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Higher Mathematics

UNIT 1

Mathematics 1

HSN21000

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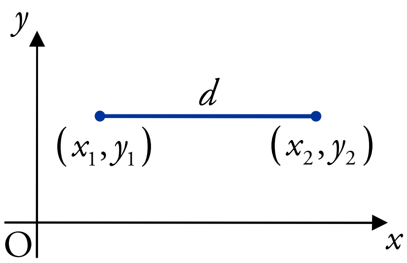
OUTCOME 1

Straight Lines

1 The Distance Between Points

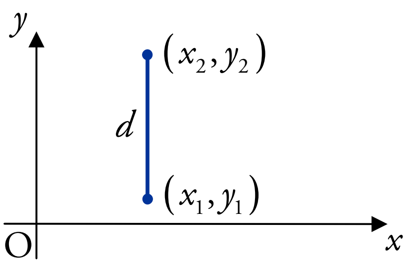
Points on Horizontal or Vertical Lines

It is relatively straightforward to work out the distance between two points which lie on a line parallel to the x- or y-axis.



In the diagram to the left, the points (x_1, y_1) and (x_2, y_2) lie on a line parallel to the x-axis, i.e. $y_1 = y_2$.

The distance between the points is simply the difference in the x-coordinates, i.e. $d = x_2 - x_1$ where $x_2 > x_1$.



In the diagram to the left, the points (x_1, y_1) and (x_2, y_2) lie on a line parallel to the y-axis, i.e. $x_1 = x_2$.

The distance between the points is simply the difference in the y-coordinates, i.e. $d = y_2 - y_1$ where $y_2 > y_1$.

EXAMPLE

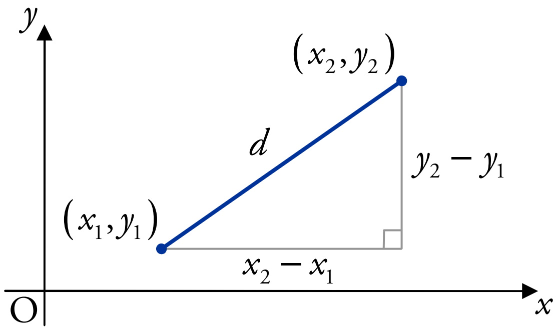
1. Calculate the distance between the points $(-7, -3)$ and $(16, -3)$.

Since both y-coordinates are 3, the distance is the difference in the x-coordinates:

$$\begin{aligned} d &= 16 - (-7) \\ &= 16 + 7 \\ &= 23 \text{ units.} \end{aligned}$$

The Distance Formula

The distance formula gives us a method for working out the length of the straight line between *any* two points. It is based on Pythagoras's Theorem.



Note
The "$y_2 - y_1$" and "$x_2 - x_1$" come from the method above.

The **distance** d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ units.}$$

EXAMPLES

2. A is the point $(-2, 4)$ and $B(3, 1)$. Calculate the length of the line AB.

$$\begin{aligned}\text{The length is } &\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{(3-(-2))^2 + (1-4)^2}\\ &= \sqrt{5^2 + (-3)^2}\\ &= \sqrt{25+9}\\ &= \sqrt{34} \text{ units.}\end{aligned}$$

3. Calculate the distance between the points $\left(\frac{1}{2}, -\frac{15}{4}\right)$ and $(-1, -1)$.

$$\begin{aligned}\text{The distance is } &\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{\left(-1-\tfrac{1}{2}\right)^2 + \left(-1+\tfrac{15}{4}\right)^2}\\ &= \sqrt{\left(-\tfrac{2}{2}-\tfrac{1}{2}\right)^2 + \left(-\tfrac{4}{4}+\tfrac{15}{4}\right)^2}\\ &= \sqrt{\left(-\tfrac{3}{2}\right)^2 + \left(\tfrac{11}{4}\right)^2}\\ &= \sqrt{\tfrac{9}{4} + \tfrac{121}{16}}\\ &= \sqrt{\tfrac{36}{16} + \tfrac{121}{16}}\\ &= \sqrt{\tfrac{157}{16}}\\ &= \frac{\sqrt{157}}{4} \text{ units.}\end{aligned}$$

Note
You need to become confident working with fractions and surds – so practise!

2 The Midpoint Formula

The point half-way between two points is called their midpoint. It is calculated as follows.

The **midpoint** of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

It may be helpful to think of the midpoint as the "average" of two points.

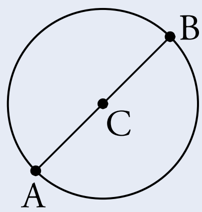
EXAMPLES

1. Calculate the midpoint of the points $(1, -4)$ and $(7, 8)$.

$$\begin{aligned}\text{The midpoint is } &\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &=\left(\frac{7+1}{2}, \frac{8+(-4)}{2}\right)\\ &=(4, 2).\end{aligned}$$

Note
Simply writing "The midpoint is (4, 2)" would be acceptable in an exam.

2. In the diagram below, $A(9, -2)$ lies on the circumference of the circle with centre $C(17, 12)$, and the line AB is a diameter of the circle. Find the coordinates of B.



