18

INVERSE TRIGONOMETRIC FUNCTIONS

In the previous lesson, you have studied the definition of a function and different kinds of functions. We have defined inverse function.

Let us briefly recall :

Let f be a one-one onto function from A to B.

Let y be an arbitary element of B. Then, f being onto, $\exists$ an element $x \in A$ such that $f(x) = y$. Also, f being one-one, then x must be unique. Thus for each $y \in B$, $\exists$ a unique element $x \in A$ such that $f(x) = y$. So we may define a function, denoted by f^{-1} as $f^{-1} : B \to A$

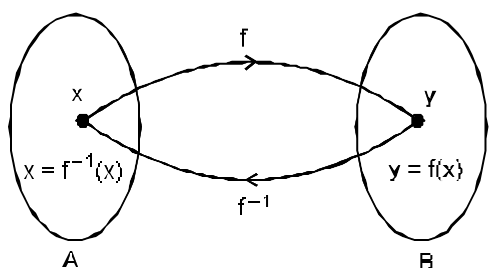


Fig. 18.1

$$\therefore \qquad f^{-1}(y) = x \Leftrightarrow f(x) = y$$

The above function f^{-1} is called the inverse of f. A function is invertiable if and only if f is one-one onto.

In this case the domain of f^{-1} is the range of f and the range of f^{-1} is the domain f.

Let us take another example.

We define a function : f : Car $\to$ Registration No.

If we write, g : Registration No. $\to$ Car, we see that the domain of f is range of g and the range of f is domain of g.

So, we say g is an **inverse function** of f, i.e., $g = f^{-1}$.

In this lesson, we will learn more about inverse trigonometric function, its domain and range, and simplify expressions involving inverse trigonometric functions.

OBJECTIVES

After studying this lesson, you will be able to :

- define inverse trigonometric functions;

- state the condition for the inverse of trigonometric functions to exist;
- define the principal value of inverse trigonometric functions;
- find domain and range of inverse trigonometric functions;
- state the properties of inverse trigonometric functions; and
- simplify expressions involving inverse trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of function and their types, domain and range of a function
- Formulae for trigonometric functions of sum, difference, multiple and sub-multiples of angles.

18.1 IS INVERSE OF EVERY FUNCTION POSSIBLE ?

Take two ordered pairs of a function (x_1, y) and (x_2, y)

If we invert them, we will get (y, x_1) and (y, x_2)

This is not a function because the first member of the two ordered pairs is the same.

Now let us take another function :

$$\left(\sin\frac{\pi}{2}, 1\right), \left(\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\sin\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

Writing the inverse, we have

$$\left(1, \sin\frac{\pi}{2}\right), \left(\frac{1}{\sqrt{2}}, \sin\frac{\pi}{4}\right) \text{ and } \left(\frac{\sqrt{3}}{2}, \sin\frac{\pi}{3}\right)$$

which is a function.

Let us consider some examples from daily life.

f : Student $\longrightarrow$ Score in Mathematics

Do you think f^{-1} will exist ?

It may or may not be because the moment two students have the same score, f^{-1} will cease to be a function. Because the first element in two or more ordered pairs will be the same. So we conclude that

every function is not invertible.

Example 18.1 If $f : R \longrightarrow R$ defined by $f(x) = x^3 + 4$. What will be f^{-1}?

Solution : In this case f is one-to-one and onto both.

$\Rightarrow$ f is invertible.

Let $\quad y = x^3 + 4$

$\therefore \quad y - 4 = x^3 \Rightarrow x = \sqrt[3]{y-4}$

So f^{-1}, inverse function of f i.e., $f^{-1}(y) = \sqrt[3]{y-4}$

The functions that are one-to-one and onto will be invertible.

Let us extend this to trigonometry :

Take $y = \sin x$. Here domain is the set of all real numbers. Range is the set of all real numbers lying between -1 and 1, including -1 and 1 i.e. $-1 \le y \le 1$.

