QUANTITATIVE REASONING

The Quantitative Reasoning domain tests your ability to use numbers and mathematical concepts to solve mathematical problems, as well as your ability to analyze data presented in a variety of ways, such as in table or graph form. Only a basic knowledge of mathematics is required (the material studied up to 9th or 10th grades in most Israeli high schools).

All of the Quantitative Reasoning problems take the form of multiple-choice questions, that is, a question followed by four possible responses, only one of which is the correct answer.

The Quantitative Reasoning sections consist of two categories of questions – Questions and Problems, and Graph or Table Comprehension.

Questions and Problems cover a variety of subjects taken from algebra and geometry. Some of the questions are presented in mathematical terms; others are word problems, which you must translate into mathematical terms before solving.

Graph or Table Comprehension questions relate to information appearing in a graph or a table. A graph presents data in graphic form, such as a bar chart, line graph or scatter plot. A table presents data arranged in columns and rows.

In general, all questions of a given type are arranged in ascending order of difficulty. The easier questions, requiring relatively less time to solve, appear first, with the questions becoming progressively more difficult and requiring more time to solve.

The figures accompanying some of the questions are not necessarily drawn to scale. Do not rely solely on the figure's appearance to deduce line length, angle size, and so forth. However, if a line in a figure appears to be straight, you may assume that it is, in fact, a straight line.

A Formula Page appears at the beginning of each Quantitative Reasoning section. This page contains instructions, general comments and mathematical formulas, which you may refer to during the test. The Formula Page also appears in the *Guide* (on the next page) and in the Quantitative Reasoning sections of the practice test. You should familiarize yourself with its contents prior to taking the test.

Pages 38-66 contain a review of basic mathematical concepts, covering much of the material upon which the questions in the Quantitative Reasoning sections are based. The actual test may, however, include some questions involving mathematical concepts and theorems that do not appear on these pages.

Pages 67-82 contain examples of different types of questions, each followed by the answer and a detailed explanation.

FORMULA PAGE

This section contains 20 questions.
The time allotted is 20 minutes.

This section consists of questions and problems involving Quantitative Reasoning. Each question is followed by four possible responses. Choose the correct answer and mark its number in the appropriate place on the answer sheet.

Note: The words appearing against a gray background are translated into several languages at the bottom of each page.

General Comments about the Quantitative Reasoning Section

* The figures accompanying some of the problems are provided to help solve the problems, but are not necessarily drawn to scale. Therefore, do not rely on the figures alone to deduce line length, angle size, and so forth.
* If a line in a figure appears to be straight, you may assume that it is in fact a straight line.
* When a geometric term (side, radius, area, volume, etc.) appears in a problem, it refers to a term whose value is greater than 0, unless stated otherwise.
* When $\sqrt{a}$ $(a > 0)$ appears in a problem, it refers to the positive root of a.
* "0" is neither a positive nor a negative number.
* "0" is an even number.
* "1" is not a prime number.

Formulas

1. **Percentages:** $a\%$ of x is equal to $\frac{a}{100} \cdot x$

2. **Exponents:** For every a that does not equal 0, and for any two integers n and m -

   a. $a^{-n} = \frac{1}{a^n}$

   b. $a^{m+n} = a^m \cdot a^n$

   c. $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$ $(0 < a, 0 < m)$

   d. $a^{n \cdot m} = (a^n)^m$

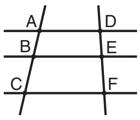
3. **Contracted Multiplication Formulas:**
   $(a \pm b)^2 = a^2 \pm 2ab + b^2$
   $(a + b)(a - b) = a^2 - b^2$

4. **Distance Problems:** $\frac{\text{distance}}{\text{time}}$ = speed (rate)

5. **Work Problems:** $\frac{\text{amount of work}}{\text{time}}$ = output (rate)

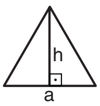
6. **Factorials:** $n! = n(n-1)(n-2) \cdot ... \cdot 2 \cdot 1$

7. **Proportions:** If $AD \parallel BE \parallel CF$ then $\frac{AB}{DE} = \frac{BC}{EF}$ and $\frac{AB}{AC} = \frac{DE}{DF}$

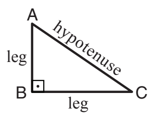


8. **Triangles:**

   a. The **area** of a triangle with base of length a and altitude to the base of length h is $\frac{a \cdot h}{2}$

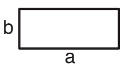


   b. **Pythagorean Theorem:** In any right triangle ABC, as in the figure, the following always holds true: $AC^2 = AB^2 + BC^2$

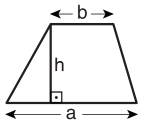


   c. In any right triangle whose angles measure 30°, 60°, 90°, the length of the leg opposite the 30° angle is equal to half the length of the hypotenuse.

9. **The area of a rectangle** of length a and width b is $a \cdot b$



10. **The area of a trapezoid** with one base length a, the other base length b, and altitude h is $\frac{(a+b) \cdot h}{2}$



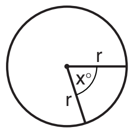
11. **The sum of the internal angles of an n-sided polygon** is $(180n - 360)$ degrees.
    In a regular n-sided polygon, **each internal angle measures** $\left(\frac{180n - 360}{n}\right) = \left(180 - \frac{360}{n}\right)$ degrees.

12. **Circle:**

    a. The **area** of a circle with radius r is πr^2 $(\pi = 3.14...)$

    b. **The circumference** of a circle is $2\pi r$

    c. **The area of a sector of a circle** with a central angle of $x°$ is $\pi r^2 \cdot \frac{x}{360}$

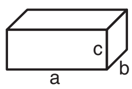


13. **Box (Rectangular Prism), Cube:**

    a. The **volume** of a box of length a, width b and height c is $a \cdot b \cdot c$

    b. The **surface area** of the box is $2ab + 2bc + 2ac$

    c. In a **cube**, $a = b = c$

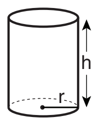


14. **Cylinder:**

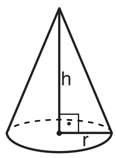
    a. The **lateral surface area** of a cylinder with base radius r and height h is $2\pi r \cdot h$

    b. The **surface area** of the cylinder is $2\pi r^2 + 2\pi r \cdot h = 2\pi r(r + h)$

    c. The **volume** of the cylinder is $\pi r^2 \cdot h$



15. **The volume of a cone** with base radius r and height h is $\frac{\pi r^2 \cdot h}{3}$



16. **The volume of a pyramid** with base area S and height h is $\frac{S \cdot h}{3}$

REVIEW OF BASIC MATHEMATICAL CONCEPTS

SYMBOLS

Symbol	Meaning of the Symbol
$a \parallel b$	lines a and b are parallel
$a \perp b$	line a is perpendicular to straight line b
⊾	90º angle (right angle)
$\sphericalangle ABC$	the angle formed by line segments AB and BC
$x = y$	x equals y
$x \neq y$	x does not equal y
$x < y$	x is less than y
$x \leq y$	x is less than or equal to y
$a < x, y$	both x and y are greater than a
$x = \pm a$	x may be equal to a or to (-a)
$\lvert x \rvert$	the absolute value of x
$x : y$	the ratio of x to y

TYPES OF NUMBERS

Integer: An integer, also called a whole number, is a number composed of whole units. An integer may be positive, negative, or 0 (zero).

Example: ... , -4 , -3 , -2 , -1 , 0 , 1 , 2 , 3 , 4 , ...

Note: 0 (zero) is an integer that is neither positive nor negative.

Non-integer: A number that cannot be expressed in whole units.

Example: $\sqrt{2}$, $-1\frac{1}{2}$, $2\frac{1}{2}$, 1.37

Consecutive numbers: Integers that follow in sequence in differences of 1. For example, 4 and 5 are consecutive numbers; 2, 3, and 4 are consecutive numbers; (-3) and (-2) are also consecutive numbers.

If n is an integer, then n and (n + 1) are consecutive numbers. This is sometimes expressed as: (n + 1) is the next consecutive integer after n.

Even number: An integer which, when divided by 2, produces an integer (in other words, it is evenly divisible by 2). If n is an integer, then 2n is an even number.

Note: 0 is an even number.

Odd number: An integer which, when divided by 2, produces a non-integer (in other words, when it is divided by 2, a remainder of one is obtained). If n is an integer, then 2n + 1 is an odd number.

Prime number: A positive integer that is evenly divisible by only two numbers – itself and the number 1.

For example, 13 is a prime number because it is evenly divisible by only 1 and 13.

Note: 1 is **not** defined as a prime number.

Opposite numbers (additive inverse):

A pair of numbers whose sum equals zero.

For example, 4 and (-4) are opposite numbers.

In general, a and $(-a)$ are opposite numbers $(a + (-a) = 0)$. In other words, $(-a)$ is the opposite number of a.

Reciprocals (multiplicative inverse:)

A pair of numbers whose product is equal to 1.

For example, 3 and $\frac{1}{3}$ are reciprocals, as are $\frac{2}{7}$ and $\frac{7}{2}$.

In general, for $a, b \neq 0$:

a and $\frac{1}{a}$ are reciprocals $\left(a \cdot \frac{1}{a} = 1\right)$. We can also say that $\frac{1}{a}$ is the reciprocal of a.

$\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals $\left(\frac{a}{b} \cdot \frac{b}{a} = 1\right)$, or in other words, $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

Absolute value: If $x > 0$, then $|x| = x$.

If $0 > x$, then $|x| = -x$.

$|0| = 0$.

ARITHMETICAL OPERATIONS WITH EVEN AND ODD NUMBERS

even	+	even	=	even
odd	+	odd	=	even
odd	+	even	=	odd
even	–	even	=	even
odd	–	odd	=	even
even	–	odd	=	odd
odd	–	even	=	odd
even	×	even	=	even
odd	×	odd	=	odd
odd	×	even	=	even

There are no similar rules for division. The quotient of two even numbers may be odd $\left(\frac{6}{2} = 3\right)$, even $\left(\frac{4}{2} = 2\right)$, or a non-integer $\left(\frac{6}{4} = 1\frac{1}{2}\right)$.

FACTORS (DIVISORS) AND MULTIPLES

A **factor (divisor)** of a positive integer is any positive integer that divides it evenly (that is, without a remainder). For example, the factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

A **common factor** of x and y is a number that is a factor of x and also a factor of y. For example, 6 is a common factor of 24 and 30.

A **prime factor** is a factor that is also a prime number. For example, the prime factors of 24 are 2 and 3. Any positive integer (greater than 1) can be written as the product of prime factors. For example, $24 = 3 \cdot 2 \cdot 2 \cdot 2 = 3 \cdot 2^3$

A **multiple** of an integer x is any integer that is evenly divisible by x. For example, 16, 32, and 88 are multiples of 8.

When the word "divisible" appears in a question, it means "evenly divisible" or "divisible without a remainder."

MATHEMATICAL OPERATIONS WITH FRACTIONS

Reduction

When the numerator and denominator of a fraction have a common factor, each of them can be divided by that common factor. The resulting fraction, which has a smaller numerator and denominator, equals the original fraction. For example, if we divide the numerator and the denominator of $\frac{16}{12}$ by 4, the result is $\frac{4}{3}$, $\left(\frac{16}{12} = \frac{4}{3}\right)$.

Multiplication

To multiply two fractions, multiply the numerators by each other and the denominators by each other.

Example: $\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$

Division

To divide a number by a fraction, multiply the number by the reciprocal of the divisor.

Example: $\frac{\frac{2}{5}}{\frac{3}{8}} = \frac{2}{5} \cdot \frac{8}{3} = \frac{2 \cdot 8}{5 \cdot 3} = \frac{16}{15}$

To multiply or divide an integer by a fraction, the integer can be regarded as a fraction whose denominator is 1.

Example: $2 = \frac{2}{1}$

Addition and Subtraction

To add or subtract fractions, they must be converted into fractions that have a common denominator. A **common denominator** is a number that is evenly divisible by the denominators of all of the original fractions. (That is, the denominators of the original fractions are all factors of the common denominator.) After finding a suitable common denominator, each of the fractions must be converted into a fraction that has this common denominator. To do so, multiply the numerator and denominator of each of the fractions by the same integer, so that the number obtained in the denominator will be the number that was chosen to be the common denominator. Multiplying the numerator and denominator by the same number is the same as multiplying by 1, and its value remains unchanged. Once the fractions have a common denominator, add or subtract the new numerators and reduce the resulting fractions to lowest terms where possible.