Since C is the centre of the circle and AB is a diameter, C is the midpoint of AB. Using the midpoint formula, we have:

$$(17, 12) = \left(\frac{9+x}{2}, \frac{-2+y}{2}\right) \quad \text{where B is the point } (x, y).$$

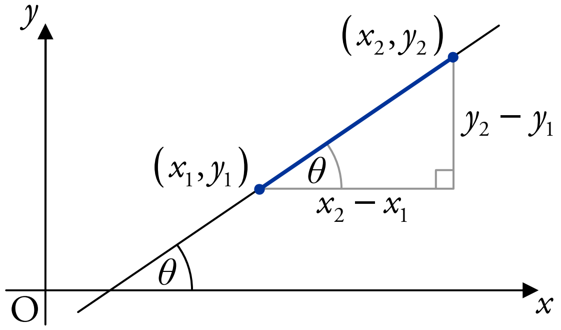
By comparing x- and y-coordinates, we have:

$$\begin{aligned}\frac{9+x}{2} &= 17 & \text{and} \qquad \frac{-2+y}{2} &= 12\\ 9+x &= 34 & -2+y &= 24\\ x &= 25 & y &= 26.\end{aligned}$$

So B is the point $(25, 26)$.

3 Gradients

Consider a straight line passing through the points (x_1, y_1) and (x_2, y_2):



Note
"θ" is the Greek letter "theta".
It is often used to stand for an angle.

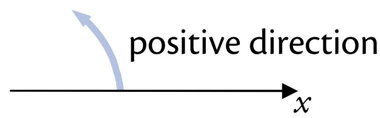
The **gradient** m of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{change in vertical height}}{\text{change in horizontal distance}} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for } x_1 \neq x_2.$$

Also, since $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{y_2 - y_1}{x_2 - x_1}$ we obtain:

$$m = \tan\theta$$

where θ is the angle between the line and the positive direction of the x-axis.



Note

As a result of the above definitions:

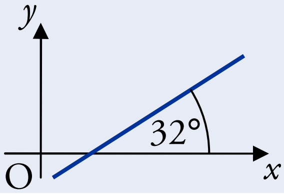
- lines with positive gradients slope up, from left to right;
- lines with negative gradients slope down, from left to right;
- lines parallel to the x-axis have a gradient of zero;
- lines parallel to the y-axis have an undefined gradient.

We may also use the fact that:

Two distinct lines are said to be **parallel** when they have the same gradient (or when both lines are vertical).

EXAMPLES

1. Calculate the gradient of the straight line shown in the diagram below.



$$m = \tan\theta$$
$$= \tan 32°$$
$$= 0{\cdot}62 \text{ (to 2 d.p.).}$$

2. Find the angle that the line joining $P(-2, -2)$ and $Q(1, 7)$ makes with the positive direction of the x-axis.

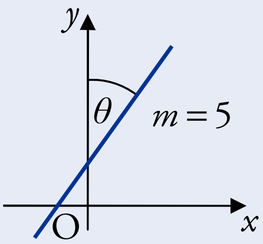
The line has gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7+2}{1+2} = 3$.

And so $m = \tan\theta$

$\tan\theta = 3$

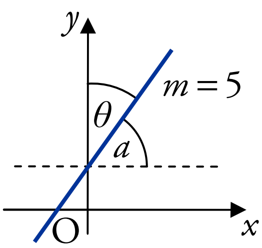
$\theta = \tan^{-1}(3) = 71{\cdot}57°$ (to 2 d.p.).

3. Find the size of angle θ shown in the diagram below.



We need to be careful because the θ in the question is *not* the θ in "$m = \tan\theta$".

So we work out the angle a and use this to find θ:



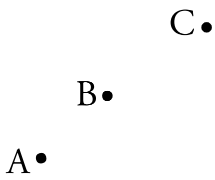
$$a = \tan^{-1}(m)$$
$$= \tan^{-1}(5)$$
$$= 78{\cdot}690°.$$

So $\theta = 90° - 78{\cdot}690° = 11{\cdot}31°$ (to 2 d.p.).

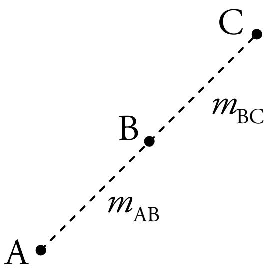
4 Collinearity

Points which lie on the same straight line are said to be **collinear**.

To test if three points A, B and C are collinear we can:

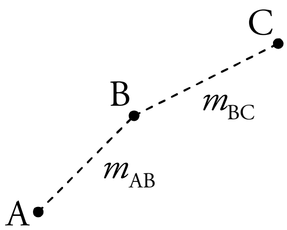


1. Work out m_{AB}.
2. Work out m_{BC} (or m_{AC}).
3. If the gradients from 1. and 2. are the same then A, B and C are collinear.



$m_{\text{AB}} = m_{\text{BC}}$ so A, B and C are collinear.

If the gradients are different then the points are not collinear.



$m_{\text{AB}} \neq m_{\text{BC}}$ so A, B and C are not collinear.

This test for collinearity can only be used in two dimensions.

EXAMPLES

1. Show that the points $\text{P}(-6, -1)$, $\text{Q}(0, 2)$ and $\text{R}(8, 6)$ are collinear.

$$m_{\text{PQ}} = \frac{2-(-1)}{0-(-6)} = \frac{3}{6} = \frac{1}{2} \qquad m_{\text{QR}} = \frac{6-2}{8-0} = \frac{4}{8} = \frac{1}{2}$$

Since $m_{\text{PQ}} = m_{\text{QR}}$ and Q is a common point, P, Q and R are collinear.