We know that there is a unique value of y for each given number x.

In inverse process we wish to know a number corresponding to a particular value of the sine.

Suppose $\qquad y = \sin x = \dfrac{1}{2}$

$$\Rightarrow \qquad \sin x = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = ...$$

x may have the values as $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6} = ...$

Thus there are infinite number of values of x.

$y = \sin x$ can be represented as

$$\left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), ...$$

The inverse relation will be

$$\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), ...$$

It is evident that it is not a function as first element of all the ordered pairs is $\dfrac{1}{2}$, which contradicts the definition of a function.

Consider $y = \sin x$, where $x \in R$ (domain) and $y \in [-1, 1]$ or $-1 \le y \le 1$ which is called range. This is many-to-one and onto function, therefore it is not invertible.

Can $y = \sin x$ be made invertible and how? Yes, if we restrict its domain in such a way that it becomes one-to-one and onto taking x as

(i) $\quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}, \qquad y \in [-1, 1] \qquad$ or

(ii) $\quad \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \qquad y \in [-1, 1] \qquad$ or

(iii) $\quad -\dfrac{5\pi}{2} \le x \le -\dfrac{3\pi}{2} \qquad y \in [-1, 1] \qquad$ etc.

Now consider the inverse function $y = \sin^{-1} x$.

We know the domain and range of the function. We interchange domain and range for the inverse of the function. Therefore,

(i)	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$x \in [-1, 1]$	or
(ii)	$\frac{3\pi}{2} \le y \le \frac{5\pi}{2}$	$x \in [-1, 1]$	or
(iii)	$-\frac{5\pi}{2} \le y \le -\frac{3\pi}{2}$	$x \in [-1, 1]$	etc.

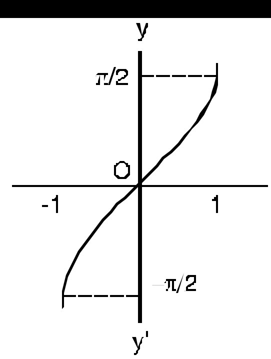
Here we take the least numerical value among all the values of the real number whose sine is x which is called the principle value of $\sin^{-1} x$.

For this the only case is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore, for principal value of $y = \sin^{-1} x$, the domain is $[-1, 1]$ i.e. $x \in [-1, 1]$ and range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

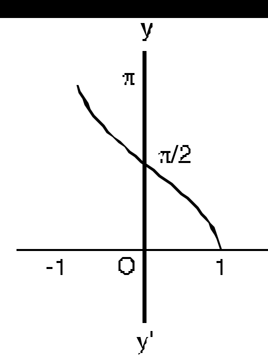
Similarly, we can discuss the other inverse trigonometric functions.

	Function	**Domain**	**Range (Principal value)**
1.	$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	R	$[0, \pi]$
5.	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y = \operatorname{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

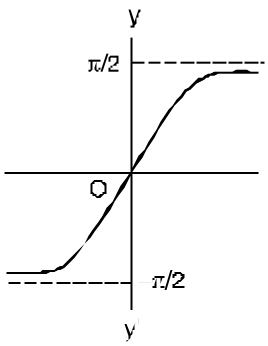
18.2 GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS



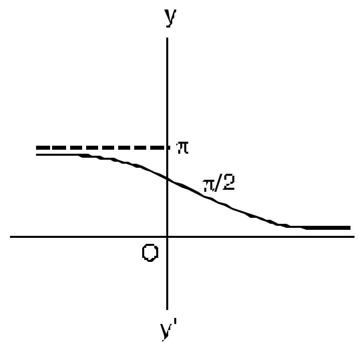
$y = \sin^{-1} x$



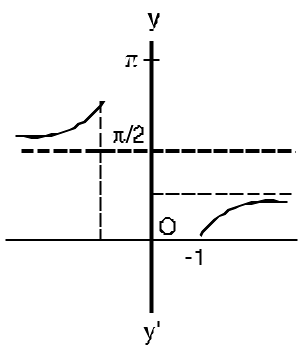
$y = \cos^{-1} x$



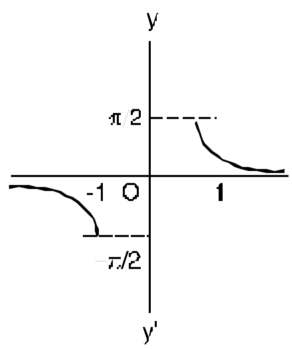
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

