Notes on Determinants and Matrix Inverse

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1 Definition of determinant

Determinant is a scalar that measures the "magnitude" or "size" of a square matrix.

Notice that conclusions presented below are focused on rows and row expansions. Similar results apply when the row-focus is changed to column-focus.

Definition (verbal): For each $n \times n$ (square) matrix A, $\det A$ is the sum of $n!$ terms, each is the product of n entries in which *one and only one* is taken from each row and column. The sum exhausts all permutations of integers $\{1, 2, \cdots, n\}$. Terms with even permutations carry a "+" sign and those with odd permutations a "-" sign.

Permutation: any rearrangement of an ordered list of numbers, say S, is a permutation of S.

Example 1.1: For $S = \{1, 2, 3\}$, there are a total of $3! = 6$ permutations:

$$\{123,\ 231,\ 312,\ 132,\ 213,\ 321\}.$$

Definition: For each permutation $(j_1 j_2 \cdots j_n)$ of the ordered list $S = \{1, 2, \cdots, n\}$,

$$\sigma(j_1 j_2 \cdots j_n) = \text{min number of pairwise exchanges to recover order } (12 \cdots n).$$

Even/odd permutations: a permutation $(j_1 j_2 \cdots j_n)$ is even/odd if the value of $\sigma(j_1 j_2 \cdots j_n)$ is even/odd.

Example 1.2: For $S = \{1, 2, 3\}$:

$$\sigma(123) = 0, \quad \sigma(231) = 2, \quad \sigma(312) = 2 \quad \text{are even;}$$

$$\sigma(132) = 1, \quad \sigma(213) = 1, \quad \sigma(321) = 1 \quad \text{are odd.}$$

Example 1.3.1: Show that the determinant of diagonal and triangular matrices is equal to the product of diagonal entries.

Answer: In each one of these matrices, there is only one way of picking one and only one entry from each row and each column that is not a zero.

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

Example 1.3.2: Show that if A has a zero column (or row), $\det A = 0$.

Answer: Since in every term of the determinant, there is one number from each row and each column. Thus, there is at least a 0 in each product which makes every term equal to 0!

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & 0 & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} = 0.$$

Definition (formula): $\forall$ $n \times n$ matrix $A = [a_{ij}]$ $(1 \leq i, j \leq n)$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where the sum is taken over all possible permutations of integers $(12 \cdots n)$ (a total of $n!$) and that

$$(-1)^{\sigma(j_1 j_2 \cdots j_n)} = \begin{cases} +1, & \text{if } \sigma(j_1 j_2 \cdots j_n) \text{ is even,} \\ -1, & \text{if } \sigma(j_1 j_2 \cdots j_n) \text{ is odd.} \end{cases}$$

A column focused formula reads

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{(i_1 i_2 \cdots i_n)} (-1)^{\sigma(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n},$$

Example 1.4: For a 3×3 matrix $A = [a_{ij}]$ $(1 \leq i, j \leq 3)$,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{n2} & a_{33} \end{vmatrix} = \sum_{(j_1 j_2 j_3)} (-1)^{\sigma(j_1 j_2 j_3)} a_{1j_1} a_{2j_2} a_{3j_3}$$

$$= (-1)^{\sigma(123)=0} a_{11} a_{22} a_{33} + (-1)^{\sigma(231)=2} a_{12} a_{23} a_{31} + (-1)^{\sigma(312)=2} a_{13} a_{21} a_{32}$$

$$(-1)^{\sigma(132)=1} a_{11} a_{23} a_{32} + (-1)^{\sigma(213)=1} a_{12} a_{21} a_{33} + (-1)^{\sigma(321)=1} a_{13} a_{22} a_{31}$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}.$$

Remark: For matrices larger than 3×3, this formula is not practical to use by hand. So, it shall be mainly employed in demonstrating important properties of determinants.

2 Row and column expansion formulas

As an engineer, you may not need to know how to prove each formula that we learn. In this section, I will focus on the “how-to” without saying “why”.

For all practical purposes, an engineer should know how to calculate the determinant of a large matrix using either row or column expansions. We have already learned how to calculate a 3×3 matrix using expansion in the first row of a matrix.