2. The points $\text{A}(1,-1)$, $\text{B}(-1,k)$ and $\text{C}(5,7)$ are collinear. Find the value of k.

Since the points are collinear $m_{\text{AB}} = m_{\text{AC}}$:

$$\frac{k-(-1)}{-1-1} = \frac{7-(-1)}{5-1}$$

$$\frac{k+1}{-2} = \frac{8}{4}$$

$$k+1 = 2\times(-2)$$

$$k = -5.$$

5 Gradients of Perpendicular Lines

Two lines at right-angles to each other are said to be **perpendicular**.

If perpendicular lines have gradients m and $m_\perp$ then

$$m \times m_\perp = -1.$$

Conversely, if $m \times m_\perp = -1$ then the lines are perpendicular.

The simple rule is: if you know the gradient of one of the lines, then the gradient of the other is calculated by inverting the gradient (i.e. "flipping" the fraction) and changing the sign. For example:

$$\text{if} \quad m = \tfrac{2}{3} \quad \text{then} \quad m_\perp = -\tfrac{3}{2}.$$

Note that this rule *cannot* be used if the line is parallel to the x- or y-axis.

- If a line is parallel to the x-axis $(m = 0)$, then the perpendicular line is parallel to the y-axis – it has an undefined gradient.
- If a line is parallel to the y-axis then the perpendicular line is parallel to the x-axis – it has a gradient of zero.

EXAMPLES

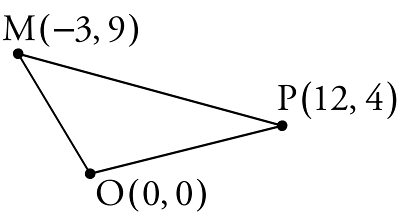
1. Given that T is the point $(1, -2)$ and S is $(-4, 5)$, find the gradient of a line perpendicular to ST.

$$m_{ST} = \frac{5-(-2)}{-4-1} = -\tfrac{7}{5}$$

So $m_\perp = \tfrac{5}{7}$ since $m_{ST} \times m_\perp = -1$.

2. Triangle MOP has vertices $M(-3, 9)$, $O(0, 0)$ and $P(12, 4)$.
   Show that the triangle is right-angled.

Sketch:



$$m_{OM} = \frac{9-0}{-3-0} = -3 \qquad m_{MP} = \frac{9-4}{-3-12} = -\tfrac{5}{15} = -\tfrac{1}{3} \qquad m_{OP} = \frac{4-0}{12-0} = \tfrac{1}{3}$$

Since $m_{OM} \times m_{OP} = -1$, OM is perpendicular to OP which means $\triangle MOP$ is right-angled at O.

Note

The converse of Pythagoras's Theorem could also be used here:

$$d_{\mathrm{OP}}^2 = 12^2 + 4^2 = 160 \qquad d_{\mathrm{MP}}^2 = (12-(-3))^2 + (4-9)^2$$
$$d_{\mathrm{OM}}^2 = (-3)^2 + 9^2 = 90 \qquad = 15^2 + (-5)^2$$
$$= 250.$$

Since $d_{\mathrm{OP}}^2 + d_{\mathrm{OM}}^2 = d_{\mathrm{MP}}^2$, triangle MOP is right-angled at O.

6 The Equation of a Straight Line

To work out the equation of a straight line, we need to know two things: the gradient of the line, and a point which lies on the line.

The straight line through the point (a, b) with gradient m has the equation

$$y - b = m(x - a).$$

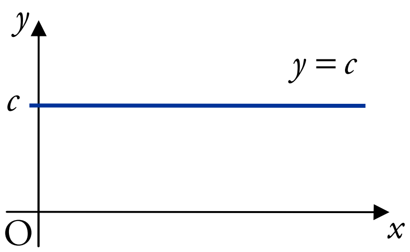
Notice that if we have a point $(0, c)$ – the y-axis intercept – then the equation becomes $y = mx + c$. You should already be familiar with this form.

It is good practice to rearrange the equation of a straight line into the form

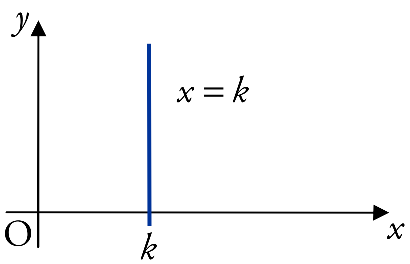
$$ax + by + c = 0$$

where a is positive. This is known as the general form of the equation of a straight line.

Lines Parallel to Axes



If a line is parallel to the x-axis (i.e. $m = 0$), its equation is $y = c$.



If a line is parallel to the y-axis (i.e. m is undefined), its equation is $x = k$.

EXAMPLES

1. Find the equation of the line with gradient $\frac{1}{3}$ passing through the point $(3,-4)$.

$$\begin{aligned} y-b &= m(x-a) \\ y-(-4) &= \tfrac{1}{3}(x-3) \qquad (\times 3) \\ 3y+12 &= x-3 \\ 3y &= x-15 \\ x-3y-15 &= 0. \end{aligned}$$

Note
It is usually easier to multiply out the fraction before expanding the brackets .