EXAMPLE

$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}=?$

We know that 4, 6, and 8 are all factors of 24. Therefore, we can use 24 as a common denominator: $\frac{24}{4}=6$, $\frac{24}{6}=4$, $\frac{24}{8}=3$

We will now convert each of the fractions into fractions with this common denominator:

$\frac{3}{4}=\frac{18}{24}$, $\frac{1}{6}=\frac{4}{24}$, $\frac{5}{8}=\frac{15}{24}$

Thus, $\frac{3}{4}+\frac{1}{6}+\frac{5}{8}=\frac{18}{24}+\frac{4}{24}+\frac{15}{24}=\frac{18+4+15}{24}=\frac{37}{24}$

PERCENTAGES

Percentages are a way of expressing hundredths: a% of x is a hundredths of x, or $\frac{a}{100}\cdot x$. In questions containing percentages, convert the percentages to hundredths, and solve as in ordinary fraction problems.

EXAMPLE

What is 60 percent of 80?

Instead of 60 percent, substitute 60 hundredths and solve it as you would for any multiplication of fractions: $\frac{60}{100}\cdot 80=\frac{60\cdot 80}{100}=6\cdot 8=48$.

Thus, 60% of 80 is 48.

Questions dealing with change expressed as a percentage refer to the percentage of the original value, unless otherwise specified.

EXAMPLE

The price of an item that cost 80 shekels was raised by 25%. What is the new price?

Since 25% was added to the old price, the new price is 125% of the old price (100% + 25%). Therefore, you must calculate 125% of 80.

Substitute hundredths for percent and solve: $\frac{125}{100}\cdot 80=100$. Thus, the new price is 100 shekels.

EXAMPLE

The price of an item dropped from 15 shekels to 12 shekels. By what percentage did the price drop?

In this example, the change in the price of an item is given, and you are asked to calculate this change as a percentage.

The difference in the price is 3 shekels out of 15 shekels. You have to calculate what percent of 15 is 3.

Convert the question into a mathematical expression: $\frac{a}{100} \cdot 15 = 3$. Solve the equation for a:
$a = \frac{3 \cdot 100}{15} = 20$

Thus, the price dropped by 20%.

RATIO

The ratio of x to y is written as x : y.

EXAMPLE

The ratio between the number of pairs of socks and the number of shirts that Eli has is 3 : 2. In other words, for every 3 pairs of socks, Eli has 2 shirts. Stating it differently, the number of socks that Eli has is $\frac{3}{2}$ greater than the number of shirts that he has.

MEAN (AVERAGE)

The **arithmetic mean (average)** of a set of numerical values is the sum of the values divided by the number of values in the set.

When the word "average" appears in a question, it refers to the arithmetic mean.

For example, the average of the set of values 1, 3, 5, 10, and 21 is 8 because
$\frac{1+3+5+10+21}{5} = \frac{40}{5} = 8$

If the average of a set of numerical values is given, their sum can be calculated by multiplying the average by the number of values.

EXAMPLE

Danny bought 5 items whose average price is 10 shekels. How much did Danny pay for all of the items?

In this question we are asked to find the sum based on the average. If we multiply the average by the number of items, we will obtain $10 \cdot 5 = 50$. Thus, Danny paid a total of 50 shekels for all of the items which he bought.

A **weighted average** is an average that takes into account the relative weight of each of the values in a set.

EXAMPLE

Robert's score on the midterm exam was 75, and his score on the final exam was 90. If the weight of the final exam is twice that of the midterm exam, what is Robert's final grade in the course?

The set of values in this case is 75 and 90, but each has a different weight in Robert's final grade for the course. The score of 75 has a weight of 1, while the score of 90 has a weight of 2. To calculate the weighted average, multiply each score by the weight assigned to it, and divide by the sum of the weights: $\frac{1 \cdot 75 + 2 \cdot 90}{1+2} = 85$. Thus, Robert's final grade in the course is 85.

This calculation is identical to the calculation of a simple average of the three numbers 75, 90 and 90.

POWERS AND ROOTS

Raising a number to the nth power (when n is a positive integer) means multiplying it by itself n times: $a^n = \underbrace{a \cdot \ldots \cdot a \cdot a}_{n \text{ times}}$.

For example: $(-3)^3 = (-3)(-3)(-3) = -27$.

The expression a^n is called a power; n is called the exponent; and a is called the base.

Any number other than zero, raised to the 0th power, equals 1. Thus, for any $a \neq 0$, $a^0 = 1$.

A power with a negative exponent is defined as the power obtained by raising the reciprocal of the base to the opposite power: $a^{-n} = \left(\frac{1}{a}\right)^n$. Example: $2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

The n^{th} root of a positive number a, expressed as $\sqrt[n]{a}$, is the positive number b, which if raised to the n^{th} power, will give a. In other words, $\sqrt[n]{a} = b$ because $b^n = a$. For example, $\sqrt[4]{81} = 3$ because $3^4 = 81$.

When the root is not specified, a 2nd-order root is intended. A 2nd-order root is also called a square root. For example, $\sqrt{81} = \sqrt[2]{81} = 9$.

A root can also be expressed as a power in which the exponent is a fraction. This fraction is the reciprocal of the order of the root: $\sqrt[n]{a} = a^{\frac{1}{n}}$ $(0 < a)$.

Note: When $\sqrt{a}$ $(0 < a)$ appears in a question, it refers to the positive root of a.

Basic rules for operations involving powers (for any m and n):

Multiplication: To multiply powers with the same base, add the exponents: $a^m \cdot a^n = a^{(m+n)}$.

Division: To divide a power by another power with the same base, subtract the exponent in the denominator from the exponent in the numerator: $\frac{a^m}{a^n} = a^{(m-n)}$.

Note: When the powers do not have the same base, the exponents cannot be added or subtracted.

Raising to a power: To raise a power to a power, multiply the exponents: $(a^m)^n = a^{(m \cdot n)}$.

Raising a product or a quotient to a power: $(a \cdot b)^m = a^m \cdot b^m$, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

Since roots can also be expressed as powers, the rules for powers can also be applied to roots.

For example, to calculate the product $\sqrt[m]{a} \cdot \sqrt[n]{a}$ $(0 < a)$, first express the roots as powers:
$\sqrt[m]{a} \cdot \sqrt[n]{a} = a^{\frac{1}{m}} \cdot a^{\frac{1}{n}}$.

Then multiply the powers, that is, add the exponents: $a^{\frac{1}{m}} \cdot a^{\frac{1}{n}} = a^{\left(\frac{1}{m}+\frac{1}{n}\right)}$.

Inequalities involving powers:

If	$0 < b < a$	and	$0 < n$	then	$b^n < a^n$
If	$0 < b < a$	and	$n < 0$	then	$a^n < b^n$
If	$1 < a$	and	$m < n$	then	$a^m < a^n$
If	$0 < a < 1$	and	$m < n$	then	$a^n < a^m$

CONTRACTED MULTIPLICATION FORMULAS

To multiply two expressions enclosed in parentheses, each of which is the sum of two terms, multiply each of the terms in the first expression by each of the terms in the second expression, then add the products.

For example, $(a + b) \cdot (c + d) = ac + ad + bc + bd$.

This general formula can be used for finding the product of any two expressions, but to save time, you might want to memorize some common formulas:

$(a + b)^2 = (a + b) \cdot (a + b) = a^2 + 2ab + b^2$

$(a - b)^2 = (a - b) \cdot (a - b) = a^2 - 2ab + b^2$

$(a - b) \cdot (a + b) = a^2 - b^2$

COMBINATORIAL ANALYSIS

Multi-Stage Experiment

EXAMPLE

If we toss a die and then toss a coin, how many different results are possible?

This experiment has two stages – tossing the die and tossing the coin.

The number of possible results of tossing a die is 6. The number of possible results of tossing a coin is 2. Thus, the number of possible results of the entire experiment is $6 \cdot 2 = 12$. One of the 12 possible results is the number 3 on the die and tails on the coin.

In effect, it makes no difference whether we first toss the die and only then toss the coin, or first toss the coin and then toss the die, or toss both at the same time. In each case, there are 12 possible results.

Now let us consider a multi-stage experiment with a set of n items, out of which an item is selected at random r times. Each selection of an item from the set constitutes a stage in the experiment, so that the experiment has a total of r stages. The number of possible results in each of the r stages depends on the sampling method by which an item is selected. The total number of possible results of the entire experiment is the product of the number of possible results obtained in each of the r stages. Each possible result in the experiment is referred to as a **sample**.

There are four basic types of multistage experiments. They are designated by the sampling method used: whether or not the order of sampling matters (called ordered and unordered) and whether or not the sampled item is returned to the original set (called with replacement and without replacement).

Ordered Samples with Replacement

Sampling method: The sampled item is immediately **returned** to the set and **the order in which the items are sampled matters**. Note: In this sampling method, an item may be sampled more than once.

Number of possible results: The number of possible results in each stage is n. Thus, the number of possible results of all r stages – that is, of the entire experiment – is $n \cdot n \cdot ... \cdot n = n^r$.

The number of ordered samples with replacement is n^r.

EXAMPLE

A box contains 9 balls, numbered 1 through 9. One ball is removed at random from the box and replaced, and this process is repeated two more times. The numbers on the balls that are removed from the box are written down in the order in which they are removed, forming a three-digit number. How many different three-digit numbers can be obtained in this way?

In this question, the order in which the results are obtained is important. For example, if balls numbered 3, 8, and 3 are removed, in that order, the number 383 is obtained; but if the order in which they are removed is 3, 3, and 8, the result is a different number – 338.

There are 3 stages in this experiment, and the number of possible results in each stage is 9. Thus, the number of possible results of the entire experiment is $9^3 = 729$. In other words, 729 different three-digit numbers can be obtained.

Ordered Samples Without Replacement

Sampling method: The sampled item is **not returned** to the set after being sampled, and **the order in which the items are sampled matters**.

Number of possible results: The number of possible results in the first stage is n; the number of possible results in the second stage is $n-1$ (because the item that was sampled in the first stage is not returned, and only $n-1$ items remain to be sampled); and so on, until the last stage, stage r, in which the number of possible results is $n-r+1$. Thus, the number of possible results of the entire experiment is $n \cdot (n-1) \cdot ... \cdot (n-r+1)$.

The number of ordered samples without replacement is $n \cdot (n-1) \cdot ... \cdot (n-r+1)$.

EXAMPLE

A box contains 9 balls, numbered 1 through 9. Three balls are removed at random from the box, one after another, and are not replaced. The numbers on the balls removed from the box are written down in the order in which they are removed, forming a three-digit number. How many different three-digit numbers can be obtained in this way?

In this experiment, too, the order in which the balls are removed is important, but unlike the previous example, in this experiment a ball that is removed from the box is not returned. Thus, the number of possible results in the first stage is 9, in the second stage, 8, and in the third stage, 7. The number of possible results of the entire experiment is $9 \cdot 8 \cdot 7 = 504$. In other words, 504 different three-digit numbers can be obtained.

Arrangements (Permutations) of an Ordered Sample

When creating an ordered sample without replacement out of all n items in a set (that is, if $r = n$), each possible result describes the arrangement of the items – which item is first, which is second, and so on. The question is: How many possible arrangements are there?

If we substitute $r = n$ in the formula for obtaining the number of ordered samples without replacement, we obtain $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$. This number is called "n factorial" and is written as $n!$.

The number of possible arrangements of n items is $n!$.

EXAMPLE

A grandmother, mother, and daughter wish to arrange themselves in a row in order to be photographed. How many different ways can they arrange themselves?

Let us regard the person on the right as the first in the set, the person in the middle as the second, and the person on the left as the third. The question is then: How many possible arrangements are there of the grandmother, mother and daughter? The grandmother, mother, and daughter can be considered a set of 3 items. Thus, the number of possible arrangements for this set is $3! = 3 \cdot 2 \cdot 1 = 6$. The possible arrangements are: grandmother-mother-daughter, grandmother-daughter-mother, mother-grandmother-daughter, mother-daughter-grandmother, daughter-grandmother-mother, daughter-mother-grandmother.

Unordered Samples

Sampling method: The sampled item is **not returned** to the set after being sampled, and **the order** in which the items are sampled **does not matter**.

When the order does not matter, all samples containing the same r items (only the sampling order is different in each sample) are regarded as the same result. The number of samples of this type is actually the number of arrangements of the r items, that is, $r!$.