As a matter of fact, the first row is not always the best to use in an expansion formula. So, we here extend that formulas to expansions in any row of a matrix.

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = a_{i1}(-1)^{i+1} \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+a_{ij}(-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} + \cdots + a_{in}(-1)^{i+n} \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

$= a_{i1}c_{i1} + \cdots + a_{ij}c_{ij} + \cdots + a_{in}c_{in}$ $\quad\leftarrow\quad$ **row expansion formula**,

where

- A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A:

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}, \qquad \det A_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

- $\det A_{ij}$ is called the $(i,j)^{th}$ *minor* of A.
- $c_{ij} = (-1)^{i+j} \det A_{ij}$ is called the $(i,j)^{th}$ *cofactor* of A.

Similarly, we can expand in the j^{th} column of a matrix.

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = a_{1j}(-1)^{1+j} \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+a_{ij}(-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} + \cdots + a_{nj}(-1)^{n+j} \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

$= a_{1j}c_{1j} + \cdots + a_{ij}c_{ij} + \cdots + a_{nj}c_{nj}.$ $\quad\leftarrow\quad$ **column expansion formula**.

Example 2.1:

$$\det A = \begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix} = ?$$

Ans: Using expansion in the first row, we obtain

$$\det A = (-1)^{1+1}2 \begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix} + (-1)^{1+3}2 \begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix} + (-1)^{1+4}4 \begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix}.$$

$$= 2 \begin{vmatrix} 0 & 3 & 2 \\ 2 & 4 & 4 \\ 0 & 6 & 2 \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 2 \end{vmatrix} - 4 \begin{vmatrix} 0 & 0 & 3 \\ 2 & 2 & 4 \\ 3 & 0 & 6 \end{vmatrix}.$$

Using expansion in the second row, we obtain

$$\det A = (-1)^{2+3}3\begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix} + (-1)^{2+4}2\begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix} = -3\begin{vmatrix} 2 & 0 & 4 \\ 2 & 4 & 4 \\ 3 & 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 6 \end{vmatrix}.$$

But the simplest is to expand in the second column,

$$\det A = (-1)^{3+2}2\begin{vmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 0 & 6 & 2 \end{vmatrix} = (-1)^{5}2\begin{vmatrix} 2 & 2 & 4 \\ 0 & 3 & 2 \\ 3 & 6 & 2 \end{vmatrix} = (-2)(12+12-36-24) = 72.$$

Remarks:

(i) Row (column) expansion is the sum of the products between all entries of that row (column) multiplied by the corresponding cofactors.

(ii) The $(i,j)^{th}$ cofactor $c_{ij} = (-1)^{i+j}\det A_{ij}$, where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of matrix A.

Derivation/proof of the expansion formula (Not required!):

$$\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} = \sum_{j=1}^{n} a_{ij} c_{ij}.$$

Proof: Based on the definition:

$$\begin{aligned}
\det A &= \sum_{(l_1 \cdots l_i \cdots l_n)} (-1)^{\sigma(l_1 \cdots l_i \cdots l_n)} \left[a_{1l_1} \cdots a_{il_i} \cdots a_{nl_n}\right] \\
&= \sum_{(l_1 \cdots l_i \cdots l_n)} (-1)^{\sigma(l_i l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n) - (i-1)} \left[a_{il_i} a_{1l_1} \cdots a_{(i-1)l_{(i-1)}} a_{(i+1)l_{(i+1)}} \cdots a_{nl_n}\right] \\
&= \sum_{(l_1 \cdots l_i \cdots l_n)} (-1)^{\sigma(j l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n) - (i-1) - (j-1)} \left[a_{ij} a_{1l_1} \cdots a_{(i-1)l_{(i-1)}} a_{(i+1)l_{(i+1)}} \cdots a_{nl_n}\right] \\
&= \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \left(\sum_{(l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n)}^{l_k \neq j \text{ for all } k} (-1)^{\sigma(l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n)} \left[a_{1l_1} \cdots a_{(i-1)l_{(i-1)}} a_{(i+1)l_{(i+1)}} \cdots a_{nl_n}\right] \right) \\
&= \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}.
\end{aligned}$$

In the derivation above, we used the following identities:

(i) $\sigma(l_1 \cdots l_i \cdots l_n) = \sigma(l_i l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n) - (i-1)$,

because it takes $(i-1)$ pairwise exchanges to turn $(l_i l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n)$ into $(l_1 \cdots l_i \cdots l_n)$.