2. Find the equation of the line passing through $A(3,2)$ and $B(-2,1)$.

To work out the equation, we must first find the gradient of the line AB:

$$\begin{aligned} m_{AB} &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{2-1}{3-(-2)} = \frac{1}{5}. \end{aligned}$$

Now we have a gradient, and can use this with one of the given points:

$$\begin{aligned} y-b &= m(x-a) \\ y-2 &= \tfrac{1}{5}(x-3) \qquad \text{using } A(3,2) \text{ and } m_{AB}=\tfrac{1}{5} \\ 5y-10 &= x-3 \\ 5y &= x+7 \\ x-5y+7 &= 0. \end{aligned}$$

3. Find the equation of the line passing through $\left(-\frac{3}{5},4\right)$ and $\left(-\frac{3}{5},5\right)$.

The gradient is undefined since the x-coordinates are equal.

So the equation of the line is $x=-\frac{3}{5}$.

Extracting the Gradient

You should already be familiar with the following fact.

The line with equation $y = mx + c$ has gradient m.

It is important to remember that you must rearrange the equation of a straight line into this form *before* extracting the gradient.

EXAMPLES

4. Find the gradient of the line with equation $3x + 2y + 4 = 0$.

We have to rearrange the equation:

$$\begin{aligned} 3x + 2y + 4 &= 0 \\ 2y &= -3x - 4 \\ y &= -\tfrac{3}{2}x - 2. \end{aligned}$$

So the gradient is $-\frac{3}{2}$.

5. The line through points $A(3,-3)$ and B has equation $5x - y - 18 = 0$.

Find the equation of the line through A which is perpendicular to AB.

First, find the gradient of AB:

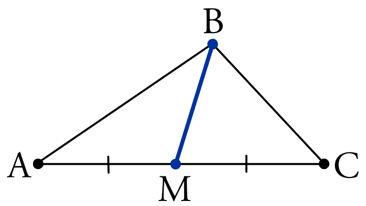
$$\begin{aligned} 5x - y - 18 &= 0 \\ y &= 5x - 18. \end{aligned}$$

So $m_{AB} = 5$ and $m_{\perp} = -\frac{1}{5}$. Therefore the equation is:

$$\begin{aligned} y + 3 &= -\tfrac{1}{5}(x - 3) \qquad \text{using } A(3,-3) \text{ and } m_{\perp} = -\tfrac{1}{5} \\ 5y + 15 &= -(x - 3) \\ 5y + 15 &= -x + 3 \\ x + 5y + 12 &= 0. \end{aligned}$$

7 Medians

A **median** of a triangle is a line through a vertex and the midpoint of the opposite side.



BM is a median of $\triangle ABC$.

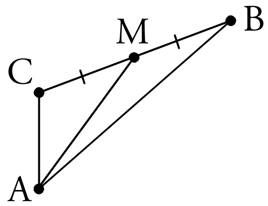
The standard process for finding the equation of a median is shown below.

EXAMPLE

Triangle ABC has vertices $A(4, -9)$, $B(10, 2)$ and $C(4, -4)$.

Find the equation of the median from A.

Start with a sketch:



Step 1

Calculate the midpoint of the relevant line.

Using $B(10, 2)$ and $C(4, -4)$:

$$\begin{aligned} M &= \left(\frac{10+4}{2}, \frac{2+(-4)}{2}\right) \\ &= \left(\frac{14}{2}, \frac{-2}{2}\right) \\ &= (7, -1). \end{aligned}$$

Step 2

Calculate the gradient of the line between the midpoint and the opposite vertex.

Using $A(4, -9)$ and $M(7, -1)$:

$$\begin{aligned} m_{AM} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-(-9)}{7-4} = \frac{8}{3}. \end{aligned}$$

Step 3

Find the equation using this gradient and either of the two points used in Step 2.

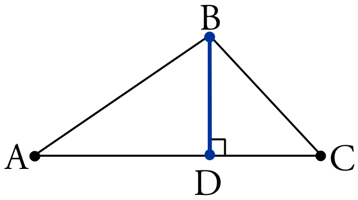
Using $A(4, -9)$ and $m_{AM} = \frac{8}{3}$:

$$\begin{aligned} y - b &= m(x-a) \\ y + 9 &= \tfrac{8}{3}(x-4) \qquad (\times 3) \\ 3y + 27 &= 8x - 32 \\ 3y &= 8x - 59 \end{aligned}$$

$$8x - 3y - 59 = 0.$$

8 Altitudes

An **altitude** of a triangle is a line through a vertex, perpendicular to the opposite side.



BD is an altitude of $\triangle ABC$.

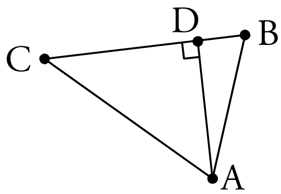
The standard process for finding the equation of an altitude is shown below.

EXAMPLE

Triangle ABC has vertices $A(3, -5)$, $B(4, 3)$ and $C(-7, 2)$.

Find the equation of the altitude from A.

Start with a sketch:



Step 1

Calculate the gradient of the side which is perpendicular to the altitude.

Using $B(4, 3)$ and $C(-7, 2)$:

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{-7-4} = \frac{1}{11}.$$

Step 2

Calculate the gradient of the altitude using $m \times m_{\perp} = -1$.

Using $m_{BC} \times m_{AD} = -1$:

$$m_{AD} = -11.$$

Step 3

Find the equation using this gradient and the point that the altitude passes through.