Fig. 18.2

Example 18.2 Find the principal value of each of the following :

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution : (i) Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

or $\sin\theta = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$ or $\theta = \frac{\pi}{4}$

(ii) Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$

$\Rightarrow$ $\cos\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$ or $\theta = \frac{2\pi}{3}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \theta$ or $-\frac{1}{\sqrt{3}} = \tan\theta$ or $\tan\theta = \tan\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $\theta = -\frac{\pi}{6}$

Example 18.3 Find the principal value of each of the following :

(a) (i) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\tan^{-1}(-1)$

(b) Find the value of the following using the principal value :

$$\sec\left[\cos^{-1}\frac{\sqrt{3}}{2}\right]$$

Solution : (a) (i) Let $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\theta$, then

$$\frac{1}{\sqrt{2}}=\cos\theta \qquad \text{or} \qquad \cos\theta=\cos\frac{\pi}{4}$$

$$\Rightarrow \qquad \theta=\frac{\pi}{4}$$

(ii) Let $\tan^{-1}(-1)=\theta$, then

$$-1=\tan\theta \qquad \text{or} \qquad \tan\theta=\tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \qquad \theta=-\frac{\pi}{4}$$

(b) Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta$, then

$$\frac{\sqrt{3}}{2}=\cos\theta \qquad \text{or} \qquad \cos\theta=\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \qquad \theta=\frac{\pi}{6}$$

$$\therefore \quad \sec\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)=\sec\theta=\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$$

Example 18.4 Simplify the following :

(i) $\cos\left[\sin^{-1}x\right]$ (ii) $\cot\left[\operatorname{cosec}^{-1}x\right]$

Solution : (i) Let $\sin^{-1}x=\theta$

$$\Rightarrow \qquad x=\sin\theta$$

$$\therefore \qquad \cos\left[\sin^{-1}x\right]=\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$$

(ii) Let $\operatorname{cosec}^{-1}x=\theta$

$$\Rightarrow \qquad x=\operatorname{cosec}\theta$$

Also $$\cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$$
$$= \sqrt{x^2 - 1}$$

CHECK YOUR PROGRESS 18.1

1. Find the principal value of each of the following :

(a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (b) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ (c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(d) $\tan^{-1}\left(-\sqrt{3}\right)$ (e) $\cot^{-1}(1)$

2. Evaluate each of the following :

(a) $\cos\left(\cos^{-1}\frac{1}{3}\right)$ (b) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$ (c) $\cos\left(\operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}\right)$

(d) $\tan\left(\sec^{-1}\sqrt{2}\right)$ (e) $\operatorname{cosec}\left[\cot^{-1}\left(-\sqrt{3}\right)\right]$

3. Simplify each of the following expressions :

(a) $\sec\left(\tan^{-1}x\right)$ (b) $\tan\left(\operatorname{cosec}^{-1}\frac{x}{2}\right)$ (c) $\cot\left(\operatorname{cosec}^{-1}x^2\right)$

(d) $\cos\left(\cot^{-1}x^2\right)$ (e) $\tan\left(\sin^{-1}\left(\sqrt{1-x}\right)\right)$

18.3 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property 1 $\sin^{-1}(\sin\theta) = \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution : Let $\sin\theta = x$