(ii) $\sigma(l_i l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n) = \sigma(j l_1 \cdots l_{(i-1)} l_{(i+1)} \cdots l_n) - (j-1)$,

because it takes $(j-1)$ pairwise exchanges to turn $(1 \cdots j \cdots n)$ into $(j1 \cdots (j-1)(j+1) \cdots l_n)$.

(iii) $(-1)^{-(i+j)-2} = (-1)^{-(i+j)} = (-1)^{i+j}$.

3 Important properties of determinants

3.1 $\det \mathbf{A} = \det \mathbf{A}^{\mathbf{T}}$

Because the "row-focused" and "column-focused" formulas are identical.

$$\begin{aligned}\det A &= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \\ &= \sum_{(i_1 i_2 \cdots i_n)} (-1)^{\sigma(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} = \det A^T.\end{aligned}$$

Example 3.1:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}.$$

3.2 $\det \mathbf{A_{k\leftrightarrow l}} = -\det \mathbf{A}$

Proof:

Suppose that matrix $A' = A_{k\leftrightarrow l}$ is obtained by swapping the k^{th} and j^{th} rows of A. Thus, $a'_{ij} = a_{ij}$ except that $a'_{kj} = a_{lj}$ and $a'_{lj} = a_{kj}$ for all $1 \le j \le n$.

Then,

$$\det A' = \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_k \cdots j_l \cdots j_n)} a'_{1j_1} a'_{2j_2} \cdots a'_{kj_k} \cdots a'_{lj_l} \cdots a'_{nj_n}$$
$$= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_k \cdots j_l \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{lj_k} \cdots a_{kj_l} \cdots a_{nj_n}$$
$$= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_k \cdots j_l \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{kj_l} \cdots a_{lj_k} \cdots a_{nj_n}$$
$$= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_l \cdots j_k \cdots j_n)+1} a_{1j_1} a_{2j_2} \cdots a_{kj_l} \cdots a_{lj_k} \cdots a_{nj_n} = -\det A.$$

Note that $\sigma(j_1 j_2 \cdots j_k \cdots j_l \cdots j_n) = \sigma(j_1 j_2 \cdots j_l \cdots j_k \cdots j_n) + 1$ because the two differ by one pairwise swap.

Example 3.2:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \qquad \text{but} \qquad \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 6 - 4 = 2.$$

3.3 $\det \mathbf{A_{k=l}} = \mathbf{0}$

If two rows (columns) of a matrix A are identical (i.e. $A = A_{k\leftrightarrow l}$), then

$$\det A = \det A_{k\leftrightarrow l} = -\det A = 0.$$

Alternatively, if two rows are identical or are constant multiples of each other, then the rows in A are not linearly independent. Thus, A is not invertible which implies $\det A = 0$.

Example 3.3:

$$\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0.$$

3.4 Determinant is linear in each row/column

Suppose a row vector is a linear combination of two row vectors, say the k^{th} row $\mathbf{a}_k = s\mathbf{b} + t\mathbf{c}$ $(k = 1, 2, \cdots, n)$, where $s,\ t$ are scalars. Then,

$$\det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ s\mathbf{b} + t\mathbf{c} \\ \vdots \\ \mathbf{a}_n \end{bmatrix} = s \det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{b} \\ \vdots \\ \mathbf{a}_n \end{bmatrix} + t \det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{c} \\ \vdots \\ \mathbf{a}_n \end{bmatrix}.$$

Equivalently, we can also express this property in the following form

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ sb_1 + tc_1 & sb_2 + tc_2 & \cdots & sb_n + tc_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = s \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + t \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Proof:

$$\det A = \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{kj_k} \cdots a_{nj_n}$$
$$= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots (sb_{j_k} + tc_{j_k}) \cdots a_{nj_n}$$
$$= s \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots b_{j_k} \cdots a_{nj_n} + t \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots c_{j_k} \cdots a_{nj_n}.$$