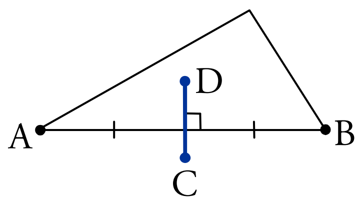
Using $A(3, -5)$ and $m_{AD} = -11$:

$$\begin{aligned} y - b &= m(x-a) \\ y + 5 &= -11(x-3) \\ y &= -11x + 28 \end{aligned}$$

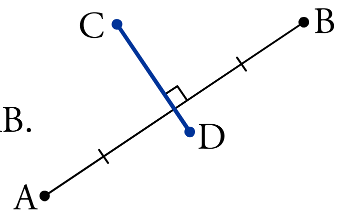
$$11x + y - 28 = 0.$$

9 Perpendicular Bisectors

A **perpendicular bisector** is a line which cuts through the midpoint of a line segment at right-angles.



In both cases, CD is the perpendicular bisector of AB.



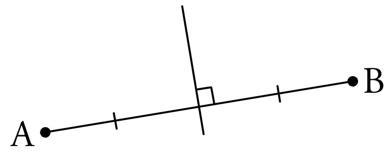
The standard process for finding the equation of a perpendicular bisector is shown below.

EXAMPLE

A is the point $(-2, 1)$ and B is the point $(4, 7)$.

Find the equation of the perpendicular bisector of AB.

Start with a sketch:



Step 1

Calculate the midpoint of the line segment being bisected.

Using $A(-2, 1)$ and $B(4, 7)$:

$$\begin{aligned}\text{Midpoint}_{AB} &= \left(\frac{-2+4}{2}, \frac{1+7}{2}\right)\\ &= (1, 4).\end{aligned}$$

Step 2

Calculate the gradient of the line used in Step 1, then find the gradient of its perpendicular bisector using $m \times m_{\perp} = -1$.

Using $A(-2, 1)$ and $B(4, 7)$:

$$\begin{aligned}m_{AB} &= \frac{7-1}{4-(-2)}\\ &= \tfrac{6}{6}\\ &= 1.\end{aligned}$$

$m_{\perp} = -1$ since $m_{AB} \times m_{\perp} = -1$.

Step 3

Find the equation of the perpendicular bisector using the point from Step 1 and the gradient from Step 2.

Using $(1, 4)$ and $m_{\perp} = -1$:

$$\begin{aligned}y - b &= m(x-a)\\ y - 4 &= -(x-1)\\ y &= -x + 1 + 4\\ y &= -x + 5\end{aligned}$$

$$x + y - 5 = 0.$$

10 Intersection of Lines

Many problems involve lines which intersect (cross each other). Once we have equations for the lines, the problem is to find values for x and y which satisfy both equations, i.e. solve simultaneous equations.

There are three different techniques and, depending on the form of the equations, one may be more efficient than the others.

We will demonstrate these techniques by finding the point of intersection of the lines with equations $3y = x + 15$ and $y = x - 3$.

Elimination

This should be a familiar method, and can be used in all cases.

$$3y = x + 15 \quad ①$$
$$y = x - 3 \quad ②$$

$$①-②: \quad 2y = 18$$
$$y = 9.$$

Put $y = 9$ into ②: $x = 9 + 3$
$\quad = 12.$

So the lines intersect at the point $(12, 9)$.

Equating

This method can be used when both equations have a common x- or y-coefficient. In this case, both equations have an x-coefficient of 1.

Make x the subject of both equations:

$$x = 3y - 15 \qquad x = y + 3.$$

Equate:

$$3y - 15 = y + 3$$
$$2y = 18$$
$$y = 9.$$

Substitute $y = 9$ into:

$$y = x - 3$$
$$x = 9 + 3$$
$$= 12.$$

So the lines intersect at the point $(12, 9)$.

Substitution

This method can be used when one equation has an x- or y-coefficient of 1 (i.e. just an x or y with no multiplier).

Substitute $y = x - 3$ into:

$$\begin{aligned} 3y &= x + 15 \\ 3(x-3) &= x + 15 \\ 3x - 9 &= x + 15 \\ 2x &= 24 \\ x &= 12. \end{aligned}$$

Substitute $x = 12$ into:

$$\begin{aligned} y &= x - 3 \\ y &= 12 - 3 \\ &= 9. \end{aligned}$$

So the lines intersect at the point $(12, 9)$.

EXAMPLE

1. Find the point of intersection of the lines $2x - y + 11 = 0$ and $x + 2y - 7 = 0$.

Eliminate y:

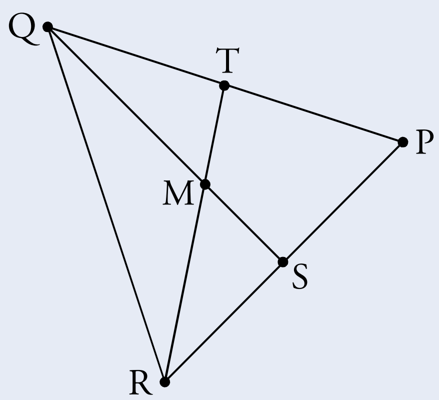
$$2x - y + 11 = 0 \quad ①$$
$$x + 2y - 7 = 0 \quad ②$$

$$2\times①+②: \quad 5x + 15 = 0$$
$$x = -3.$$

Put $x = -3$ into ①: $-6 - y + 11 = 0$
$$y = 5.$$

So the point of intersection is $(-3,5)$.

