dx = (dy / dt) / (dx / dt) = d / dt g(t) / d / dt f(t) = g' (t) / f' (t)$$

5. Differential Coefficient Using Inverse Trigonometrical Substitutions

Sometimes the given function can be deduced with the help of inverse Trigonometrical substitution and then to find the differential coefficient is very easy.

$$(i) 2 \sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$$

$$(ii) 2 \cos^{-1} x = \cos^{-1} (2x^2 - 1) \text{ or } \cos^{-1} (1 - 2x^2)$$

$$(iii) 2 \tan^{-1} x = \begin{cases} \sin^{-1} \left(\frac{2x}{1+x^2} \right) \\ \tan^{-1} \left(\frac{2x-x^2}{1} \right) \\ \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \end{cases}$$

$$(iv) 3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$$

$$(v) 3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$$

$$(vi) 3 \tan^{-1} x = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$$

$$(vii) \cos^{-1} x + \sin^{-1} x = \pi/2$$

$$(viii) \tan^{-1} x + \cot^{-1} x = \pi/2$$

$$(ix) \sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$$

$$(x) \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right]$$

$$(xi) \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} \left[xy \mp \sqrt{(1-x^2)(1-y^2)} \right]$$

$$(xii) \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left[\frac{x \pm y}{1 \mp xy} \right]$$

Logarithmic Differentiation Function

(i) If a function is the product and quotient of functions such as $y = f_1(x) f_2(x) f_3(x) \dots / g_1(x) g_2(x) g_3(x) \dots$, we first take algorithm and then differentiate.

(ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$, we first take logarithm and then differentiate.

Differentiation of a Function with Respect to Another Function

Let $y = f(x)$ and $z = g(x)$, then the differentiation of y with respect to z is

$$dy / dz = dy / dx \cdot dx / dz = f'(x) / g'(x)$$

Successive Differentiations

If the function $y = f(x)$ be differentiated with respect to x , then the result dy / dx or $f'(x)$, so obtained is a function of x (may be a constant).

Hence, dy / dx can again be differentiated with respect of x .

The differential coefficient of dy / dx with respect to x is written as $d / dx (dy / dx) = d^2y / dx^2$ or $f''(x)$. Again, the differential coefficient of d^2y / dx^2 with respect to x is written as

$$d / dx (d^2y / dx^2) = d^3y / dx^3 \text{ or } f'''(x) \dots\dots$$

Here, dy / dx , d^2y / dx^2 , d^3y / dx^3 , ... are respectively known as first, second, third, ... order differential coefficients of y with respect to x . These alternatively denoted by $f'(x)$, $f''(x)$, $f'''(x)$, ... or y_1 , y_2 , y_3 ..., respectively.

$$\text{Note } dy / dx = (dy / d\theta) / (dx / d\theta) \text{ but } d^2y / dx^2 \neq (d^2y / d\theta^2) / (d^2x / d\theta^2)$$

Leibnitz Theorem

If u and v are functions of x such that their n th derivative exist, then

$$D^n(u \cdot v) = {}^nC_0(D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + {}^nC_3(D^{n-3}u)(D^3v) + \dots + {}^nC_r D^{n-r}u \cdot D^r v + \dots + {}^nC_n(D^n u)v$$

n th Derivative of Some Functions

- (i) $\frac{d^n}{dx^n} [\sin(ax + b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$
- (ii) $\frac{d^n}{dx^n} [\cos(ax + b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$
- (iii) $\frac{d^n}{dx^n} (ax + b)^m = \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}$
- (iv) $\frac{d^n}{dx^n} [\log(ax + b)] = \frac{(-1)^{n-1} (n-1)! a^n}{(ax + b)^n}$
- (v) $\frac{d^n}{dx^n} (e^{ax}) = a^n e^{ax}$
- (vi) $\frac{d^n}{dx^n} (a^x) = a^x (\log a)^n$

$$(vii) (a) \frac{d^n}{dx^n} [e^{ax} \sin(bx + c)] = r^n e^{ax} \sin(bx + c + n\phi)$$

$$(b) \frac{d^n}{dx^n} [e^{ax} \cos(bx + c)] = r^n e^{ax} \cos(bx + c + n\phi)$$

$$\text{where, } r = \sqrt{a^2 + b^2} \text{ and } \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