This property can also be expressed in form of two related properties:

$$
(1) \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ sa_{k1} & sa_{k2} & \cdots & sa_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = s \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},
$$

$$
(2) \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.
$$

Example 3.4:

$$
\begin{vmatrix} a & b \\ 3 & 5 \end{vmatrix} = 5a - 3b, \qquad \begin{vmatrix} 2a & 2b \\ 3 & 5 \end{vmatrix} = 10a - 6b = 2(5a - 3b) = 2 \begin{vmatrix} a & b \\ 3 & 5 \end{vmatrix}.
$$

$$
\begin{vmatrix} a + c & b + d \\ 3 & 5 \end{vmatrix} = \begin{vmatrix} a & b \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} c & d \\ 3 & 5 \end{vmatrix}.
$$

3.5 Adding a scalar multiple of one row to another

Let matrix A' be identical to A except that k^{th} row multiplied by scalar s ($\neq 0$) is added to l^{th} row $(1 \leq k < l \leq n)$.

Thus, $a'_{ij} = a_{ij}$ for all $1 \leq i, j \leq n$ except that $a'_{lj} = a_{lj} + sa_{kj}$. Then,

$$\det A' = \det A.$$

Proof:

$$\begin{aligned}
\det A' &= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a'_{1j_1} a'_{2j_2} \cdots a'_{lj_k} \cdots a'_{nj_n} \\
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} \cdots a_{kj_k} \cdots (a_{lj_l} + sa_{kj_l}) \cdots a_{nj_n} \\
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} \cdots a_{kj_k} \cdots a_{lj_l} \cdots a_{nj_n} + s \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} a_{1j_1} \cdots a_{kj_k} \cdots a_{kj_l} \cdots a_{nj_n} \\
&= \det A + 0 = \det A.
\end{aligned}$$

Example 3.5:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a & b \\ c+2a & d+2b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + 2 \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

3.6 $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

Example 3.6:

$$\det A = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5, \qquad \det B = \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} = 3 - (-4) = 7.$$

$$AB = \begin{bmatrix} -2 & 5 \\ -13 & 15 \end{bmatrix}, \qquad \det AB = \begin{vmatrix} -2 & 5 \\ -13 & 15 \end{vmatrix} = -30 - (-65) = 35 = \det A \det B.$$

Proof(1): (Not required!)

Case I: If at least one of A and B is noninvertible, i.e. either $\det A = 0$ or $\det B = 0$ or both, then AB as the composite transformation of the two must also be noninvertible. Thus,

$$\det(AB) = 0 = \det A \det B.$$

Case II: If both A and B are invertible, then based on the results on elementary matrices:

$$A = A_1 A_2 \cdots A_k, \qquad B = B_1 B_2 \cdots B_l,$$

where $A_1, \ \cdots, \ A_k$ and $B_1, \ \cdots, \ B_l$ are all elementary matrices. Therefore,

$$\det(AB) = \det(A_1 A_2 \cdots A_k B_1 B_2 \cdots B_l) = (\det A_1 \cdots \det A_k)(\det B_1 \cdots \det B_l) = \det A \det B.$$

Proof(2): (Not required!)

Note that $AB = [(AB)_{ij}]$, where $(AB)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$.

$$
\begin{aligned}
\det AB &= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} (AB)_{1j_1} (AB)_{2j_2} \cdots (AB)_{nj_n} \\
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} \left(\sum_{l_1=1}^{n} a_{1l_1} b_{l_1 j_1} \right) \left(\sum_{l_2=1}^{n} a_{2l_2} b_{l_2 j_2} \right) \cdots \left(\sum_{l_n=1}^{n} a_{nl_n} b_{l_n j_n} \right)
\end{aligned}
$$

$$
\xrightarrow[\text{because an identical term with opposite sign occurs when sumed over all permutations of } (\mathrm{j_1 j_2 \cdots j_n})]{\text{All non}-\text{selfavoiding terms cancel (i.e. } \mathrm{l_1, l_2, \cdots, l_n} \text{ must be distinct)}}
$$

