Graphics 2011/2012, 4th quarter

Lecture 4

Matrices, determinants

$m \times n$ matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called an $m \times n$ matrix
with m rows and n columns.

The a_{ij}'s are called the coefficients of the matrix,
and $m \times n$ is its dimension.

Special cases

A square matrix (for which $m = n$) is called a diagonal matrix if all elements a_{ij} for which $i \neq j$ are zero.

If all elements a_{ii} are one, then the matrix is called an identity matrix, denoted with I_m (depending on the context, the subscript m may be left out).

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

If all matrix entries are zero (i.e. $a_{ij} = 0$ for all i, j), then the matrix is called a zero matrix, denoted with 0.

Matrix addition

For an $m_A \times n_A$ matrix A and an $m_B \times n_B$ matrix B, we can define addition as

$$A + B = C, \text{ with } c_{ij} = a_{ij} + b_{ij}$$

for all $1 \leq i \leq m_A, m_B$ and $1 \leq j \leq n_A, n_B$.

For example:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 8 & 14 \\ 10 & 16 \\ 12 & 18 \end{pmatrix}$$

Notice, that the dimensions of the matrices A and B have to fulfill the following conditions: $m_A = m_B$ and $n_A = n_B$.
Otherwise, addition is not defined.

Matrix subtraction

Similarly, we can define subtraction between an $m_A \times n_A$ matrix A and an $m_B \times n_B$ matrix B as

$$A - B = C, \text{ with } c_{ij} = a_{ij} - b_{ij}$$

for all $1 \leq i \leq m_A, m_B$ and $1 \leq j \leq n_A, n_B$.

For example:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 9 & 12 \\ 8 & 11 \\ 7 & 10 \end{pmatrix} = \begin{pmatrix} -8 & -8 \\ -6 & -6 \\ -4 & -4 \end{pmatrix}$$

Again, for the dimensions of the matrices A and B we must have $m_A = m_B$ and $n_A = n_B$.

Multiplication with a scalar

Multiplying a matrix with a scalar is defined as follows:

$$cA = B, \text{ with } b_{ij} = ca_{ij}$$

for all $1 \leq i \leq m_A$ and $1 \leq j \leq n_A$.

For example:

$$2 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

Obviously, there are no restrictions in this case (other than c being a scalar value, of course).

Matrix multiplication

The multiplication of two matrices with dimensions $m_A \times n_A$ and $m_B \times n_B$ is defined as

$$AB = C \text{ with } c_{ij} = \sum_{k=1}^{n_A} a_{ik} b_{kj}$$

For example:

$$\begin{pmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 2 \\ 37 & 5 & 16 \end{pmatrix}$$

Again, we see that certain conditions have to be fulfilled, i.e.

$$n_A = m_B.$$

The dimensions of the resulting matrix C are $m_A \times n_B$.

Matrix multiplication

Useful notation when doing this on paper:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{pmatrix}$$

Properties of matrix multiplication

Matrix multiplication is distributive over addition:

$$\begin{aligned} A(B+C) &= AB+AC \\ (A+B)C &= AC+BC \end{aligned}$$

and it is associative:

$$(AB)C = A(BC)$$

However, it is not commutative, i.e. in general,

AB is not the same as BA.

Properties of matrix multiplication

Proof that matrix multiplication is not commutative,
i.e. that in general, $AB \neq BA$.

Proof:

Properties of matrix multiplication

Proof that matrix multiplication is not commutative, i.e. that in general, $AB \neq BA$.

Alternative proof (proof by counterexample):

Assume two matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$.

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Identity and zero matrix revisited

Identity matrix I_m:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

With matrix multiplication we get $IA = AI = A$ (hence the name "identity matrix").

Identity and zero matrix revisited

Zero matrix 0:

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

With matrix multiplication we get $0A = A0 = 0$.