2. Triangle PQR has vertices $\mathrm{P}(8,3)$, $\mathrm{Q}(-1,6)$ and $\mathrm{R}(2,-3)$.



(a) Find the equation of altitude QS.
(b) Find the equation of median RT.
(c) Hence find the coordinates of M.

(a) Find the gradient of PR:

$$m_{\text{PR}} = \frac{3-(-3)}{8-2} = \tfrac{6}{6} = 1.$$

Note
This is the standard method for finding the equation of an altitude.

So the gradient of QS is $m_{\text{QS}} = -1$, since the $m_{\text{PR}} \times m_{\text{QS}} = -1$.

Find the equation of QS using $\text{Q}(-1,6)$ and $m_{\text{QS}} = -1$:

$$\begin{aligned} y-6 &= -1(x+1) \\ y-6 &= -x-1 \\ x+y-5 &= 0. \end{aligned}$$

(b) Find the coordinates of T, the midpoint of PQ:

Note
This is the standard method for finding the equation of a median.

$$\text{T} = \left(\frac{8-1}{2}, \frac{3+6}{2}\right) = \left(\tfrac{7}{2}, \tfrac{9}{2}\right).$$

Find the gradient of RT using $\text{R}(2,-3)$ and $\text{T}\left(\tfrac{7}{2}, \tfrac{9}{2}\right)$:

$$m_{\text{RT}} = \frac{\tfrac{9}{2}-(-3)}{\tfrac{7}{2}-2} = \frac{\tfrac{15}{2}}{\tfrac{3}{2}} = \tfrac{15}{3} = 5.$$

Find the equation of RT using $\text{R}(2,-3)$ and $m_{\text{RT}} = 5$:

$$\begin{aligned} y+3 &= 5(x-2) \\ y+3 &= 5x-10 \\ 5x-y-13 &= 0. \end{aligned}$$

(c) Now solve the equations simultaneously to find M.

Eliminate y:

$$\begin{aligned} x+y-5 &= 0 \quad \text{①} \\ 5x-y-13 &= 0 \quad \text{②} \end{aligned}$$

Note
Any of the three techniques could have been used here.

①+②: $6x-18=0$

$x = 3.$

Put $x=3$ into ①: $3+y-5=0$

$y = 2.$

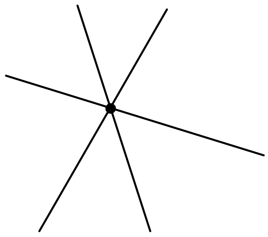
So the point of intersection is $\text{M}(3,2)$.

11 Concurrency

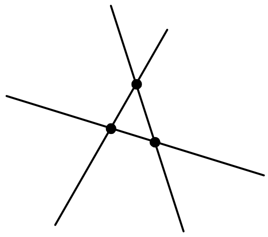
Any number of lines are said to be **concurrent** if there is a point through which they all pass.

So in the previous section, by finding a point of intersection of two lines we showed that the two lines were concurrent.

For *three* lines to be concurrent, they must all pass through a single point.

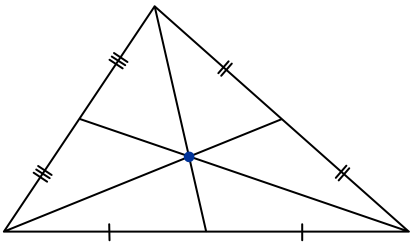


The three lines are concurrent

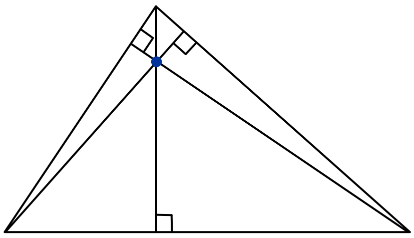


The three lines are *not* concurrent

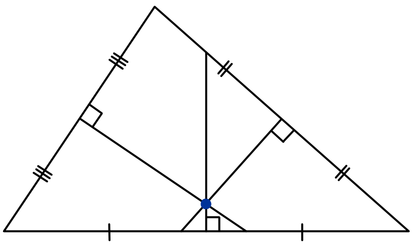
A surprising fact is that the following lines in a triangle are concurrent.



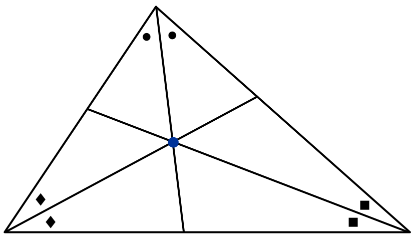
The three **medians** of a triangle are concurrent.



The three **altitudes** of a triangle are concurrent.



The three **perpendicular bisectors** in a triangle are concurrent.



The three **angle bisectors** of a triangle are concurrent.

OUTCOME 2

Functions and Graphs

1 Sets

In order to study functions and graphs, we use set theory. This requires some standard symbols and terms, which you should become familiar with.

A **set** is a collection of objects (usually numbers).

For example, $S = \{5, 6, 7, 8\}$ is a set (we just list the objects inside curly brackets).

We refer to the objects in a set as its **elements** (or members), e.g. 7 is an element of *S*. We can write this symbolically as $7 \in S$. It is also clear that 4 is *not* an element of *S*; we can write $4 \notin S$.

Given two sets *A* and *B*, we say *A* is a **subset** of *B* if all elements of *A* are also elements of *B*. For example, $\{6, 7, 8\}$ is a subset of *S*.