Derivatives of Special Types of Functions

$$(i) \text{ If } y = f(x)^{f(x)}, \text{ then } \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)\{1 - y \log f(x)\}}$$

$$(ii) \text{ If } e^{g(y)} - e^{-g(y)} = 2f(x), \text{ then } \frac{dy}{dx} = \frac{f'(x)}{g'(y)} \cdot \frac{1}{\sqrt{1 + \{f(x)\}^2}}$$

$$(iii) \text{ If } y = \sqrt{\frac{1+g(x)}{1-g(x)}}, \text{ then } \frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2} \cdot \sqrt{\frac{1-g(x)}{1+g(x)}}$$

$$(iv) \text{ If } y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}}, \text{ then } \frac{dy}{dx} = \frac{f'(x)}{2y - 1}$$

$$(v) \text{ If } \{f(x)\}^{g(y)} = e^{f(x) - g(y)}, \text{ then } \frac{dy}{dx} = \frac{f'(x) \log f(x)}{g'(y)\{1 + \log f(x)\}^2}$$

$$(vi) \text{ If } \{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}, \text{ then}$$

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)} \left[\frac{f(x) \log g(y) - g(y)}{g(y) \log f(x) - f(x)} \right]$$

(vii) Differentiation of a Determinant

$$\text{If } y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}, \text{ then}$$

$$\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$$

(viii) **Differentiation of Integrable Functions** If $g_1(x)$ and $g_2(x)$ are defined in $[a, b]$, Differentiable at $x \in [a, b]$ and $f(t)$ is continuous for $g_1(a) \leq f(t) \leq g_2(b)$, then

$$\text{Thus, } \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\text{and } \frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Partial Differentiation

The partial differential coefficient of $f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ when y is regarded as a constant. It is written as $\partial f / \partial x$ or $D_x f$ or f_x .

$$\frac{d}{dx} \int_{g_1(x)}^{g_2(x)} f(t) dt = f[g_2(x)] \frac{d}{dx} [g_2(x)] - f[g_1(x)] \frac{d}{dx} [g_1(x)].$$

e.g., If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^4y + x + 2y$

Then, $\partial z / \partial x$ or $\partial f / \partial x$ or $f_x = 4x^3 + 3y^2 + 2xy + 1$ (here, y is considered as constant)

$\partial z / \partial y$ or $\partial f / \partial y$ or $f_y = 4y^3 + 6xy + x^2 + 2$ (here, x is considered as constant)

Higher Partial Derivatives

Let $f(x, y)$ be a function of two variables such that $\partial f / \partial x$, $\partial f / \partial y$ both exist.

- (i) The partial derivative of $\partial f / \partial y$ w.r.t. 'x' is denoted by $\partial^2 f / \partial x^2$ or f_{xx} .
- (ii) The partial derivative of $\partial f / \partial y$ w.r.t. 'y' is denoted by $\partial^2 f / \partial y^2$ or f_{yy} .
- (iii) The partial derivative of $\partial f / \partial x$ w.r.t. 'y' is denoted by $\partial^2 f / \partial y \partial x$ or f_{xy} .
- (iv) The partial derivative of $\partial f / \partial x$ w.r.t. 'x' is denoted by $\partial^2 f / \partial y \partial x$ or f_{yx} .

Note $\partial^2 f / \partial x \partial y = \partial^2 f / \partial y \partial x$

These four are second order partial derivatives.

Euler's Theorem on Homogeneous Function

If $f(x, y)$ be a homogeneous function in x, y of degree n , then

$$x (\partial f / \partial x) + y (\partial f / \partial y) = nf$$

Deduction Form of Euler's Theorem

If $f(x, y)$ is a homogeneous function in x, y of degree n , then

- (i) $x (\partial^2 f / \partial x^2) + y (\partial^2 f / \partial x \partial y) = (n - 1) \partial f / \partial x$
- (ii) $x (\partial^2 f / \partial y \partial x) + y (\partial^2 f / \partial y^2) = (n - 1) \partial f / \partial y$
- (iii) $x^2 (\partial^2 f / \partial x^2) + 2xy (\partial^2 f / \partial x \partial y) + y^2 (\partial^2 f / \partial y^2) = n(n - 1) f(x, y)$

Important Points to be Remembered

If α is m times repeated root of the equation $f(x) = 0$, then $f(x)$ can be written as

$$f(x) = (x - \alpha)^m g(x), \text{ where } g(\alpha) \neq 0.$$

From the above equation, we can see that

$$f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \dots, f^{(m-1)}(\alpha) = 0.$$

Hence, we have the following proposition

$$f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \dots, f^{(m-1)}(\alpha) = 0.$$

Therefore, α is m times repeated root of the equation $f(x) = 0$.