To calculate the possible number of results in an unordered sample, calculate the number of possible results as if the order matters, and divide it by the number of arrangements of the r items.

$$\textbf{The number of unordered samples} = \frac{\textbf{the number of ordered samples without replacement}}{\textbf{the number of arrangements in the sample}}$$

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-r+1)}{r!}$$

EXAMPLE

A box contains 9 balls, numbered 1 through 9. Three balls are removed at random from the box, one after another, and are not replaced. The balls that were removed are placed in a hat. How many possible different combinations are there of the balls in the hat?

In this question, the composition of the balls in the hat is important and not the order in which they are removed from the box. For example, if the balls are removed in the order of 5, 1, and 4, the composition of the balls in the hat is 1, 4, and 5. This would be the composition of the balls in the hat even if they are removed from the box in the order of 4, 5, and 1 or in any other of the 3! possible orders: 1-4-5, 1-5-4, 4-1-5, 4-5-1, 5-1-4, and 5-4-1. (Actually, the fact that the balls are removed one after another is irrelevant; they could be removed at the same time without affecting the result.)

Thus, the number of possible combinations is $\frac{9 \cdot 8 \cdot 7}{3!}$, which equals 84 different possible combinations of the balls in the hat.

PROBABILITY

Probability theory is a mathematical model for phenomena (experiments) whose occurrence is not certain. Each possible outcome of an experiment is called a simple event, and the collection of outcomes is called an event. (For the sake of brevity, we will use the term "event" to also denote a simple event.) Each event is assigned a number from 0 to 1, which represents the probability (likelihood) that the event will occur. The higher the probability, the greater the chance that the event will occur. An event that is certain to occur has a probability of 1, and an event that has no possibility of occurring has a probability of 0.

The sum of the probabilities of all the simple events in an experiment is 1.

When each of the n-possible results of a particular experiment has an equal likelihood of occurring, we say that they have equal probabilities. In such a case, each result has a probability of $\frac{1}{n}$.

EXAMPLE

Experiment: The tossing of a coin.

Possible results: Either side of the coin. They are marked 1 or 0 (heads or tails).

If we are tossing a **fair coin**, both results are equally likely. In other words, the probability of "1" coming up is equal to the probability of "0" coming up. Thus, the probability of each possible result is $\frac{1}{2}$.

EXAMPLE

Experiment: The tossing of a fair die.

Possible results: The numbers 1, 2, 3, 4, 5, and 6 which are marked on the faces of the die.

If we are tossing a **fair die**, the probability of obtaining each of the possible results is $\frac{1}{6}$.

When all possible results are equally likely,

the probability that an event will occur is:

$$\frac{\textbf{the number of possible results of a particular event}}{\textbf{the total number of possible results of the experiment}}$$

EXAMPLE

Experiment: The tossing of a fair die.

Event: The result is less than 4.

Results of this event: the numbers 1, 2, and 3.

Probability of the event: $\frac{3}{6} = \frac{1}{2}$.

EXAMPLE

Experiment: The removal of a ball from a box containing 5 white balls and 5 black balls.

Event: The removal of a black ball.

Probability of the event: $\frac{\text{the number of black balls}}{\text{the total number of balls in the box}} = \frac{5}{10} = \frac{1}{2}$.

The probability that two events will occur

When two events occur at the same time or one after another, there are two possible scenarios:

A. The events are independent, that is, the probability of one event occurring is not affected by the occurrence of the other event.

B. The events are dependent, that is, the probability of one event is affected by the occurrence of another event. In other words, the probability of a particular event occurring after (or given that) another event has occurred, is different from the probability of that particular event occurring independent of the other event.

EXAMPLE

There are 10 balls in a box, 5 white and 5 black. Two balls are removed from the box, one after another.
The first ball that is removed is black.
What is the probability that the second ball that is removed is also black?

(a) The first ball is returned to the box.

Since the first ball is returned to the box, there is no change either in the total number of balls in the box or in the number of black balls.

The probability of removing a second black ball is $\frac{5}{10} = \frac{1}{2}$ and is equal to the probability that the first ball that was removed is black.

The fact that a black ball was removed the first time does not change the probability of removing a black ball the second time. In other words, the two events are independent.

(b) The first ball is not returned to the box.

After removing a black ball, a total of 9 balls remain in the box, of which 4 are black. Thus, the probability of removing a second black ball is $\frac{4}{9}$.

This probability is different from the probability of removing a black ball the first time. Thus, the second event is dependent on the first event.

The probability of two independent events occurring (in parallel or one after another) is equal to the product of the probabilities of each individual event occurring.

EXAMPLE

Experiment: The tossing of two fair dice, one red and the other yellow.

Event A: obtaining a number that is less than 3 on the red die. The probability of event A is $\frac{2}{6} = \frac{1}{3}$.

Event B: obtaining an even number on the yellow die. The probability of event B is $\frac{3}{6} = \frac{1}{2}$.

Since the result of tossing one die does not affect the probability of the result obtained by tossing the other die, event A and event B are independent events.

The probability of both event A and event B occurring (together) is thus,

$\left(\text{the probability of event A} \times \text{the probability of event B}\right)$, that is, $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

Let us define two dependent events, A and B (in any given experiment).

The probability of event B occurring given that event A has occurred is:

$$\frac{\textbf{the number of results common to both A and B}}{\textbf{the number of results of A}}$$

EXAMPLE

Experiment: The tossing of a die.
What is the probability of obtaining a result that is less than 4 if we know that the result obtained was an even number?

Event A: an even result is obtained.
Event B: a result that is less than 4 is obtained.

We will rephrase the question in terms of the events: What is the probability of B, given that we know that A occurred?

Event A has 3 results: 2, 4, and 6.
Event B has 3 results: 1, 2, and 3.

But, if we know that event A occurred, there is only one possible result for B – 2.

In other words, the result 2 is the only result that is common to both A and B.

Thus, the probability of B, given that we know that A occurred is $\frac{1}{3}$.

This probability is different from the probability of B occurring (without our knowing anything about A), which is equal to $\frac{1}{2}$.

Distance Problems: Distance, Speed (Rate), Time

The speed (rate) at which an object moves is the distance that the object covers in a unit of time.

The formula for the relationship between the speed, the distance the object covers, and the amount of time it requires to cover that distance is:

$v = \frac{s}{t}$ where v = speed (rate)
s = distance
t = time

All possible relationships between distance, speed and time can be derived from this formula: $t = \frac{s}{v}$ and $s = v \cdot t$.

EXAMPLE

A train traveled 240 kilometers (km) at a speed of 80 kilometers per hour (kph). How long did the journey take?

We are given v (80 kph) and s (240 km), and we have to calculate t.

Since the speed is given in kilometers per hour, the traveling time must be calculated in hours.

Substituting the given information into the formula $t = \frac{s}{v}$, we get $t = \frac{240}{80} = 3$.

Thus, the journey took 3 hours.

The unit of measurement of two of the variables determines the unit of measurement of the third variable.

For example, if the distance is expressed in kilometers (km) and the time in hours, then the speed will be expressed in kilometers per hour (kph).

If the distance is expressed in meters and the time in seconds, then the speed will be expressed in meters per second.

Meters can be converted to kilometers and seconds to hours, and vice versa. There are 1,000 meters in every kilometer (1 meter = $\frac{1}{1,000}$ kilometer).

In every hour, there are 3,600 seconds, which equal 60 minutes (1 second = $\frac{1}{3,600}$ hour).

A speed of 1 meter per hour is equal to a speed of $\frac{5}{18}$ meters per second $\left(\frac{1,000}{3,600} = \frac{5}{18}\right)$.

A speed of 1 meter per second is equal to a speed of 3.6 kilometers per hour $\left(\frac{\frac{1}{1,000}}{\frac{1}{3,600}} = \frac{3,600}{1,000} = 3.6\right)$

Work Problems: Output (Rate), Work, Time

Output is the amount of work per unit of time.

The formula for the relationship between output, amount of work and the time needed to do the work is $p = \frac{w}{t}$, where p = output (rate)

w = amount of work

t = time

All possible relationships between output, amount of work and time can be derived from this formula: $t = \frac{w}{p}$ and $w = p \cdot t$.

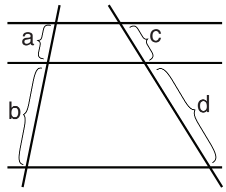
EXAMPLE

A builder can finish building one wall in 3 hours. How many hours would be needed for two builders working at the same rate to finish building 5 walls?

We are given the amount of work of one builder (1 wall), and the amount of time he spent working (3 hours). Therefore his output is $\frac{1}{3}$ of a wall in an hour. Since the question involves two builders, the output of the two of them together is $2 \cdot \frac{1}{3} = \frac{2}{3}$ walls per hour.

We are also given the amount of work which the two builders must do – 5 walls. We can therefore calculate the amount of time they will need: $t = \frac{5}{\frac{2}{3}} = 5 \cdot \frac{3}{2} = \frac{15}{2} = 7\frac{1}{2}$. Thus, they will need $7\frac{1}{2}$ hours to build the walls.

PARALLEL (STRAIGHT) LINES

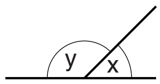


Parallel lines that intersect any two lines divide those lines into segments that are proportional in length.
Thus, in the figure, $\frac{a}{c} = \frac{b}{d}$, $\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{a+b} = \frac{c}{c+d}$.
Other relationships between the segments can be deduced based on the above relationships.

ANGLES

A right angle is a 90° angle.
In all of the figures, right angles are marked by ⊾.
An acute angle is an angle that is less than 90°.
An obtuse angle is an angle that is greater than 90°.
A straight angle is a 180° angle.

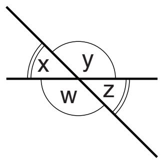
Adjacent Supplementary Angles



The two angles that are formed between a straight line and a ray that extends from a point on the straight line are called adjacent supplementary angles. Together they form a straight angle and their sum therefore equals 180°.

For example, in the figure, x and y are adjacent supplementary angles; thus, $x + y = 180°$.

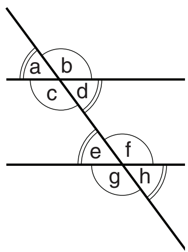
Vertical Angles



When two straight lines intersect, they form four angles. Each pair of non-adjacent angles are called vertical angles and they are equal in size.

For example, in the figure, x and z are vertical angles, as are w and y. Therefore, $x = z$ and $w = y$.

When two parallel lines are intersected by another line (called a transversal), eight angles are formed. In the figure, these angles are designated a, b, c, d, e, f, g, and h .



Corresponding Angles

Angles located on the same side of the transversal and on the same side of the parallel lines are referred to as corresponding angles. Corresponding angles are equal.

Thus, in the figure, $a = e$, $b = f$, $c = g$, $d = h$.

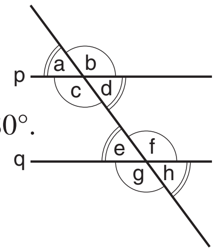
Alternate Angles

Angles located on opposite sides of the transversal and on opposite sides of the parallel lines are called alternate angles. Alternate angles are equal.

Thus, in the figure, $a = h$, $b = g$, $c = f$, $d = e$.

EXAMPLE

Given: Straight lines p and q are parallel.



$d + f = ?$

c and d are adjacent supplementary angles. Therefore, $d + c = 180°$.

c and f are alternate angles. Therefore, $c = f$.

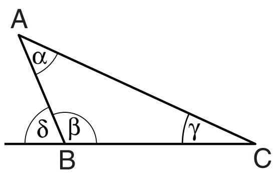
Thus, $d + f = d + c = 180°$, and the answer is 180°.

TRIANGLES

Angles of a Triangle

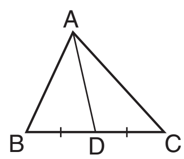
The sum of the interior angles of any triangle is 180°. For example, in the figure, $\alpha + \beta + \gamma = 180°$.

An angle formed by the extension of one side of a triangle and the adjacent side is called an exterior angle, and it equals the sum of the other two angles of the triangle. For example, in the figure, δ is the angle adjacent to β, and therefore $\delta = \alpha + \gamma$.



In any triangle, the longer side lies opposite the larger angle. For example, in the figure, if $\gamma < \alpha < \beta$, it follows that side AC (which is opposite angle β) is longer than side BC (which is opposite angle α), and side BC is longer than side AB (which is opposite angle γ).