$$
\begin{aligned}
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} \sum_{(l_1 l_2 \cdots l_n)} (a_{1l_1} b_{l_1 j_1})(a_{2l_2} b_{l_2 j_2}) \cdots (a_{nl_n} b_{l_n j_n}) \\
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} \sum_{(l_1 l_2 \cdots l_n)} (a_{1l_1} a_{2l_2} \cdots a_{nl_n})(b_{l_1 j_1} b_{l_2 j_2} \cdots b_{l_n j_n}) \\
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} \sum_{(l_1 l_2 \cdots l_n)} (a_{1l_1} a_{2l_2} \cdots a_{nl_n})(-1)^{\sigma(l_1 l_2 \cdots l_n)} (b_{1j_1} b_{2j_2} \cdots b_{nj_n}) \\
&= \sum_{(j_1 j_2 \cdots j_n)} (-1)^{\sigma(j_1 j_2 \cdots j_n)} (b_{1j_1} b_{2j_2} \cdots b_{nj_n}) \sum_{(l_1 l_2 \cdots l_n)} (-1)^{\sigma(l_1 l_2 \cdots l_n)} (a_{1l_1} a_{2l_2} \cdots a_{nl_n}) \\
&= \det B \det A.
\end{aligned}
$$

The key step in the proof above is to eliminate all the non-selfavoiding terms. Here, a "non-selfavoiding term" refers to those terms in which $l_i = l_j$ $(i \neq j)$ happens for at least one pair of $i,\ j$ $(1 \leq i, j \leq n)$. To see this more clearly, we show the case when only two subscripts exists for 2×2 matrices. In this case,

$$
\begin{aligned}
\det AB &= \sum_{(j_1 j_2)}^{(12) \text{ or } (21)} (-1)^{\sigma(j_1 j_2)} (AB)_{1j_1} (AB)_{2j_2} \\
&= \sum_{(j_1 j_2)}^{(12) \text{ or } (21)} (-1)^{\sigma(j_1 j_2)} \left(\sum_{l_1=1}^{2} a_{1l_1} b_{l_1 j_1} \right) \left(\sum_{l_2=1}^{2} a_{2l_2} b_{l_2 j_2} \right) \\
&= \sum_{(j_1 j_2)}^{(12) \text{ or } (21)} (-1)^{\sigma(j_1 j_2)} \left(a_{11} b_{1j_1} + a_{12} b_{2j_1} \right) \left(a_{21} b_{1j_2} + a_{22} b_{2j_2} \right)
\end{aligned}
$$

$$
= \left(a_{11} b_{11} + a_{12} b_{21} \right) \left(a_{21} b_{12} + a_{22} b_{22} \right) - \left(a_{11} b_{12} + a_{12} b_{22} \right) \left(a_{21} b_{11} + a_{22} b_{21} \right)
$$

$$
\xrightarrow[\text{Self}-\text{avoiding colors!}]{\text{Terms with identical color cancel each other.}}
$$

$$= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11}$$

$$\begin{aligned}
&= \sum_{l_1,l_2=1,\ (l_1\neq l_2)}^{2} (a_{1l_1}b_{l_11}a_{2l_2}b_{l_22} - a_{1l_1}b_{l_12}a_{2l_2}b_{l_21}) \\
&= \sum_{l_1,l_2=1,\ (l_1\neq l_2)}^{2} \sum_{(j_1j_2)} (-1)^{\sigma(j_1j_2)} a_{1l_1}b_{l_1j_1}a_{2l_2}b_{l_2j_2} = \sum_{(l_1l_2)} \sum_{(j_1j_2)} (-1)^{\sigma(j_1j_2)} a_{1l_1}b_{l_1j_1}a_{2l_2}b_{l_2j_2} \\
&= \sum_{(l_1l_2)} \sum_{(j_1j_2)} (-1)^{\sigma(j_1j_2)} (a_{1l_1}a_{2l_2})(b_{l_1j_1}b_{l_2j_2}) = \sum_{(l_1l_2)} \sum_{(j_1j_2)} (-1)^{\sigma(j_1j_2)} (a_{1l_1}a_{2l_2})(-1)^{\sigma(l_1l_2)}(b_{1j_1}b_{2j_2}) \\
&= \sum_{(l_1l_2)} (-1)^{\sigma(l_1l_2)} (a_{1l_1}a_{2l_2}) \sum_{(j_1j_2)} (-1)^{\sigma(j_1j_2)} (b_{1j_1}b_{2j_2}) \\
&= \det A \det B.
\end{aligned}$$