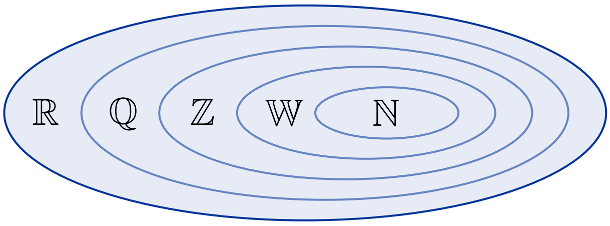
The **empty set** is the set with no elements. It is denoted by $\{\ \}$ or $\varnothing$.

Standard Sets

There are common sets of numbers which have their own symbols. Note that numbers can belong to more than one set.

$\mathbb{N}$	natural numbers	counting numbers, i.e. $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$.
$\mathbb{W}$	whole numbers	natural numbers including zero, i.e. $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$.
$\mathbb{Z}$	integers	positive and negative whole numbers, i.e. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
$\mathbb{Q}$	rational numbers	can be written as a fraction of integers, e.g. $-4, \frac{1}{3}, 0{\cdot}25, -\frac{1}{3}$.
$\mathbb{R}$	real numbers	all points on the number line, e.g. $-6, -\frac{1}{2}, \sqrt{2}, \frac{1}{12}, 0{\cdot}125$.

Notice that $\mathbb{N}$ is a subset of $\mathbb{W}$, which is a subset of $\mathbb{Z}$, which is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$. These relationships between the standard sets are illustrated in the "Venn diagram" below.



EXAMPLE

List all the numbers in the set $\mathrm{P}=\{x\in\mathbb{N}:1<x<5\}$.

P contains natural numbers which are strictly greater than 1 and strictly less than 5, so:

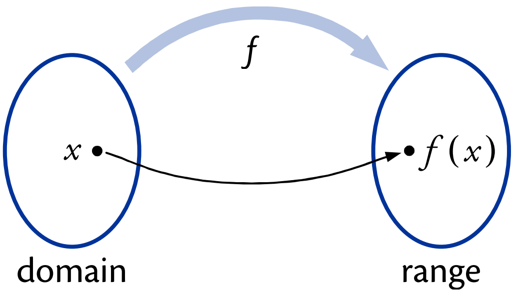
$$\mathrm{P}=\{2, 3, 4\}.$$

Note
In set notation, a colon (:) means "such that".

2 Functions

A **function** relates a set of inputs to a set of outputs, with each input related to exactly one output.

The set of inputs is called the **domain** and the resulting set of outputs is called the **range**.



A function is usually denoted by a lower case letter (e.g. f or g) and is defined using a formula of the form $f(x)=\ldots$. This specifies what the output of the function is when x is the input.

For example, if $f(x)=x^2+1$ then f squares the input and adds 1.

Restrictions on the Domain

The domain is the set of all possible inputs to a function, so it must be possible to evaluate the function for any element of the domain.

We are free to choose the domain, provided that the function is defined for all elements in it. If no domain is specified then we assume that it is as large as possible.

Division by Zero

It is impossible to divide by zero. So in functions involving fractions, the domain must exclude numbers which would give a denominator (bottom line) of zero.

For example, the function defined by:

$$f(x) = \frac{3}{x-5}$$

cannot have 5 in its domain, since this would make the denominator equal to zero.

The domain of f may be expressed formally as $\{x \in \mathbb{R} : x \neq 5\}$. This is read as "all x belonging to the real set such that x does not equal five".

Even Roots

Using real numbers, we cannot evaluate an even root (i.e. square root, fourth root etc.) of a negative number. So the domain of any function involving even roots must exclude numbers which would give a negative number under the root.

For example, the function defined by:

$$f(x) = \sqrt{7x-2}$$

must have $7x - 2 \geq 0$. Solving for x gives $x \geq \frac{2}{7}$, so the domain of f can be expressed formally as $\left\{x \in \mathbb{R} : x \geq \frac{2}{7}\right\}$.

EXAMPLE

1. A function g is defined by $g(x) = x - \dfrac{6}{x+4}$.

   Define a suitable domain for g.

We cannot divide by zero, so $x \neq -4$. The domain is $\{x \in \mathbb{R} : x \neq -4\}$.

Identifying the Range

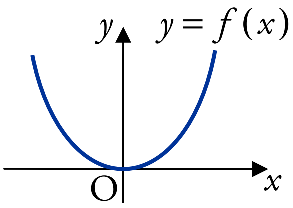
Recall that the range is the set of possible outputs. Some functions cannot produce certain values so these are not in the range.

For example:

$$f(x) = x^2$$

does not produce negative values, since any number squared is either positive or zero.

Looking at the graph of a function makes identifying its range more straightforward.



If we consider the graph of $y = f(x)$ (shown to the left) it is clear that there are no negative y-values.

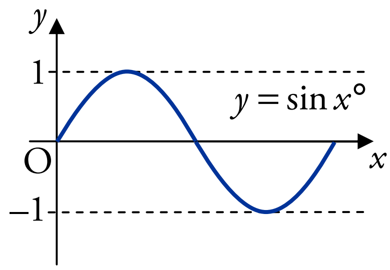
The range can be stated as $f(x) \geq 0$.

Note that the range also depends on the choice of domain. For example, if the domain of $f(x) = x^2$ is chosen to be $\{x \in \mathbb{R} : x \geq 3\}$ then the range can be stated as $f(x) \geq 9$.