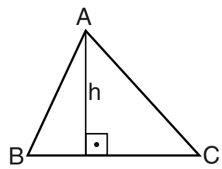
The **median of a triangle** is a line segment that joins a vertex of a triangle to the midpoint of the opposite side.



For example, in the triangle in the figure, AD is the median to side BC (BD = DC).

Altitude of a Triangle

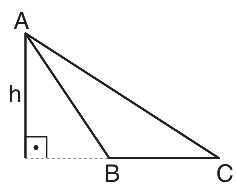
The altitude to a side of a triangle is a line that is drawn from a vertex of the triangle to the opposite side (or its extension) and is perpendicular to that side.



For example, in the triangles in the figures, h is the altitude to side BC.

Area of a Triangle

The area of a triangle equals half the length of one of the sides multiplied by the altitude to that side.



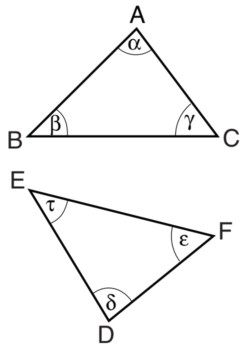
For example, the area of each triangle ABC in the above figures is $\frac{BC \cdot h}{2}$.

Inequality in a Triangle

In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.

For example, in the triangles in the figures, $AB + BC > AC$.

Congruent Triangles



Two geometric figures are congruent if one of them can be placed on the other in such a way that the two coincide. **Congruent triangles** are one example of congruent geometric figures. In congruent triangles, the corresponding sides and angles are equal.

For example, if triangles ABC and DEF in the figure are congruent, then their corresponding sides are equal: AB = DE, BC = EF, and AC = DF, and their corresponding angles are equal: $\alpha = \delta$, $\beta = \tau$, and $\gamma = \varepsilon$.

Each of the following four theorems enables us to deduce that two triangles are congruent:

(a) Two triangles are congruent if two sides of one triangle equal the two corresponding sides of the other triangle and the angle between these sides in one triangle equals the corresponding angle in the other triangle (Side-Angle-Side – SAS).

For example, if AB = DE, AC = DF, and $\alpha = \delta$, then the two triangles in the figure are congruent.

(b) Two triangles are congruent if two angles of one triangle equal the two corresponding angles of another triangle, and the length of the side between these angles in one triangle equals the length of the corresponding side in the other triangle (Angle-Side-Angle – ASA).

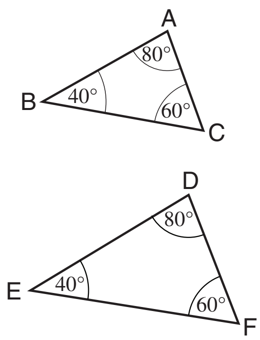
For example, if $\alpha = \delta$, $\beta = \tau$, and AB = DE, then the two triangles in the figure are congruent.

(c) Two triangles are congruent if the three sides of one triangle equal the three sides of the other triangle (Side-Side-Side – SSS).

(d) Two triangles are congruent if two sides of one triangle equal the corresponding two sides of the other triangle, and the angle opposite the longer of the two sides of one triangle is equal to the corresponding angle in the other triangle (Side-Side-Angle – SSA).

For example, the triangles in the figure above are congruent if AB > AC and DE > DF; and AB = DE, AC = DF, and $\gamma = \varepsilon$.

Similar Triangles



Two triangles are similar if the three angles of one triangle are equal to the three angles of the other triangle.

In similar triangles, the ratio between any two sides of one triangle is the same as the ratio between the corresponding two sides of the other triangle.

For example, in the figure, triangles ABC and DEF are similar. Therefore, $\frac{AB}{AC} = \frac{DE}{DF}$.

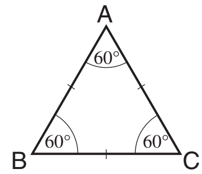
It also follows that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

Congruent triangles are necessarily also similar triangles.

TYPES OF TRIANGLES

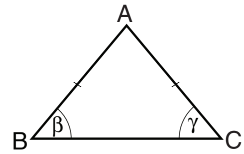
An **equilateral triangle** is a triangle whose sides are all of equal length. For example, in the figure, AB = BC = AC. In a triangle of this type, all of the angles are also equal (60°).

If the length of the side of such a triangle is a, then its altitude is $a \cdot \frac{\sqrt{3}}{2}$ and its area is $a^2 \cdot \frac{\sqrt{3}}{4}$.



An **isosceles triangle** is a triangle with two sides of equal length. For example, in the figure, AB = AC. The third side of an isosceles triangle is called the base.

The two angles opposite the equal sides are equal. For example, in the figure, β = γ.



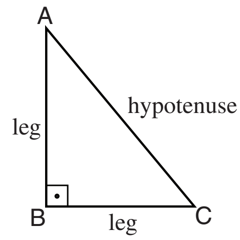
An **acute triangle** is a triangle in which all the angles are acute.

An **obtuse triangle** is a triangle with one obtuse angle.

A **right triangle** is a triangle with one right angle (90°). The side opposite the right angle (side AC in the figure) is called the **hypotenuse**, and the other two sides are called **legs** (sides AB and BC in the figure).

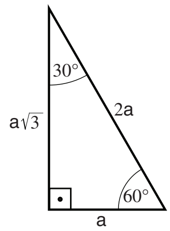
According to the Pythagorean theorem, in a right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.
For example, in the figure, $AC^2 = AB^2 + BC^2$.

This formula can be used to find the length of any side if the lengths of the other two sides are given.



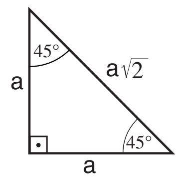
In a right triangle whose angles measure 30°, 60° and 90°, the length of the leg opposite the 30° angle equals half the length of the hypotenuse.

For example, in the figure, the length of the hypotenuse is 2a. Therefore, the length of the leg opposite the 30° angle is a. It follows from the Pythagorean theorem that the length of the leg opposite the 60° angle is $a\sqrt{3}$.



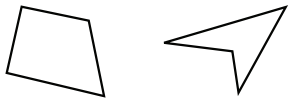
In an isosceles right triangle, the angles measure 45°, 45°, and 90°, the two legs are of equal length, and the length of the hypotenuse is $\sqrt{2}$ times greater than the length of the legs (based on the Pythagorean theorem).

For example, in the figure, the length of each leg is a, and therefore the length of the hypotenuse is $a\sqrt{2}$.



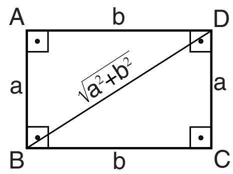
QUADRILATERALS

A quadrilateral is any four-sided polygon. For example:



Rectangles and Squares

A **rectangle** is a quadrilateral whose angles are all right angles. In a rectangle, each pair of opposite sides is equal in length.

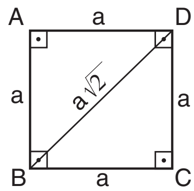


The **perimeter of the rectangle** in the figure is $2a + 2b = 2(a + b)$.

The length of a diagonal of a rectangle in the figure is $\sqrt{a^2+b^2}$ (based on the Pythagorean theorem).

The area of a rectangle is the product of the lengths of two adjacent sides. The area of the rectangle in the figure is $a \cdot b$.

A **square** is a rectangle whose sides are all of equal length.



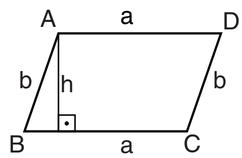
The **perimeter of the square** in the figure is $4a$.

The length of a diagonal of the square in the figure is $\sqrt{a^2+a^2} = a\sqrt{2}$.

The area of a square equals the square of the length of its side.
The area of the square in the figure is a^2.

Parallelograms and Rhombuses

A **parallelogram** is a quadrilateral in which each pair of opposite sides is parallel and of equal length. For example, in the parallelogram in the figure: $AB \parallel DC$, $AD \parallel BC$
$AB = DC$, $AD = BC$



The **diagonals of a parallelogram** bisect each other.

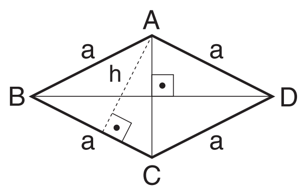
The **perimeter of the parallelogram** in the figure is $2a + 2b$.

The **altitude of a parallelogram** is a line that connects two opposite sides (or their extensions) and is perpendicular to them.

The **area of a parallelogram** equals the product of a side multiplied by the altitude to that side. For example, the area of the parallelogram in the figure is $a \cdot h$.

A **rhombus** is a quadrilateral whose four sides are all equal. Each pair of opposite sides in a rhombus is parallel, and it can therefore be regarded as a parallelogram with equal sides.

Diagonals of a rhombus
Since a rhombus is a type of parallelogram, its diagonals also bisect each other. In a rhombus, the diagonals are also perpendicular to each other.



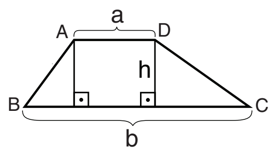
The perimeter of the rhombus in the figure is 4a.

Area of a rhombus
Since a rhombus is a type of parallelogram, its area, too, can be calculated by multiplying a side by the altitude to that side. For example, the area of the rhombus in the figure is $a \cdot h$.

The area of a rhombus can also be calculated as half the product of the length of its diagonals. For example, the area of the rhombus in the figure is $\frac{AC \cdot BD}{2}$.

Trapezoid

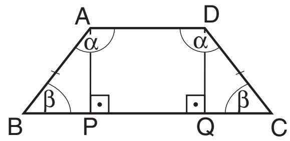
A trapezoid is a quadrilateral with **only one** pair of parallel sides. The parallel sides are called **bases**, and the other two sides are called **legs**. The bases of a trapezoid are not equal, and are therefore referred to as the long base and the short base.



The altitude of a trapezoid is a line that joins the bases of the trapezoid and is perpendicular to them.

The **area of a trapezoid** equals half the sum of the base lengths multiplied by the altitude.
For example, the area of the trapezoid in the figure is $\frac{h \cdot (a+b)}{2}$.

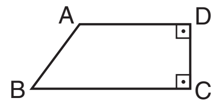
An **isosceles trapezoid** is a trapezoid whose legs are of equal length. For example, in the figure, AB = DC.



The angles of the long base of an isosceles trapezoid are equal, as are the angles of the short base. For example, in the figure, $\sphericalangle BAD = \sphericalangle CDA = \alpha$, and $\sphericalangle ABC = \sphericalangle DCB = \beta$.

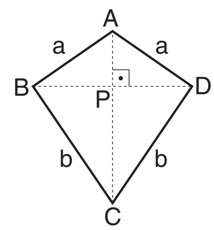
In an isosceles trapezoid, if two altitudes are drawn from the ends of the short base to the long base, a rectangle and two congruent right triangles are obtained (ABP and DCQ).

A **right trapezoid** is a trapezoid in which one of the legs forms a right angle with each of the bases.



Kite (Deltoid)

A kite is a quadrilateral formed by two isosceles triangles joined at their bases. For example, kite ABCD in the figure is composed of triangles ABD and BCD (AB = AD, CB = CD).



The diagonal joining the vertices of the two isosceles triangles bisects the diagonal that is the base of these two triangles and is perpendicular to it. For example, in the figure, AC bisects BD (BP = PD) and $AC \perp BD$.

The **perimeter of the kite** in the figure is 2a + 2b.

The **area of a kite** equals half the product of the lengths of the diagonals.

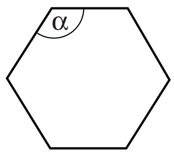
For example, the area of the kite in the figure is $\frac{AC \cdot BD}{2}$.

REGULAR POLYGON

A **regular polygon** is a polygon whose sides are all of equal length and whose interior angles are all equal.

Examples: A regular octagon is a regular polygon with 8 sides.
A regular pentagon is a regular polygon with 5 sides.
A square is a regular polygon with 4 sides.
An equilateral triangle is a regular polygon with 3 sides.

The size of the interior angle α of a regular polygon with n sides can be calculated using the formula $\alpha = \left(180° - \frac{360°}{n}\right) = \left(\frac{180°n - 360°}{n}\right)$.



For example, in a regular hexagon, such as in the figure, each of the interior angles is 120°: $\alpha = 180° - \frac{360°}{6} = 120°$.

CIRCLE

A **radius** is a line segment that joins the center of a circle to a point on its circumference.

A **chord** of a circle is a line segment that passes though the circle and joins two different points on its circumference.

A **diameter** of a circle is a chord that passes through its center. The length of a circle's diameter is twice the length of its radius. If the radius of a circle is r, the diameter of the circle is $2r$.