Here, non-selfavoiding terms are terms like $a_{11}b_{11}a_{21}b_{12}$ and $a_{12}b_{21}a_{22}b_{22}$ in which entries from the same column or row of a matrix occur twice in the same product. They all cancel out in the summation. Therefore, "self-avoiding" means in each term of the expression, there must be *one and only one* entries from each row and each column of each matrix. The same conclusion works in a similar way when the matrix is bigger. For any term that contains at least one pair of non-selfavoiding entries, we can use the same argument for this particular pair of entries in exactly the same way as we treat the 2×2 case.

4 Formula for the inverse of $n \times n$ matrices

Definition of minor and cofactor:

$\forall\ n \times n$ matrices $A = [a_{ij}]$, the $(i,j)^{th}-$minor and $(i,j)^{th}-$cofactor are

$$m_{ij} = \det A_{ij}, \quad c_{ij} = (-1)^{i+j} m_{ij},$$

where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A.

Definition of cofactor matrix:

$\forall\ n \times n$ matrices $A = [a_{ij}]$, its cofactor matrix is defined as

$$C = [c_{ij}] = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix},$$

where $c_{ij} = (-1)^{i+j} \det A_{ij}$ is the $(i,j)^{th}-$cofactor of matrix A.

Formula for A^{-1}:

$\forall$ invertible $n \times n$ matrices A with a cofactor matrix C:

$$A^{-1} = \left(\frac{1}{\det A}\right) C^T.$$

Example 4.1: Consider the matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$.

(a) Show that it is invertible using the row expansion formula;

(b) Find A^{-1} using the cofactor formula.

Ans:

(a) First calculate the cofactors for the expansion in row 1:

$$c_{11} = + \begin{vmatrix} 1 & 0 \\ -5 & 2 \end{vmatrix} = 2; \quad c_{12} = - \begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} = 2; \quad c_{13} = + \begin{vmatrix} -1 & 1 \\ 2 & -5 \end{vmatrix} = 3.$$

$$\Longrightarrow \qquad \det A = (0)c_{11} + (2)c_{12} + (1)c_{13} = 4 + 3 = 7 \neq 0!$$

(b) Then, calculate all the other cofactors:

$$c_{21} = - \begin{vmatrix} 2 & 1 \\ -5 & 2 \end{vmatrix} = -9; \quad c_{22} = + \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = -2; \quad c_{23} = - \begin{vmatrix} 0 & 2 \\ 2 & -5 \end{vmatrix} = 4.$$

$$c_{31} = + \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1; \quad c_{32} = - \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = -1; \quad c_{33} = + \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = 2.$$

Therefore,

$$C = \begin{bmatrix} 2 & 2 & 3 \\ -9 & -2 & 4 \\ -1 & -1 & 2 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \left(\frac{1}{\det A} \right) C^T = \frac{1}{7} \begin{bmatrix} 2 & -9 & -1 \\ 2 & -2 & -1 \\ 3 & 4 & 2 \end{bmatrix}.$$

Verify,

$$A^{-1}A = \frac{1}{7} \begin{bmatrix} 2 & -9 & -1 \\ 2 & -2 & -1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Derivation/proof of the cofactor formula for inverse: (Not required!)

$$A^{-1} = \left(\frac{1}{\det A}\right) C^T.$$

Proof: Based on the definition of $\det A$:

$$\det A = \sum_{j=1}^{n} a_{ij}c_{ij} \qquad \Longrightarrow \qquad \sum_{j=1}^{n} a_{ij}\frac{c_{ij}}{\det A} = 1.$$