$\Rightarrow$ $$\theta = \sin^{-1}x$$
$$= \sin^{-1}(\sin\theta) = \theta$$

Also $\sin\left(\sin^{-1}x\right) = x$

Similarly, we can prove that

(i) $\cos^{-1}(\cos\theta) = \theta,\ 0 \le \theta \le \pi$

(ii) $\tan^{-1}(\tan\theta) = \theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Property 2 (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ (ii) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Solution : (i) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \quad x = \operatorname{cosec}\theta$

$\Rightarrow \quad \frac{1}{x} = \sin\theta$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) Let $\cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\therefore \quad \frac{1}{x} = \cos\theta$ or $\theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Property 3 (i) $\sin^{-1}(-x) = -\sin^{-1} x$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Solution : (i) Let $\sin^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \sin\theta$ or $x = -\sin\theta = \sin(-\theta)$

$\therefore \quad -\theta = \sin^{-1} x$ or $\theta = -\sin^{-1} x$

or $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) Let $\tan^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \tan\theta$ or $x = -\tan\theta = \tan(-\theta)$

$\therefore \quad \theta = -\tan^{-1} x$ or $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) Let $\cos^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \cos\theta$ or $x = -\cos\theta = \cos(\pi - \theta)$

$\therefore \quad \cos^{-1} x = \pi - \theta$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$

Property 4 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Soluton : (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Let $\sin^{-1} x = \theta \quad \Rightarrow \quad x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

or $\qquad \cos^{-1} x = \left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \qquad \theta + \cos^{-1} x = \frac{\pi}{2} \qquad$ or $\qquad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) Let $\cot^{-1} x = \theta \quad \Rightarrow x = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\therefore \qquad \tan^{-1} x = \frac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \tan^{-1} x = \frac{\pi}{2}$

or $\qquad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

(iii) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \qquad$ ___ $x = \operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

$\therefore \qquad \sec^{-1} x = \frac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \sec^{-1} x = \frac{\pi}{2}$

$\Rightarrow \operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Property 5 (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Solution : (i) Let $\tan^{-1} x = \theta$, $\tan^{-1} y = \phi \quad \Rightarrow x = \tan\theta, y = \tan\phi$

We have to prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

By substituting that above values on L.H.S. and R.H.S., we have

$$\text{L.H.S.} = \theta + \phi \text{ and } \quad \text{R.H.S.} = \tan^{-1}\left[\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right]$$

$$= \tan^{-1}\left[\tan(\theta + \phi)\right] = \theta + \phi = \text{L.H.S.}$$

$\therefore$ The result holds.
Simiarly (ii) can be proved.

Property 6 $$2\tan^{-1}x=\sin^{-1}\left[\frac{2x}{1+x^2}\right]=\cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]=\tan^{-1}\left[\frac{2x}{1-x^2}\right]$$

(i) (ii) (iii) (iv)

Let $x=\tan\theta$

Substituting in (i), (ii), (iii), and (iv) we get

$$2\tan^{-1}x=2\tan^{-1}(\tan\theta)=2\theta \qquad \text{......(i)}$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$

$$=\sin^{-1}(2\sin\theta\cos\theta)$$

$$=\sin^{-1}(\sin 2\theta)=2\theta \qquad \text{.....(ii)}$$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$=\cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$=\cos^{-1}(\cos^2\theta-\sin^2\theta)$$

$$=\cos^{-1}(\cos 2\theta)=2\theta \qquad \text{......(iii)}$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 2\theta)=2\theta \qquad \text{.....(iv)}$$

From (i), (ii), (iii) and (iv), we get

$$2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Property 7

(i) $$\sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$=\sec^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$=\cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$=\operatorname{cosec}^{-1}\left[\frac{1}{x}\right]$$

(ii) $$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \text{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{x}\right]$$

Proof : Let $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$

(i) $\cos\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$, $\sec\theta = \frac{1}{\sqrt{1-x^2}}$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$ and $\text{cosec}\,\theta = \frac{1}{x}$

$$\therefore \quad \sin^{-1} x = \theta = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \text{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) Let $\cos^{-1} x = \theta \Rightarrow \quad x = \cos\theta$