The **circumference** of a circle with radius r is $2\pi r$. (The value of π is approximately 3.14.)

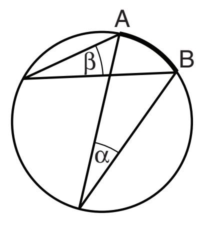
The **area** of a circle with radius r is πr^2.

An **arc** is a part of the circle's circumference bounded by two points.

A **sector** is a part of the area of a circle bounded by two radii and an arc.

Inscribed Angle

An inscribed angle is an angle whose vertex lies on the circumference of a circle and whose sides are chords of the circle. Inscribed angles intercepting the same arc are equal.

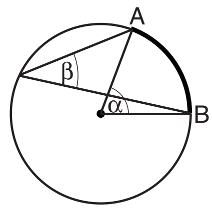


For example, in the figure, angles α and β are inscribed angles, both of which intercept arc AB; therefore, $\alpha = \beta$.

An inscribed angle that lies on the diameter of a circle (that is, it intercepts an arc whose length equals half the circle's circumference) is a right angle.

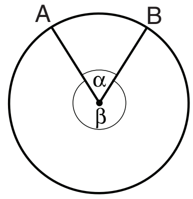
Central Angle

A central angle is an angle whose vertex is the center of the circle and whose sides are radii of the circle. A central angle is twice the size of any inscribed angle that intercepts the same arc. For example, in the figure, α is a central angle and β is an inscribed angle, and both intercept the same arc AB. Therefore, $\alpha = 2\beta$.



Arc Length

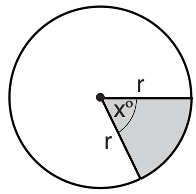
Two points on the circumference of a circle define two arcs. For example, in the figure, points A and B define two arcs – one corresponding to central angle α and the other corresponding to central angle β. The minor arc AB corresponds to α, the smaller of the two angles.



If r is the radius of the circle, then the length of this arc is $2\pi r \cdot \frac{\alpha}{360}$.

Area of a Sector

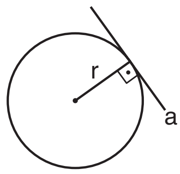
The angle formed between the two radii that bound a sector is a central angle.



For example, the shaded region in the figure is the sector of a circle with central angle x°. The area of the sector of the circle is $\pi r^2 \cdot \frac{x}{360}$.

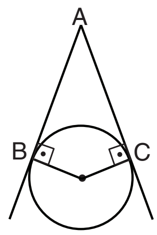
Tangent to a Circle

A tangent to a circle is a line that touches the circumference of a circle at only one point, called the point of tangency. The angle formed by the radius and the tangent at the point of tangency is a right angle.



For example, in the figure, line segment a is tangent to the circle whose radius is r.

Two tangents to the same circle that intersect at a particular point are said to originate at the same point. The length of each tangent is the length of the segment that joins the tangents' point of intersection and the point of tangency.



Tangents to a circle that originate at the same point are equal in length.

For example, in the figure, A is the point of intersection, B and C are the points of tangency, and therefore AB = AC.

Polygon Circumscribing a Circle

A polygon that circumscribes a circle is a polygon whose sides are all tangent to the circle.

Polygon Inscribed in a Circle

A polygon inscribed in a circle is a polygon whose vertices all lie on the circumference of the circle.

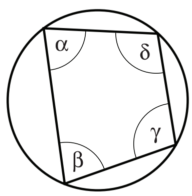
Inscribed Triangle

Any triangle can be inscribed in a circle.
Every triangle has only one circle that circumscribes it.

If an inscribed triangle is a right triangle, the center of the circle that circumscribes it is the midpoint of the triangle's hypotenuse.

Quadrilateral Inscribed in a Circle

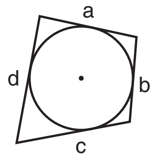
Not every quadrilateral can be inscribed in a circle.



The sum of the opposite angles of a quadrilateral inscribed in a circle always equals 180°.

For example, in the quadrilateral in the figure, $\alpha + \gamma = 180°$
$\beta + \delta = 180°$

Quadrilateral Circumscribing a Circle

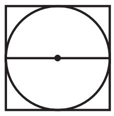


Not every quadrilateral can circumscribe a circle.

When a quadrilateral circumscribes a circle, the sums of the lengths of each pair of opposite sides is equal.

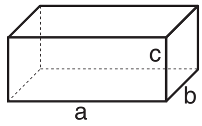
For example, in the quadrilateral in the figure, $a + c = b + d$.

When a square circumscribes a circle, the length of the side of the square equals the length of the diameter of the circle.



SOLIDS

Box (Rectangular Prism) and Cube



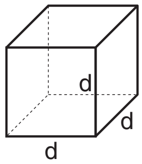
A **box** is a three-dimensional figure with six rectangular faces. The box's three dimensions are its length, width and height (a, b and c respectively, in the figure).

Every face of a box is perpendicular to the faces adjacent to it.

The **surface area** of a box is the sum of the areas of its faces.
The surface area of the box in the figure is
$ab + ac + bc + ab + ac + bc = 2ab + 2ac + 2bc$.

The **volume** of a box is the product of its length, width and height.
The volume of the box in the figure is $a \cdot b \cdot c$.

A **cube** is a box whose length, width and height are all equal.



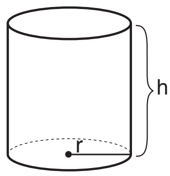
All of the faces of a cube are congruent squares.

The area of each face of the cube in the figure is d^2.

Therefore, the **surface area** of the cube is $6d^2$.

The **volume** of the cube in the figure is d^3.

Cylinder



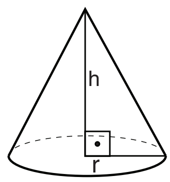
A **cylinder** is a three-dimensional figure whose two bases are congruent circles on parallel planes joined by a lateral surface. The line joining the centers of the circles is perpendicular to each of the bases.

The **lateral surface area** of a cylinder with base radius of length r and height h equals the circumference of the base multiplied by the height, that is, $2\pi r \cdot h$.

The **total surface area** of a cylinder is the sum of the areas of the bases and the lateral surface. The area of each base is πr^2 and the lateral surface area is $2\pi r \cdot h$. Thus, the total surface area is $2\pi r \cdot h + 2\pi r^2 = 2\pi r \cdot (h + r)$.

The **volume** of a cylinder equals the area of one of the bases multiplied by the height, that is, $\pi r^2 \cdot h$.

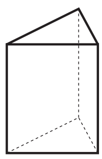
Cone



A **right cone** is a three-dimensional figure formed by joining the points on the circumference of a circle to a point lying outside the plane of the circle. This point is called the cone's vertex and it lies on a line that passes through the center of the circle and is perpendicular to the plane of the circle (see figure).

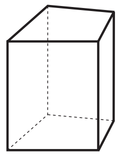
The **volume** of a cone with base radius r and height h is $\frac{\pi r^2 \cdot h}{3}$.

Prism



A **right prism** is a three-dimensional figure whose two bases are congruent polygons on parallel planes and whose lateral faces are rectangles. A prism is referred to by the number of sides of its base. For example, a triangular prism has three-sided bases, a quadrangular prism has four-sided bases, and so on (see figures).

The **height** of a prism is the length of the segment that joins the bases of the prism and is perpendicular to them. This is the distance between the bases of the prism.

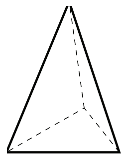


The **lateral surface area** of the prism is the sum of the areas of all the lateral faces. The lateral surface can also be calculated by multiplying the perimeter of the prism's base by the height of the prism.

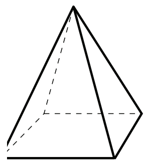
The **total surface area** of a prism is the sum of the lateral surface area and the areas of the two bases.

The **volume** of a prism equals the area of one of the bases multiplied by the height of the prism.

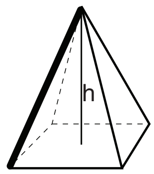
Pyramid



A **right pyramid** is a three-dimensional figure formed by joining the vertices of any regular polygon to a point outside the plane of the polygon. The polygon is called the base of the pyramid and the point is the pyramid's vertex or apex.



The lateral faces of a pyramid are triangles. A pyramid is referred to by the number of sides of its base. For example, a triangular pyramid has a three-sided base, a quadrangular pyramid has a four-sided base, and so on (see figure).



The **height** of a pyramid is the length of the line segment extending perpendicularly from the pyramid's vertex to its base. This is the distance between the pyramid's vertex and base (see figure).

If S is the area of the pyramid's base and h is the pyramid's height, then the pyramid's **volume** is $\frac{S \cdot h}{3}$

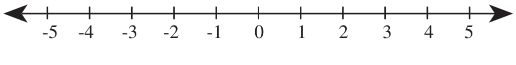
Edge

The edge of a three-dimensional figure is the straight line formed where two faces meet. The bold line in the pyramid in the above figure is one of its edges.

A box has 12 edges.

NUMBER LINE (AXIS)

A number line is a geometric representation of the relationships between numbers.



The numbers along the axis increase to the right.

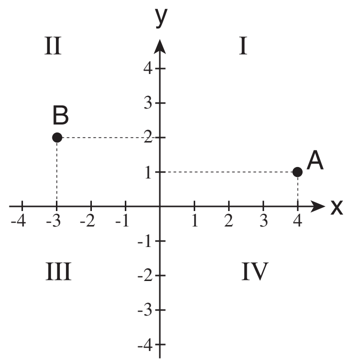
The distance between points on the axis is proportional to the difference between the numerical values corresponding to the points.

For example, the distance between the points corresponding to the values (-4) and (-2) is equal to the distance between the points corresponding to the values 3 and 5.

CARTESIAN COORDINATE SYSTEM

In a cartesian coordinate system in a plane, there are two number lines (axes) that are perpendicular to each other. The horizontal line is called the x-axis and the vertical line is called the y-axis. The numbers along the x-axis increase to the right. The numbers along the y-axis increase upwards.

The axes divide the plane into four quadrants, designated in the figure by Roman numerals I, II, III, and IV.



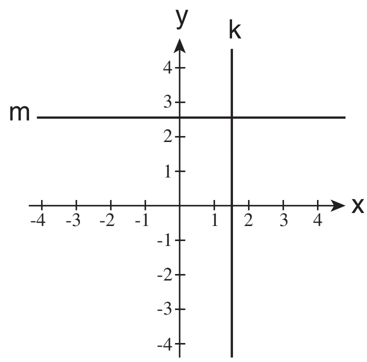
Each point in the coordinate plane corresponds to a pair of x and y values which describe their location relative to the axes.

For example, in the figure, the x-value of point A is 4, and its y-value is 1. The x-value of point B in the figure is (-3) and its y-value is 2.

It is customary to write the x- and y-values of the points in parentheses, with the x-value to the left of the y-value: (x , y). Sometimes, the values of a point are written next to the letter representing the point, for example, A(4 , 1) and B(-3 , 2).

The x- and y-values of a point are sometimes called the **coordinates** of that point.

The point in the plane corresponding to (0 , 0) is the point of intersection of the two axes and is called the **origin**.



All points on a line parallel to the x-axis have the same y-coordinate, and all points on a line parallel to the y-axis have the same x-coordinate.

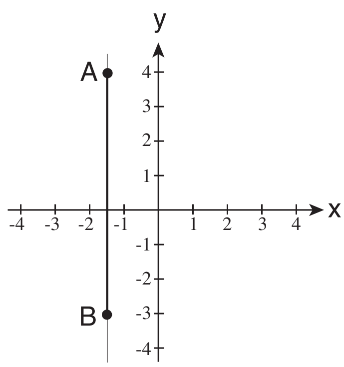
For example, in the figure, line k is parallel to the y-axis. Thus, all of the points on line k have the same x-coordinate (in the figure, x = 1.5).

Line m is parallel to the x-axis. Thus, all of the points on line m have the same y-coordinate (in the figure, y = 2.5).

Only one line can be drawn through any two points on a plane. The part of the line that is located between the two points is called a line segment.

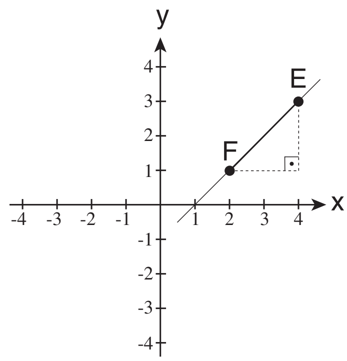
If a line segment is parallel to the y-axis, its length is the difference (in absolute value) between the y-coordinates of the two points.