Transpose of a matrix

The transpose A^T of an $m \times n$ matrix A is an $n \times m$ matrix that is obtained by interchanging the rows and columns of A, so a_{ij} becomes a_{ji} for all i, j:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Transpose of a matrix

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

For the transpose of the product of two matrices we have

$$(AB)^T = B^T A^T$$

Transpose of a matrix

For the transpose of the product of two matrices we have $(AB)^T = B^T A^T$.
Let's look at the left side first:

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n_A 1} & \cdots & b_{n_A j} & \cdots & b_{n_A n_B} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n_A} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in_A} \\ \vdots & \ddots & \vdots \\ a_{m_A 1} & \cdots & a_{m_A n_A} \end{pmatrix}, \quad AB = \begin{pmatrix} c_{11} & \cdots & \cdots & \cdots & c_{1n_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & c_{ij} & \cdots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m_A 1} & \cdots & \cdots & \cdots & c_{m_A n_B} \end{pmatrix}$$

So, for AB, we get $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in_A}b_{n_A j}$
for all $1 \leq i \leq m_A$ and $1 \leq j \leq n_B$

Transpose of a matrix

Now let's look at the right side of the equation:

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{i1} & \cdots & a_{m_A 1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1n_A} & \cdots & a_{in_A} & \cdots & a_{m_A n_A} \end{pmatrix}$$

$$B^T = \begin{pmatrix} b_{11} & \cdots & b_{n_A 1} \\ \vdots & \ddots & \vdots \\ b_{1j} & \cdots & b_{n_A j} \\ \vdots & \ddots & \vdots \\ b_{1n_B} & \cdots & b_{n_A n_B} \end{pmatrix}, \quad B^T A^T = \begin{pmatrix} d_{11} & \cdots & \cdots & \cdots & d_{1n_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & d_{ji} & \cdots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{m_A 1} & \cdots & \cdots & \cdots & d_{m_A n_B} \end{pmatrix}$$

So, for $B^T A^T$ we get $d_{ji} = b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{n_A j}a_{in_A}$
for all $1 \leq i \leq m_A$ and $1 \leq j \leq n_B$

Comment

Notice the differences in the two proofs:

- For $AB \neq BA$, we showed that the general statement $AB = BA$ is **not** true.

  Hence, it was enough to give one example where this general statement was not fulfilled.

- For $(AB)^T = B^T A^T$, we needed to show that this general statement is true.

  Hence, it is **not** enough to just give one (or more) examples, but we have to prove that it is fulfilled for all possible values.

Inverse matrices

The inverse of a matrix A is a matrix A^{-1} such that

$$AA^{-1} = I$$

Only square matrices possibly have an inverse.

Note that the inverse of A^{-1} is A, so we have

$$AA^{-1} = A^{-1}A = I$$

The dot product revisited

If we regard (column) vectors as $n \times 1$ matrices, we see that the dot product of two vectors can be written as $u \cdot v = u^T v$:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 32$$

(A 1×1 matrix is simply a number, and the brackets are omitted.)

Note: Remember the different vector notations, e.g.

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (v_1, v_2, v_3) = (v_1 \ v_2 \ v_3)^T$$

Linear equation systems (LES)

The system of m linear equations in n variables $x_1, x_2, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written as a matrix equation by $Ax = b$, or in full

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

LESs in graphics

Suppose we want to solve the following system:

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 17 \\ 2x & + & y & + & z & = & 15 \\ x & + & 2y & + & 3z & = & 26 \end{array}$$

Q: what is the geometric interpretation of this system?

LESs in graphics

Suppose we want to solve the following system:

$$\begin{array}{rcrcrcrcl} x & + & y & + & 2z & - & 17 & = & 0 \\ 2x & + & y & + & z & - & 15 & = & 0 \\ x & + & 2y & + & 3z & - & 26 & = & 0 \end{array}$$

Q: what is the geometric interpretation of this system?

Gaussian elimination

If an LES has a unique solution, it can be solved with Gaussian elimination. Matrices are not necessary for this, but very convenient, especially augmented matrices.

The augmented matrix corresponding to a system of m linear equations is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

LESs in graphics

Example:

The previous LES:

$$\begin{array}{rcrcrcrcl} x & + & y & + & 2z & - & 17 & = & 0 \\ 2x & + & y & + & z & - & 15 & = & 0 \\ x & + & 2y & + & 3z & - & 26 & = & 0 \end{array}$$

And its related augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right)$$

Gaussian elimination

Basic idea of Gaussian elimination:
Apply certain operations to the matrix that do not change the solution, in order to bring the matrix into a from where we can immediately "see" the solution.

These permitted operations in Gaussian elimination are

- interchanging two rows.
- multiplying a row with a (non-zero) constant.
- adding a multiple of another row to a row.