$\therefore \quad \sin\theta = \sqrt{1-x^2}, \quad \tan\theta = \frac{\sqrt{1-x^2}}{x}, \quad \sec\theta = \frac{1}{x}, \quad \cot\theta = \frac{x}{\sqrt{1-x^2}}$

and $\quad \text{cosec}\,\theta = \frac{1}{\sqrt{1-x^2}}$

$$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{x}\right)$$

Property 8

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$

(iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$

Proof (i) : Let $x = \sin\theta$, $y = \sin\phi$, then

L.H.S. = $\theta + \phi$

R.H.S. $= \sin^{-1}(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= \sin^{-1}\left[\sin(\theta+\phi)\right] = \theta + \phi$

$\therefore$ L.H.S. = R.H.S.

(ii) Let $x = \cos\theta$ and $y = \cos\phi$

L.H.S. = $\theta + \phi$

R.H.S.= $\cos^{-1}(\cos\theta\cos\phi - \sin\theta\sin\phi)$

$= \cos^{-1}\left[\cos(\theta+\phi)\right] = \theta + \phi$

$\therefore$ L.H.S. = R.H.S.

(iii) Let $x = \sin\theta$, $y = \sin\phi$

L.H.S. = $\theta - \phi$

R.H.S. = $\sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

$= \sin^{-1}\left[\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta}\right]$

$= \sin^{-1}[\sin\theta\cos\phi - \cos\theta\sin\phi]$

$= \sin^{-1}\left[\sin(\theta-\phi)\right] = \theta - \phi$

$\therefore$ L.H.S. = R.H.S.

(iv) Let $x = \cos\theta$, $y = \cos\phi$

$\therefore$ L.H.S = $\theta - \phi$

R.H.S. = $\cos^{-1}[\cos\theta\cos\phi + \sin\theta\sin\phi]$

$= \cos^{-1}\left[\cos(\theta-\phi)\right] = \theta - \phi$

$\therefore$ L.H.S. = R.H.S.

Example 18.5 Evaluate :

$$\cos\left[\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right]$$

Soluton : Let $\sin^{-1}\left(\frac{3}{5}\right)=\theta$ and $\sin^{-1}\left(\frac{5}{13}\right)=\phi$, then

$$\sin\theta=\frac{3}{5} \text{ and } \sin\phi=\frac{5}{13}$$

$$\Rightarrow \qquad \cos\theta=\frac{4}{5} \text{ and } \cos\phi=\frac{12}{13}$$

$\therefore$ The given expression becomes $\cos[\theta+\phi]$

$$=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$=\frac{4}{5}\cdot\frac{12}{13}-\frac{3}{5}\cdot\frac{5}{13} = \frac{33}{65}$$

Example 18.6 Prove that

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

Solution : Applying the formula :

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ we have}$$

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)=\tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

Example 18.7 Prove that

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$

Applying the property

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), \text{ we have}$$

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right)$$

$$=\cos^{-1}\left(\frac{33}{65}\right)$$

Example 18.8 Prove that

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

Solution : Let $\sqrt{x} = \tan\theta$ then

$$\text{L.H.S.} = \theta \text{ and R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \frac{1}{2}\times 2\theta = \theta$$

$\therefore$ L.H.S. = R.H.S.