For example, in the figure, line segment AB is parallel to the y-axis. The y-coordinate of point A is 4 and the y-coordinate of point B is (-3). The difference between the values of the y-coordinates of the two points is 4 – (-3) = 7. Therefore, the length of line segment AB is 7.



The length of a line segment parallel to the x-axis is calculated in the same way.

If a line segment is not parallel to either of the axes (for example, line segment EF in the figure), its length can be calculated using the Pythagorean theorem. Draw a right triangle such that the line segment is the hypotenuse and the legs are parallel to the x-axis and the y-axis. The length of the leg parallel to the x-axis equals the difference between the x-coordinates of points E and F ($4 - 2 = 2$). The length of the leg parallel to the y-axis equals the difference between the y-coordinates of points E and F ($3 - 1 = 2$).



Using the Pythagorean theorem, we can calculate the length of the hypotenuse: $EF = \sqrt{2^2 + 2^2} = \sqrt{8}$.

QUESTIONS AND PROBLEMS

The algebra questions cover a variety of topics, such as equations, distance problems, work problems, combinatorial analysis, and probability. The geometry questions deal with characteristics of geometric shapes, such as area, volume, and angles. Some of the questions are word problems, which you must first convert into mathematical expressions; others are non-word problems that are already presented in the form of mathematical expressions. Below are some sample questions, together with solutions and explanations.

Note: The examples in the *Guide* are arranged by type, but in the actual exam they do not appear in any particular order.

ALGEBRA WORD PROBLEMS

1. A driver traveled from Haifa to Eilat. He covered a third of the distance at a speed of 75 kph (kilometers per hour), a fifth of the remaining distance in one hour, and the rest of the distance at a speed of 80 kph. The distance between Haifa and Eilat is 450 kilometers. If the driver had driven the entire distance at a constant speed, at what speed would he have needed to drive so that the journey from Haifa to Eilat would take exactly the same amount of time?

(1) 70 kph
(2) 75 kph
(3) 80 kph
(4) 90 kph

This question is presented as a word problem; therefore, the first step is to convert it into a mathematical expression. Start by clearly defining what you are asked to find: the constant **speed** at which one would have to drive in order to cover the **distance** between Haifa and Eilat in the same amount of **time** that it took the driver in the question. Thus, this is a distance problem. The formula $v = \frac{s}{t}$, which defines the relationship between distance, speed, and time, can be applied since the distance (s) is given, the time (t) can be calculated, and the speed (v) is the unknown that you have to find.

The question provides the information that the distance between Haifa and Eilat is 450 kilometers. The total amount of time it took the driver to cover the entire distance between Haifa and Eilat can be calculated as follows:

The distance is divided into three segments. Calculate the time it took the driver to cover each segment:

a. A third of the distance is **150 kilometers**, because $450 \cdot \frac{1}{3}$ kilometers equals 150 kilometers. It took the driver **two hours** to cover this segment, since it takes two hours to travel 150 kilometers at a speed of 75 kph $\left(\frac{150}{75} = 2\right)$.

b. A fifth of the remaining distance is **60 kilometers**, since the remaining distance is $450 - 150 = 300$ kilometers, and $300 \cdot \frac{1}{5}$ kilometers equals 60 kilometers. The question provides the information that the driver covered this segment of the journey in **one hour**.

c. The rest of the distance is **240 kilometers**, since $450 - 150 - 60 = 240$. The driver covered this segment in **three hours**, since it takes three hours to travel 240 kilometers at a speed of 80 kph $\left(\frac{240}{80} = 3\right)$.

Thus, the journey from Haifa to Eilat took a total of 6 hours (two hours, plus one hour, plus three hours).

By inserting the data into the above formula, you can now compute the constant speed at which it is necessary to drive in order to cover 450 kilometers in **6 hours**: $v = \frac{s}{t} = \frac{450}{6} = 75$.

Thus, the speed is 75 kph, and the correct response is (**2**).

2. When it was 10 days old, a baby elephant ate 5 candies. From then onwards, its appetite grew, and each day it ate twice the number of candies it had eaten the previous day.

How many candies did the baby elephant eat when it was 14 days old?

(1) 40
(2) 80
(3) 100
(4) 120

When it was 10 days old, the baby elephant ate 5 candies. Each day after that it ate twice the number of candies that it had eaten the previous day. Thus, when it was 11 days old, it ate 10 candies ($5 \cdot 2$), when it was 12 days old, it ate 20 candies ($5 \cdot 2 \cdot 2$), and so on.

In general, if n is a positive integer, then on day $(10+n)$ the baby elephant ate $5 \cdot 2^n$ candies.

Thus, when it was 14 days old, it ate 80 candies ($5 \cdot 2^4 = 80$), and the correct response is (**2**).

3. For a business lunch in a certain restaurant, you may choose one of 3 different first courses and one of 4 different main courses. In addition to the first course and the main course, you have a choice of soup or dessert.

How many different combinations of a three-course business lunch does this restaurant offer?

(1) 12
(2) 14
(3) 18
(4) 24

There are **three** possible choices for the first course, and **four** different main courses that can be added to each choice of first course. Thus, there are $4 \cdot 3$ different combinations of first course and main course. Either soup or dessert can be added to each of these 12 combinations. In other words, there are a total of $12 \cdot 2$ different combinations of the three courses, which equals 24 possibilities.

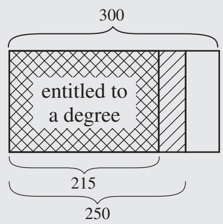
The correct response is therefore (**4**).

4. Students receive a B.A. degree only after passing all their tests and submitting all their papers. Out of 300 students, 250 passed all their tests and 215 submitted all their papers. How many students received a B.A. degree?

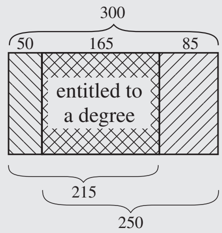
(1) at least 215
(2) no more than 185
(3) exactly 215
(4) at least 165

We can identify two groups of students: those who submitted all their papers and those who passed all their tests. Any student belonging to both groups is entitled to receive a B.A. The amount of overlap between the two groups is not known, but there are two possible extremes. We will use diagrams to illustrate them:

– In a situation of **maximum overlap** between the two groups, a maximum number of students would be entitled to a degree. There would be maximum overlap if all 215 students who submitted all their papers also passed all their tests. In other words, **at most** 215 students would be entitled to a degree.



– In a situation of **minimum overlap** between the two groups, a minimum number of students would be entitled to a degree. Fifty students (300 – 250) are not entitled to a degree because they did not pass all their tests, and 85 students (300 – 215) are not entitled to a degree because they did not submit all their papers. In other words, the maximum number of students who would not be entitled to a degree for one of the reasons is $50+85=135$. Thus, the minimal number of students entitled to a degree is 300 – 135 = 165. In other words, **at least** 165 students would be entitled to a degree.



Hence, the number of students entitled to a B.A. degree could range from 165 to 215. The correct response is therefore (**4**).

5. A factory manufacturing at a steady rate produces 20 cars in 4 days. How many cars could 3 such factories produce in 6 days, if they were working at the same rate?

(1) 60
(2) 80
(3) 90
(4) 120

This is a work problem. One way of solving such problems is by determining the output of one work unit (in this case, one factory) per one time unit (in this case, one day), and then multiplying the output that the problem asks about by the number of work units (3 factories) and by the number of time units (6 days). If a factory produces 20 cars in 4 days, then it produces 5 cars per day $\left(\frac{20}{4}=5\right)$.

Therefore, in 6 days, 3 factories will produce $5 \cdot 3 \cdot 6$ cars, which equals 90 cars. The correct response is (**3**).

6. There are 20 white hats and 13 black hats in a box. Jack randomly took 3 hats (in succession) out of the box and did not put them back in the box. All three hats were black.
What is the probability that the fourth hat that he takes out at random will also be black?

(1) $\frac{13}{33}$

(2) $\frac{10}{33}$

(3) $\frac{1}{3}$

(4) $\frac{1}{33}$

You are asked to calculate the probability of Jack taking out a black hat after previously taking out three black hats. The probability is the number of black hats remaining in the box divided by the total number of hats (black and white) remaining in the box. After three black hats were taken out of the box, 10 black hats and 20 white hats remained in the box. In other words, out of the 30 hats still in the box, 10 are black. Thus, the probability of Jack now taking out a black hat is $\frac{10}{30} = \frac{1}{3}$. Therefore, the correct response is (**3**).

NON-WORD ALGEBRA PROBLEMS

1. Given: $2^x \cdot 2^y = 32$
$x + y = ?$

(1) 8
(2) 7
(3) 5
(4) 4

According to the laws of exponents, when multiplying powers with the same base, we add the exponents. Therefore, $2^x \cdot 2^y = 2^{x+y}$. According to the information provided, $2^{x+y} = 32$. In order to find the value of $x + y$, 32 must be expressed as a power of base 2, that is, $32 = 2^5$. It follows that $2^{x+y} = 2^5$. When two powers are equal and have the same base, their exponents are also equal, and we can therefore deduce that $x + y = 5$.

Thus, the correct response is (**3**).

2. The average of the three numbers x, y, and z is $x \cdot y$.

$z = ?$

(1) $3 \cdot x \cdot y - x - y$
(2) $x \cdot y - x - y$
(3) $3 \cdot x \cdot y + x + y$
(4) $3 \cdot x \cdot y - (x - y)$

An average (arithmetic mean) is the sum of the terms divided by the number of terms. Thus, the average of x, y, and z equals $\frac{x+y+z}{3}$.

Substitute the information in the question into the equation: $\frac{x+y+z}{3} = x \cdot y$. Multiply both sides by 3: $x + y + z = 3 \cdot x \cdot y$. Solve for z: $z = 3 \cdot x \cdot y - x - y$.

Thus, the correct response is (**1**).

3. For any two numbers a and b, the operation $ is defined as follows:

$(a , b) = a · (a + b)

$($(2 , 0) , 1) = ?

(1) 20
(2) 12
(3) 10
(4) 4

You are asked to find the value of the expression $($(2 , 0) , 1), where a = $(2 , 0) and b = 1.

According to the definition of the operation, $($(2 , 0) , 1) = $(2 , 0) · $(2 , 0) +1).

To calculate the value of the above expression, first calculate $(2 , 0).

According to the definition of the operation, $(2 , 0) = 2 · (2 + 0) = 4.

Substitute the value you have obtained for $(2 , 0) into the original expression and you will obtain $($(2 , 0) , 1) = $(4 , 1).

According to the definition of the operation, $(4 , 1) = 4 · (4 + 1) = 20, and the correct response is (**1**).

4. Given: $B < C$
$B < D < A$

Which of the following expressions is **necessarily** true?

(1) $C < D$
(2) $D < C$
(3) $C < A$
(4) None of the above expressions is necessarily true.

It is impossible to make any deductions about the relationship of C to A and D from the information provided. Based on the information, there are three possible situations:

(a) $B < C < D < A$
(b) $B < D < C < A$
(c) $B < D < A < C$

The expression in response (1) is true in situation (a) but not in situations (b) and (c). The expression in response (2) is true in situations (b) and (c) but not in situation (a). The expression in response (3) is true in situations (a) and (b) but not in situation (c). Thus, each of the expressions is true in certain situations and not true in others. Therefore, none of the expressions in responses (1)-(3) is **necessarily** true, and the correct response is (**4**).

5. K is an even number and P is an odd number.
Which of the following statements is **not** correct?

(1) $P - K - 1$ is an odd number.
(2) $P + K + 1$ is an even number.
(3) $P \cdot K + P$ is an odd number.
(4) $P^2 + K^2 + 1$ is an even number.

Let us examine each of the statements:

(1) The difference between an odd number (P) and an even number (K) is an odd number. Therefore, $P - K$ is an odd number. If we subtract 1 from this odd number, we get an even number. Therefore, $P - K - 1$ is an **even** number, and the statement is **not** correct.

(2) The sum of an odd number (P) and an even number (K) is an odd number. Therefore, $P + K$ is an odd number. If we add 1 to this odd number, we get an even number. Therefore, $P + K + 1$ is an **even** number, and the statement is **correct**.

(3) The product of an even number and any integer is always an even number; therefore, $P \cdot K$ is an even number. If we add an odd number to this even product, we get an odd number. Therefore, $P \cdot K + P$ is an **odd** number and the statement is **correct**.