Let $A' = [a'_{ij}]$ be a matrix obtained by replacing the i^{th} row of A by its l^{th} row. Then,

$$\det A' = \sum_{j=1}^{n} a'_{ij}c_{ij} = \sum_{j=1}^{n} a_{lj}c_{ij}.$$

If $l = i$ (no change), then $A' = A$

$$\det A' = \sum_{j=1}^{n} a'_{ij}c_{ij} = \sum_{j=1}^{n} a_{lj}c_{ij} = \sum_{j=1}^{n} a_{ij}c_{ij} = \det A \qquad \Longrightarrow \qquad \sum_{j=1}^{n} a_{lj}\frac{c_{ij}}{\det A} = 1 \quad (\text{ for } l = i).$$

If $l \neq i$, then A' has two identical rows i and l, thus

$$\det A' = 0 = \sum_{j=1}^{n} a'_{ij}c_{ij} = \sum_{j=1}^{n} a_{lj}c_{ij} \qquad \Longrightarrow \qquad \sum_{j=1}^{n} a_{lj}\frac{c_{ij}}{\det A} = 0 \quad (\text{ for } l \neq i).$$

Therefore,

$$\sum_{j=1}^{n} a_{lj}\frac{c_{ij}}{\det A} = \begin{cases} 1, & \text{if } l = i, \\ 0, & \text{if } l \neq i. \end{cases}$$

Introduce a new matrix $B = [b_{ij}]$ such that

$$b_{ij} = \frac{c_{ji}}{\det A}.$$

Now, let $D = AB = [d_{ij}]$. Based on definition of matrix multiplication

$$d_{li} = \sum_{j=1}^{n} a_{lj} b_{ji} = \sum_{j=1}^{n} a_{lj} \frac{c_{ij}}{\det A} = \begin{cases} 1, & \text{if } l = i, \\ 0, & \text{if } l \neq i. \end{cases}$$

Therefore, $D = I = AB \Rightarrow$

$$A^{-1} = B = [b_{ij}] = [\frac{c_{ji}}{\det A}] = \left(\frac{1}{\det A}\right) C^T.$$

5 Cramer's rule (not in syllabus)

Cramer's Rule on component-wise solution of matrix equation $A\mathbf{x} = \mathbf{b}$:

Let $A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$ where $\mathbf{c}_j$ $(j = 1, 2, \dots, n)$ are its column vectors.

If $\det A \neq 0$, then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}\mathbf{b} = \frac{1}{\det A} C^T \mathbf{b}.$$

Thus, component-wise we have

$$x_i = \frac{1}{\det A} \sum_{j=1}^{n} c_{ij}^T b_j = \frac{1}{\det A} \sum_{j=1}^{n} b_j c_{ji}, \qquad (i = 1, 2, \dots, n),$$

where

$\displaystyle\sum_{j=1}^{n} b_j c_{ji}$ $\quad - \quad$ is detA in form of expansion in i^{th} column with a_{ji} replaced by b_j!

Thus,

$$\sum_{j=1}^{n} b_j c_{ji} = \det A_{(i=\mathbf{b})} = \det[\mathbf{c}_1\ \cdots\ \mathbf{c}_{i-1}\ \mathbf{b}\ \mathbf{c}_{i+1}\ \cdots\ \mathbf{c}_n] = \begin{vmatrix} a_{11} & \cdots & a_{1(i-1)} & b_1 & a_{1(i+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(i-1)} & b_n & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix},$$

where $A_{(i=\mathbf{b})}$ is obtained by replacing the i^{th} column of A by the vector $\mathbf{b}$!

Therefore,

$$x_i = \frac{\det A_{(i=\mathbf{b})}}{\det A}, \qquad (i = 1, 2, \dots, n).$$

Example 5.1: Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ component-wise using Cramer's Rule, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

Ans: Using Cramer's Rule

$$x_1 = \frac{\det A_{(1=\mathbf{b})}}{\det A} = \frac{\begin{vmatrix} 10 & 1 \\ 20 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{10-20}{2-1} = -10.$$

$$x_2 = \frac{\det A_{(2=\mathbf{b})}}{\det A} = \frac{\begin{vmatrix} 2 & 10 \\ 1 & 20 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{40-10}{2-1} = 30.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix}.$$