Example 18.9 Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, x > 0$$

Solution : Let $x = \tan\theta$, then

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{1}{2}\theta$$

$$\Rightarrow \quad \frac{\pi}{4} - \theta = \frac{1}{2}\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}\times\frac{2}{3} = \frac{\pi}{6}$$

$$\therefore \quad x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Example 18.10 Show that

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)$$

Solution : Let $x^2 = \cos 2\theta$, then

$$2\theta = \cos^{-1}(x^2)$$

$$\Rightarrow \quad \theta = \frac{1}{2}\cos^{-1}x^2$$

Substituting $x^2 = \cos 2\theta$ in L.H.S. of the given equation, we have

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(x^2\right)$$

CHECK YOUR PROGRESS 18.2

1. Evaluate each of the following :

(a) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cot\left(\tan^{-1}\alpha+\cot^{-1}\alpha\right)$

(c) $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$

(d) $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ (e) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$

2. If $\cos^{-1}x+\cos^{-1}y=\beta$, prove that

$x^2-2xy\cos\beta+y^2=\sin^2\beta$

3. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that

$x^2+y^2+z^2+2xyz=1$

4. Prove each of the following :

(a) $\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{2}{\sqrt{5}}=\frac{\pi}{2}$ (b) $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

(c) $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$ (d) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

LET US SUM UP

- Inverse of a trigonometric function exists if we restrict the domain of it.

(i) $\sin^{-1} x = y$ if $\sin y = x$ where $-1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $\cos^{-1} x = y$ if $\cos y = x$ where $-1 \le x \le 1, 0 \le y \le \pi$

(iii) $\tan^{-1} x = y$ if $\tan y = x$ where $x \in R, -\frac{\pi}{2} < y < \frac{\pi}{2}$

(iv) $\cot^{-1} x = y$ if $\cot y = x$ where $x \in R, 0 < y < \pi$

(v) $\sec^{-1} x = y$ if $\sec y = x$ where $x \ge 1, 0 \le y < \frac{\pi}{2}$ or $x \le -1, \frac{\pi}{2} < y \le \pi$

(vi) $\text{cosec}^{-1} x = y$ if $\text{cosec}\, y = x$ where $x \ge 1, 0 < y \le \frac{\pi}{2}$

or $x \le -1, -\frac{\pi}{2} \le y < 0$

- Graphs of inverse trigonometric functions can be represented in the given intervals by interchanging the axes as in case of y = sin x, etc.
- **Properties :**

(i) $\sin^{-1}(\sin\theta) = \theta$, $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}\theta) = \theta$ and $\sin(\sin^{-1}\theta) = \theta$

(ii) $\text{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$, $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) $\sin^{-1}(-x) = -\sin^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(iv) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(vi) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\sin^{-1} x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$$

(viii) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(ix) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Prove each of the following :

   (a) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\sin^{-1}\left(\frac{77}{85}\right)$ (b) $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{9}\right)=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

   (c) $\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right)$

2. Prove each of the following :

   (a) $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{23}{11}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}2$

   (c) $\tan^{-1}\left(\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{1}{3}\right)$

3. (a) Prove that $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

   (b) Prove that $2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right)$

   (c) Prove that $\cos^{-1}x=2\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)=2\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

4. Prove the following :

   (a) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$ (b) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\frac{\pi}{4}-x$

   (c) $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=0$

5. Solve each of the following :

   (a) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ (b) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x)$

   (c) $\cos^{-1}x+\sin^{-1}\left(\frac{1}{2}x\right)=\frac{\pi}{6}$ (d) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12},\ x>0$

ANSWERS

CHECK YOUR PROGRESS 18.1

1. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$ (e) $\frac{\pi}{4}$

2. (a) $\frac{1}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{1}{2}$ (d) 1 (e) -2

3. (a) $\sqrt{1+x^2}$ (b) $\frac{2}{\sqrt{x^2-4}}$ (c) $\sqrt{x^4-1}$ (d) $\frac{x^2}{\sqrt{x^4+1}}$ (e) $\sqrt{\frac{1-x}{x}}$

CHECK YOUR PROGRESS 18.2

1. (a) 1 (b) 0 (c) $\frac{x+y}{1-xy}$ (d) $\frac{5}{12}$ (e) $-\frac{7}{17}$

TERMINAL EXERCISE

5. (a) $\frac{1}{6}$ (b) $\frac{\pi}{4}$ (c) ± 1 (d) $\sqrt{3}$