Gaussian elimination: example

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 3 & 19 \\ 0 & 1 & 1 & 9 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & -2 & -10 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Permitted operations:
(1) exchange rows
(2) multiply row with scalar
(3) add multiple of 1 row to another

Gaussian elimination: example

Remember that the augmented matrix represents the linear equation system $Ax = b$, i.e.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right) \text{ is equivalent to } \begin{array}{ccccccc} 1x & + & 0y & + & 0z & = & 3 \\ 0x & + & 1y & + & 0z & = & 4 \\ 0x & + & 0y & + & 1z & = & 5 \end{array}$$

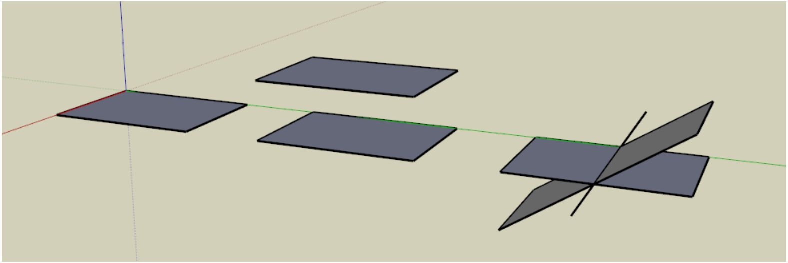
which directly gives us the solution $x = 3$, $y = 4$, and $z = 5$

Q: what is the geometric interpretation of this solution?

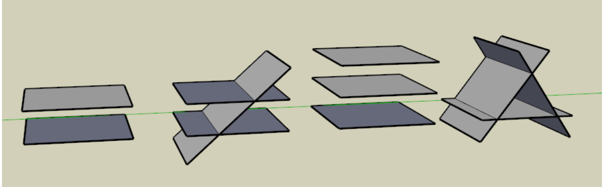
Intersection of two planes

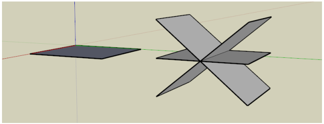
Q: what is the geometric interpretation of this system?

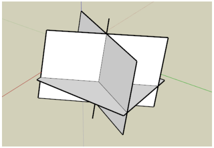
1st: intersection of 2 planes



Intersection of three planes







Gaussian elimination

Q: what about the other cases (e.g. line or no intersection)?
Q: and what if an LGS can not be reduced to a triangular form?

Three possible situations:

- We get a line $0x + 0y + 0z = b$ (with $b \neq 0$)
- Or we get one line like $0x + 0y + 0z = 0$
- Or we get two lines like $0x + 0y + 0z = 0$

Gaussian elimination

Note: In the literature you also find a slightly different procedure

1st: *"Put 0's in the lower triangle ..."* (forward step)
2nd: *".. then work your way back up"* (backward step)

$$\left(\begin{array}{ccccc|c} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{array}\right)$$

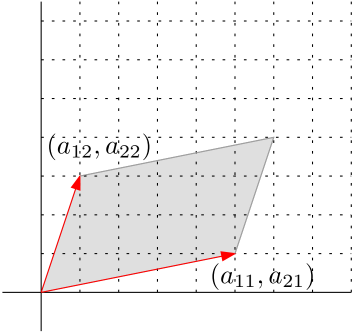
Another note: Gaussian elimination can also be used to invert matrices (and you will do this in the tutorials)

Determinants

The determinant of an $n \times n$ matrix is the signed volume spanned by its column vectors. The determinant $det\,A$ of a matrix A is also written as $|A|$. For example,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$det\,A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

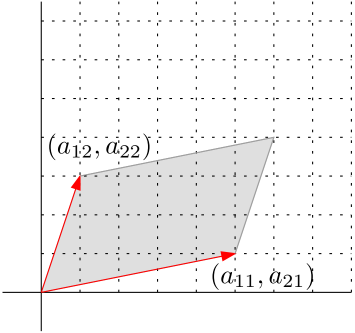


Determinants: geometric interpretation

In 2D: $det A$ is the oriented area of the parallelogram defined by the 2 vectors.

$$det\ A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

In 3D: $det A$ is the oriented area of the parallelepiped defined by the 3 vectors.



Computing determinants

Using Laplace's expansion determinants can be computed as follows:

The determinant of a matrix is the sum of the products of the elements of any row or column of the matrix with their cofactors

If only we knew what cofactors are. . .

Cofactors

Take a deep breath...