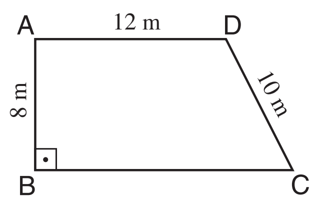
(4) The square of an odd number (P^2) is an odd number because it is the product of an odd number multiplied by an odd number ($P \cdot P$). The square of an even number (K^2) is an even number because it is the product of an even number multiplied by an even number ($K \cdot K$). The sum of the two squared numbers ($P^2 + K^2$) is odd because it is the sum of an odd number and an even number. Therefore, if we add 1 to this sum, we get an even number. Thus, $P^2 + K^2 + 1$ is an **even** number, and the statement is **correct**.

In this question, you are asked to choose the statement that is **not** correct; therefore, (**1**) is the correct response.

GEOMETRY PROBLEMS

1. The accompanying figure shows a right trapezoid ($AD \parallel BC$).

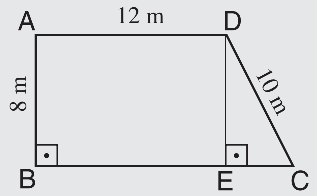
Based on this information and the information in the figure, what is the area of the trapezoid (in m^2)?



(1) 150
(2) 120
(3) 108
(4) 96

The formula for calculating the area of a trapezoid with bases a and b and height h is $S = \frac{(a+b)\cdot h}{2}$.

The trapezoid in the figure is a right trapezoid; therefore, the leg perpendicular to the bases equals the height of the trapezoid. The figure gives the length of the short base and the height, but not the length of the long base. In order to calculate the length of the long base, drop a perpendicular from point D to base BC (DE in the accompanying figure). The resulting rectangle ABED is 12 meters long and 8 meters wide. Thus, BE = 12 and DE = 8. Next, you must calculate the length of segment EC in order to find the length of the trapezoid's long base. The length of segment EC can be calculated using the Pythagorean theorem.



In right triangle DEC, $DE^2 + EC^2 = DC^2$.

Solve for EC: $EC = \sqrt{DC^2 - DE^2}$.

Substitute the information $EC = \sqrt{10^2 - 8^2} = 6$.

The length of the long base is thus 18 meters (12 meters + 6 meters).

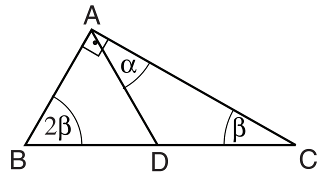
Now you can calculate the area of the trapezoid: $S = \frac{(12+18)\cdot 8}{2} = 120$.

The area of the trapezoid is thus 120 m^2, and the correct response is (**2**).

2. The accompanying figure shows right triangle ABC and isosceles triangle ABD (AB = AD).

Based on this information and the information in the figure,

$\alpha = ?$



(1) 60°
(2) 45°
(3) 30°
(4) 25°

The sum of the angles of a triangle is 180°. Therefore, in triangle ABC, $90° + 2\beta + \beta = 180°$. Solving this equation, we get $\beta = 30°$.

We are given the information that triangle ABD is an isosceles triangle. It follows that $\sphericalangle ADB = \sphericalangle ABD$.

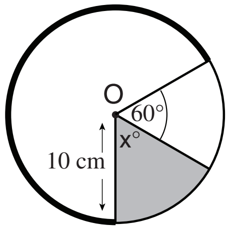
$\sphericalangle ABD = 2\beta = 60°$, and therefore $\sphericalangle ADB = 60°$.

In triangle ABD, $\sphericalangle BAD + \sphericalangle ABD + \sphericalangle ADB = 180°$. In other words, $\sphericalangle BAD = 180° - \sphericalangle ABD - \sphericalangle ADB$.

Substituting the values of the angles that we have calculated, $\sphericalangle BAD = 180° - 60° - 60° = 60°$.

According to the figure, $\sphericalangle BAD + \alpha = \sphericalangle BAC$. Substituting the known values of the angles, we get $60° + \alpha = 90°$. Thus, $\alpha = 30°$ and the correct response is (**3**).

3. The accompanying figure shows a circle whose center is O and whose radius is 10 centimeters. The shaded sector equals $\frac{1}{6}$ of the area of the circle.
Based on this information and the information in the figure, what is the length (in centimeters) of the arc shown in bold?



(1) 30π

(2) $\frac{40\pi}{3}$

(3) $\frac{20\pi}{3}$

(4) 20π

The length of the arc shown in bold is equal to the circumference of the entire circle minus the length of the arc that is not in bold. To find the length of the arc not in bold, you must determine the size of the central angle that intercepts this arc. The size of this angle is $60° + x°$ (as shown in the figure). x is the central angle of the shaded sector. The size of the central angle of the shaded sector can be found using the formula for calculating the area of a sector of a circle: $\pi r^2 \cdot \frac{x}{360}$.

According to the information given, the area of the shaded sector equals $\frac{1}{6}$ of the area of the circle, i.e., $\frac{\pi r^2}{6}$ (since the area of the entire circle equals πr^2).

We will thus obtain the equation $\pi r^2 \cdot \frac{x}{360} = \frac{\pi r^2}{6}$. Divide the two sides by πr^2 to obtain $\frac{x}{360} = \frac{1}{6}$ and solve for x: $x = \frac{360}{6} = 60°$. Thus, the size of the central angle that intercepts the arc that is not in bold is $x° + 60° = 60° + 60° = 120°$. The length of the arc that intercepts this angle is $2\pi r \cdot \frac{120}{360} = 2\pi r \cdot \frac{1}{3}$, that is $\frac{1}{3}$ of the circumference of the circle.

Thus, the length of the arc shown in bold is $\frac{2}{3}$ of the circumference of the circle.

The circumference of the circle (in cm) is $2\pi r = 2\pi \cdot 10 = 20\pi$.

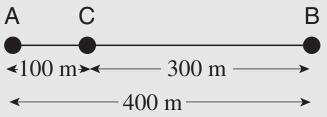
Thus, $\frac{2}{3}$ of the circumference of the circle is $\frac{2}{3} \cdot 20\pi = \frac{40\pi}{3}$.

In other words, the length of the arc shown in bold is $\frac{40\pi}{3}$ cm, and the correct response is (**2**).

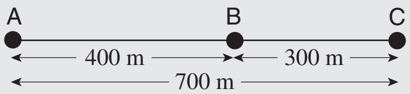
4. The distance between points A and B is 400 meters.
The distance between points B and C is 300 meters.
It follows that the distance between points A and C is **necessarily** -

(1) 100 meters
(2) 500 meters
(3) 700 meters
(4) It cannot be determined from the information given.

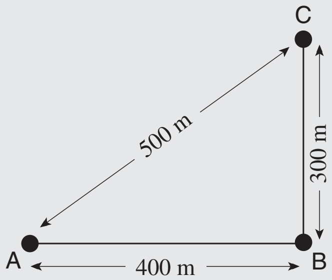
The information given in this question does not tell us about the relative positions of the three points. They can be arranged in many ways, such as:



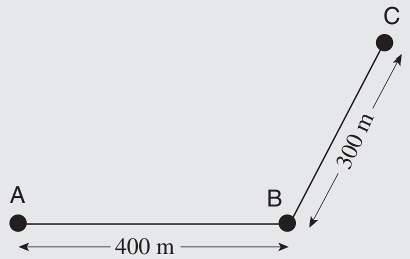
Appropriate for response (1)



Appropriate for response (3)



Appropriate for response (2)

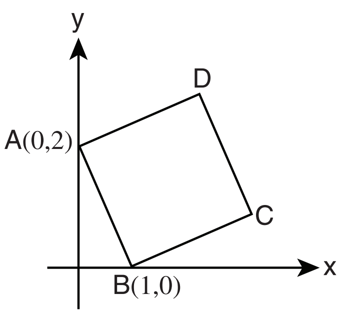


Not appropriate for any of responses (1) - (3)

All of these arrangements are possible, as are many others, and none of them is **necessarily** the case. Therefore, the correct response is (**4**).

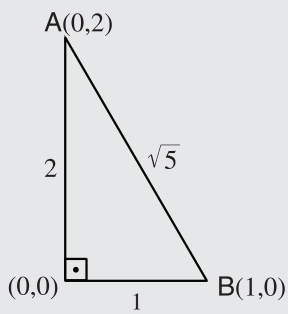
5. The accompanying coordinate system shows square ABCD.

Based on this information and the information in the figure, what is the area of the square?



(1) It cannot be determined from the information given.
(2) 6
(3) 5
(4) 4

To calculate the area of the square, we must determine the length of its side. The length of the side is the distance between any two adjacent vertices, such as A and B. Since segment AB is not parallel to either one of the axes, we will determine its length using the Pythagorean theorem.



The origin and points A and B form a right triangle whose hypotenuse is AB. The length of one leg is the distance between the origin (0 , 0) and point A (0 , 2), which equals 2; the length of the other leg is the distance between the origin (0 , 0) and point B (1 , 0), which equals 1.

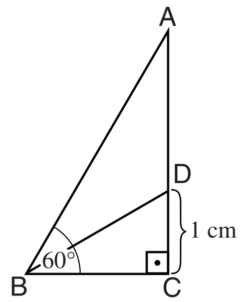
According to the Pythagorean theorem, the length of hypotenuse AB is $\sqrt{2^2+1^2}=\sqrt{4+1}=\sqrt{5}$.

Thus, the length of a side of the square is $\sqrt{5}$, and it follows that the area of the square is $(\sqrt{5})^2=5$.

Therefore, the correct response is (**3**).

6. The accompanying figure shows right triangle ABC.
BD bisects ∢ABC.

Based on this information and the information in the figure,
AD = ?



(1) 1 cm

(2) 2 cm

(3) $\sqrt{3}$ cm

(4) $\frac{4}{\sqrt{3}}$ cm

Based on the sum of the angles in triangle ABC, it can be determined that ∢BAD = 30°. Knowing that BD bisects ∢ABC, it follows that ∢ABD = 30°. In triangle ADB, ∢BAD = ∢ABD. Therefore, triangle ADB is an isosceles triangle in which AD = BD.

BD is also the hypotenuse of triangle BDC. This is a 30° 60° 90° triangle, and therefore BD = 2 · DC = 2 · 1 = 2 centimeters. Since AD = BD, AD = 2 centimeters. Thus, the correct response is (**2**).

7. All of the liquid filling a rectangular container that measures 2 cm x 10 cm x 20 cm is poured into a cylindrical container with a base radius of 5 cm.

What height (in cm) will the surface of the liquid reach in the cylindrical container?

(1) $\frac{16}{\pi}$

(2) $\frac{40}{\pi}$

(3) 8π

(4) 8

The volume of a rectangular container equals the product of its three dimensions. Thus, the volume of the liquid in the rectangular container is 2 · 10 · 20 cubic centimeters, which equals 400 cubic centimeters. The volume of this liquid remains the same after it is poured into a cylindrical container. You must now find the height of a cylinder whose base radius is 5 centimeters and whose volume is 400 cubic centimeters. This is the height that the water will reach in the cylinder.

The formula for the volume of a cylinder is $V = \pi r^2 \cdot h$. You must find h when r = 5 centimeters and V = 400 cubic centimeters.

To calculate the volume, substitute the numbers into the formula: $400 = \pi \cdot 5^2 \cdot h = \pi \cdot 25 \cdot h$.

To solve for h, divide both sides of the equation by 25π to obtain $h = \frac{400}{25\pi} = \frac{16}{\pi}$. Thus, the correct response is (**1**).

GRAPH AND TABLE COMPREHENSION QUESTIONS

These questions are based on information appearing in a graph or table. The graph or table is usually accompanied by a short explanation. The data in a table are arranged in columns and rows, whereas in a graph they are presented in graphic form, e.g. a bar chart. Below are samples of a graph and a table, each followed by questions and explanations.