The cofactor of an entry a_{ij} in an $n \times n$ matrix A is the determinant of the $(n-1) \times (n-1)$ matrix A' that is obtained from A by removing the i-th row and the j-th column, multiplied by -1^{i+j}.

Right: long live recursion!

Cofactors

Example: for a 4×4 matrix A, the cofactor of the entry a_{13} is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \qquad a_{13}^c = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

and $|A| = a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c + a_{14}a_{14}^c$

Determinants and cofactors

Example:

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} &= 0\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \\
&= 0 \\
&\quad -1(3|8|(-1)^{(1+1)} + 5|6|(-1)^{(1+2)}) \\
&\quad +2(4|6|(-1)^{(1+2)} + 7|3|(-1)^{(2+2)}) \\
&= 0 - 1(24 - 30) + 2(21 - 24) \\
&= 0.
\end{aligned}$$

Computing determinants for 3×3 matrices

3x3 matrices:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \dots \\ &= (aei + bfg + cdh) - (ceg + afh + bdi) \end{aligned}$$

An easy way to do this on paper (Rule of Sarrus):

$$\begin{vmatrix} a & b & c & a & b & c \\ d & e & f & d & e & f \\ g & h & i & g & h & i \end{vmatrix}$$

Computing determinants for 2×2 matrices

It also works for 2x2 matrices:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

But unfortunately not for higher dimensions than 3!

The cross product revisited

Let's look at another example:

$$\begin{vmatrix} x & y & z \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

From

$$\begin{vmatrix} x & y & z & x & y & z \\ v_1 & v_2 & v_3 & v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 & w_1 & w_2 & w_3 \end{vmatrix}$$

we get

$$\begin{array}{ccccc} (v_2 w_3 & - & v_3 w_2) & x & + \\ (v_3 w_1 & - & v_1 w_3) & y & + \\ (v_1 w_2 & - & v_2 w_1) & z & \end{array}$$

Systems of linear equations and determinants

Consider our system of linear equations again:

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

Such a system of n equations in n unknowns can be solved by using determinants using Cramer's rule:

If we have $Ax = b$, then $x_i = \frac{|A^i|}{|A|}$

where A^i is obtained from A by replacing the i-th column with b.

Systems of linear equations and determinants

So for our system

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 15 \\ 26 \end{pmatrix}$$

we have

$$x = \frac{\begin{vmatrix} 17 & 1 & 2 \\ 15 & 1 & 1 \\ 26 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 1 & 17 & 2 \\ 2 & 15 & 1 \\ 1 & 26 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 1 & 1 & 17 \\ 2 & 1 & 15 \\ 1 & 2 & 26 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}$$

Systems of linear equations and determinants

$x_i = \frac{|A^i|}{|A|}$. Why? In 2D:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

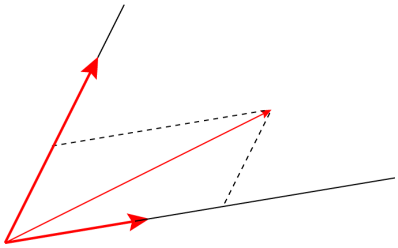
From the image we see:

$$|x_1\vec{a_1}\vec{a_2}| = |\vec{b}\vec{a_2}|$$

(because shearing a parallelogram doesn't change its volume) and

$$x_1|\vec{a_1}\vec{a_2}| = |\vec{b}\vec{a_2}|$$

(because scaling one side of a parallelogram changes its volume by the same factor)



Systems of linear equations and determinants

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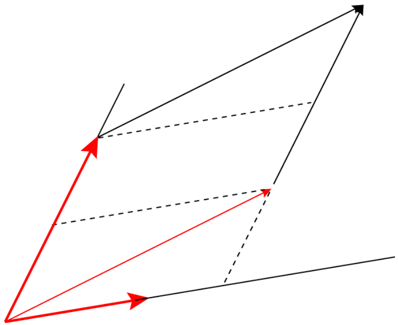

Determinants and inverse matrices

Determinants can also be used to compute the inverse A^{-1} of an invertible matrix A:

$$A^{-1} = \frac{\tilde{A}}{|A|}$$

where $\tilde{A}$ is the adjoint of A, which is the transpose of the cofactor matrix of A.

The cofactor matrix of A is obtained from A by replacing every entry a_{ij} by its cofactor a^c_{ij}.