GRAPH COMPREHENSION

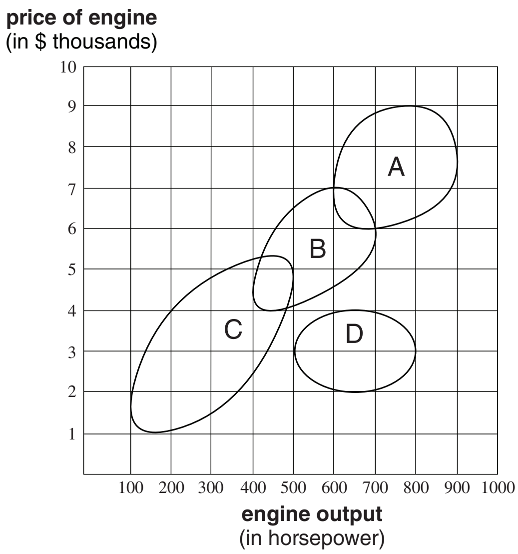
Study the graph below and answer the questions that follow.

The accompanying graph presents information on 4 different technologies used for producing a certain type of engine.

Each technology is marked by a letter (A-D) and is represented on the graph by a closed area. Each point in that area represents the horsepower and price of an engine that can be manufactured using that technology.

For example, using technology A, it is possible to manufacture a 750-horsepower engine at a price of $8,500, but it is not possible to manufacture an engine with the same horsepower at a price of $5,000.

Note: Technologies A and B have an area that overlaps, as do technologies B and C.



Note: In answering each question, disregard the information appearing in the other questions.

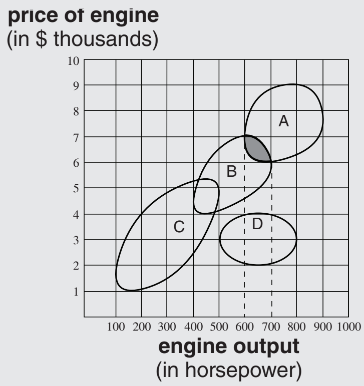
QUESTIONS AND EXPLANATIONS:

1. What is the range of engine outputs (in horsepower) that can be obtained using technology A as well as technology B?

(1) 400–500
(2) 500–600
(3) 600–700
(4) None of the above

To solve graph comprehension problems, rephrase the question in the terms used in the graph and then look for the relevant information in the graph. This question deals with engines that can be manufactured using technology A as well as technology B. These engines are depicted in the graph by the region in which the areas representing the two technologies overlap (the shaded region in figure I). Now find the range of outputs of these engines. The boundaries of the shaded region relative to the horizontal axis represent the range of outputs of engines that can be manufactured using either one of these technologies. As can be seen in the figure, the boundaries are between 600 and 700 horsepower. In other words, engines that can be produced using technology A as well as technology B will have outputs ranging from 600 to 700 horsepower, and the correct response is (**3**).

Figure I

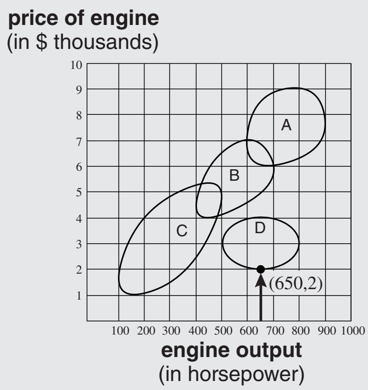


2. What is the minimum price at which an engine with an output of 650 horsepower can be manufactured?

(1) $1,000
(2) $2,000
(3) $1,500
(4) $2,500

First note that we are dealing here with an engine that has an output of 650 horsepower. Engine outputs are represented on the graph's horizontal axis. Therefore, the first step is to locate the specified engine output on the horizontal axis. The second step is to find the minimum price of an engine with this output. Draw a vertical line from the point on the horizontal axis that represents 650 horsepower to the first point at which it meets one of the technology areas (see figure II). This point of contact represents the lowest possible price for an engine with an output of 650 horsepower. The lowest point of contact intercepts the boundary of the area of technology D and represents a price of $2,000. This is the minimum price for manufacturing an engine with the desired output. Therefore, the correct response is (**2**).

Figure II

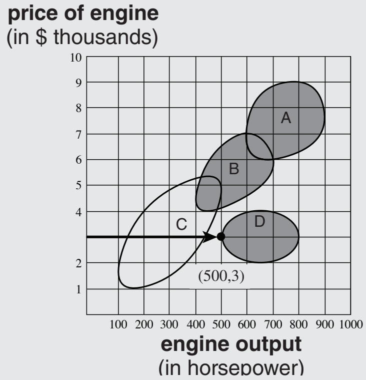


3. One of the companies that manufactures engines decided it will no longer use technology C. After it implements the decision, what will be the minimum output (in horsepower) of an engine priced at $3,000 that the company produces?

(1) 500
(2) 400
(3) 300
(4) It is impossible to produce an engine of this kind.

Since the problem states that the company will no longer use technology C, we can ignore this technology area and relate only to the other technology areas (the shaded regions in figure III). Note that in this question we are dealing with an engine whose price is $3,000. Engine prices are represented on the graph's vertical axis; start by locating the point on the vertical axis that represents a price of $3,000. The further we move to the right of this point, the greater the output. Thus, if we draw a horizontal line from this point (see figure III), the first point of contact with one of the technology areas will represent the **lowest** possible output for an engine priced at $3,000. The first point of contact is with technology area D. This point lies on the vertical line representing 500 horsepower on the horizontal axis. This is now the minimum output for an engine priced at $3,000. Therefore, the correct response is (**1**).

Figure III



4. A certain company is not allowed to produce engines with an output of over 550 horsepower. Which technologies can the company use to manufacture its engines?

(1) C only
(2) B and C only
(3) C and D only
(4) B, C and D only

Note that we are dealing here with an engine that has an output of 550 horsepower. Find the point on the horizontal axis that represents an output of 550 horsepower and draw a vertical line from this point up the entire length of the graph (see figure IV). All engines to the right of this line have an output of over 550 horsepower, and all engines to the left of this line have an output of less than 550 horsepower. The company is only allowed to manufacture engines with outputs of less than 550 horsepower. It can therefore use only those technologies represented by areas that lie at least partly to the **left** of the line (the shaded areas in figure IV). To the left of the line we find the entire area of technology C, part of the area of technology B, and part of the area of technology D. Thus, the company can use technologies B, C and D for manufacturing engines with an output of less than 550 horsepower, and the correct response is (4).

Figure IV

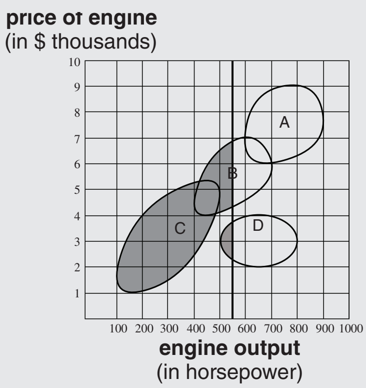


TABLE COMPREHENSION

Study the table below and answer the questions that follow.

The table below contains data on 10 companies from different industries. The companies are designated by the letters A through J.

For each company, the table shows the industry to which it belongs, its sales volume, the current year's profits, its asset value, and the number of workers it employs.

For example, Company E deals in electronics, employs 400,000 workers, and has assets valued at \$90 million. The company's sales volume totaled \$70 billion this year (9% higher than last year's sales volume), and its profits amounted to \$6,000 million (60% higher than last year's profits).

An example of how to calculate percentage of change: If a certain company's sales volume totaled \$40 billion last year, and this year the volume increased to \$50 billion, then the percentage of change compared to last year is 25% $\left(\frac{50-40}{40}\cdot 100\right)$.

		Sales		Profits			
Name of company	**Industry**	**Sales** (in \$ billions)	**Percentage of change compared to last year**	**Profits** (in \$ millions)	**Percentage of change compared to last year**	**Asset value** (in \$ millions)	**Number of workers** (in thousands)
A	Automobile	125	-1.5	-2,000	-150	180	750
B	Oil	110	25	6,500	0	100	150
C	Oil	105	22	5,000	40	390	100
D	Automobile	100	1.5	900	-80	180	350
E	Electronics	70	9	6,000	60	90	400
F	Automobile	65	7	3,000	15	55	100
G	Metals	60	25	1,000	-20	not given	400
H	Oil	60	20	3,000	-15	60	120
I	Oil	55	15	2,000	7	40	70
J	Electronics	50	6	4,500	10	150	300

Note: In answering each question, disregard the information appearing in the other questions.

QUESTIONS AND EXPLANATIONS

1. Which of the companies in the automobile industry has the **lowest** asset value?

(1) A
(2) D
(3) F
(4) A and D

The industry to which each company belongs is listed in the second column from the left. It shows that companies A, D and F are the only companies in the automobile industry. Examining the asset value (second column from the right) of each of these companies, we see that the asset value of Company A is $180 million, which is also the asset value of Company D. The asset value of Company F is $55 million. Thus, Company F has the lowest asset value of the companies in the automobile industry, and the correct response is (**3**).

2. Assuming that profits are divided equally among all the workers in a company, which of the following companies shows the greatest profit per **individual** worker?

(1) H
(2) B
(3) C
(4) F

The amount of profit per individual worker is not specified in the table but can be calculated from the information that does appear in it. The table shows the profit and number of workers for each company. The profit per individual worker of a particular company is the total profit of that company divided by the number of workers.

The profits of each company are given in millions of dollars, and the number of workers in thousands. We can compare companies by simply relating to the numbers appearing in the table, and show the profit per worker as follows:

H	B	C	F
$\frac{3,000}{120}$	$\frac{6,500}{150}$	$\frac{5,000}{100}$	$\frac{3,000}{100}$

It is possible, of course, to calculate the profit per worker and determine the company that obtains the highest value. But the expressions can be compared without performing this calculation:

Companies F and H show the same profit (3,000), but in Company F it is divided among fewer workers (100 < 120); thus, the profit per worker is greater in Company F.

Companies F and C employ the same number of workers (100), but Company C shows greater total profits (5,000 > 3,000); thus, the profit per worker is greater in Company C.

Companies B and C differ both in terms of the number of workers they employ and in terms of their total profits. The number of workers employed in Company B is 1.5 times that of Company C (150 vs. 100). If the total profit of Company B were also 1.5 times greater than that of Company C, the profit per individual worker would be the same for both companies – that is, if Company B's profit were 5,000 · 1.5 = 7,500. But the total profit of Company B is less than this amount (6,500 < 7,500). Thus, the profit per worker in Company B is less than the profit per worker in Company C.

Hence, Company C has the greatest profit per individual worker, and the correct response is (**3**).

It is possible, of course, to calculate the profit per individual worker in Companies B and C:

In Company C, the profit per individual worker is 50 $\left(\frac{5,000}{100} = 50\right)$ while in Company B it is less than 50 $\left(\frac{6,500}{150} < 50\right)$. Therefore, the profit per individual worker in Company C is greater.

3. What was Company G's sales volume last year (in $ billions)?

(1) 48
(2) 50
(3) 64
(4) 76

Last year's sales volume does not appear in the table, but can be calculated using the current year's sales volume and the percent of change compared with last year. The table shows that Company G's sales this year amounted to $60 billion and that its sales increased by 25% over last year. In other words, its sales volume last year is a value which, if 25% is added to it, gives 60. This can be expressed in the following equation, where x is last year's sales volume:

$x + \frac{25}{100} \cdot x = 60$

Simplifying the equation: $\frac{125}{100} \cdot x = 60$.

Solving for x: $x = 60 \cdot \frac{100}{125}$, $x = 60 \cdot \frac{4}{5} = 48$

Thus, last year's sales volume for Company G was $48 billion, and the correct response is (**1**).

4. A company's expenditure in a particular year is defined as follows:

(sales in that year) – (profits in that year) = (expenditure for that year)

The company with the greatest expenditure this year belongs to which industry?

(1) Automobile
(2) Oil
(3) Electronics
(4) Metals

To calculate a company's expenditure for this year, subtract profits from sales. The table shows sales in $ billions, while profits are shown in $ millions. To subtract one from the other, both must be converted to the same units. Multiplying the sales volume appearing in the table by 1,000 gives us a sales volume in $ millions.

Thus, for example, Company C's sales amount to $105,000 million. Its profits amount to $5,000 million. Thus, its expenditure is $100,000 million. Using this method, it is possible to calculate the expenditure of each of the companies appearing in the table and find the company with the greatest expenditure.

However, this calculation is unnecessary. Given the formula that defines expenditure, it follows that the higher the sales and the lower the profits, the higher the expenditure. It therefore makes sense to first examine those companies with the highest sales or lowest profits. Looking at the table, we see that Company A has the highest sales and also the lowest profits (and it is also the only company showing a negative profit, that is, a loss). Thus, it is clearly the company with the greatest expenditure. Company A belongs to the automobile industry, and therefore, the correct response is (**1**).