EXAMPLE

2. A function f is defined by $f(x) = \sin x°$ for $x \in \mathbb{R}$. Identify its range.

Sketching the graph of $y = f(x)$ shows that $\sin x°$ only produces values from –1 to 1 inclusive.



This can be written as $-1 \leq f(x) \leq 1$.

3 Composite Functions

Two functions can be "composed" to form a new **composite function**.

For example, if we have a squaring function and a halving function, we can compose them to form a new function. We take the output from one and use it as the input for the other.

$$x \longrightarrow \boxed{\text{square}} \longrightarrow x^2 \longrightarrow \boxed{\text{halve}} \longrightarrow \frac{x^2}{2}$$

The order is important, as we get a different result in this case:

$$x \longrightarrow \boxed{\text{halve}} \longrightarrow \frac{x}{2} \longrightarrow \boxed{\text{square}} \longrightarrow \frac{x^2}{4}$$

Using function notation we have, say, $f(x) = x^2$ and $g(x) = \frac{x}{2}$.

The diagrams above show the composite functions:

$$\begin{aligned} g(f(x)) &= g\left(x^2\right) \\ &= \frac{x^2}{2} \end{aligned} \qquad\qquad \begin{aligned} f(g(x)) &= f\left(\tfrac{x}{2}\right) \\ &= \left(\tfrac{x}{2}\right)^2 = \tfrac{x^2}{4}. \end{aligned}$$

EXAMPLES

1. Functions f and g are defined by $f(x) = 2x$ and $g(x) = x - 3$. Find:
   (a) $f(2)$ (b) $f(g(x))$ (c) $g(f(x))$

(a) $\begin{aligned} f(2) &= 2(2) \\ &= 4. \end{aligned}$ (b) $\begin{aligned} f(g(x)) &= f(x-3) \\ &= 2(x-3). \end{aligned}$ (c) $\begin{aligned} g(f(x)) &= g(2x) \\ &= 2x - 3. \end{aligned}$

2. Functions f and g are defined on suitable domains by $f(x) = x^3 + 1$ and $g(x) = \frac{1}{x}$.
   Find formulae for $h(x) = f(g(x))$ and $k(x) = g(f(x))$.

$$\begin{aligned} h(x) &= f(g(x)) \\ &= f\left(\tfrac{1}{x}\right) \\ &= \left(\tfrac{1}{x}\right)^3 + 1. \end{aligned} \qquad\qquad \begin{aligned} k(x) &= g(f(x)) \\ &= g\left(x^3 + 1\right) \\ &= \frac{1}{x^3 + 1}. \end{aligned}$$

4 Inverse Functions

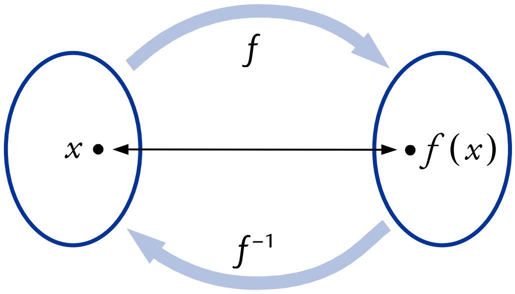
The idea of an **inverse function** is to reverse the effect of the original function. It is the "opposite" function.

You should already be familiar with this idea – for example, doubling a number can be reversed by halving the result. That is, multiplying by two and dividing by two are inverse functions.

The inverse of the function f is usually denoted f^{-1} (read as "f inverse").

The functions f and g are said to be **inverses** if $f\left(g(x)\right)=g\left(f(x)\right)=x$.

This means that when a number is worked through a function f then its inverse f^{-1}, the result is the same as the input.



For example, $f(x)=4x-1$ and $g(x)=\dfrac{x+1}{4}$ are inverse functions since:

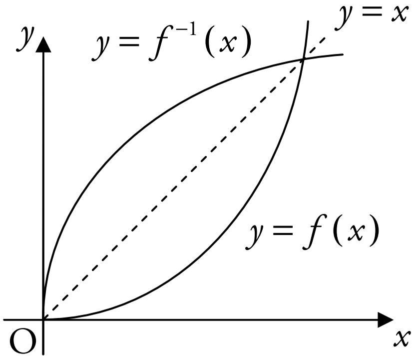
$$\begin{aligned} f\left(g(x)\right) &= f\left(\frac{x+1}{4}\right) \\ &= 4\left(\frac{x+1}{4}\right)-1 \\ &= x+1-1 \\ &= x \end{aligned}$$

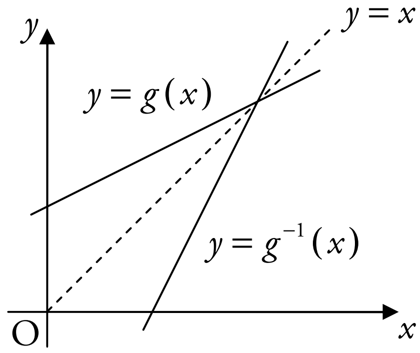
$$\begin{aligned} g\left(f(x)\right) &= g(4x-1) \\ &= \frac{(4x-1)+1}{4} \\ &= \frac{4x}{4} \\ &= x. \end{aligned}$$

Graphs of Inverses

If we have the graph of a function, then we can find the graph of its inverse by reflecting in the line $y = x$.

For example, the diagrams below show the graphs of two functions and their inverses.





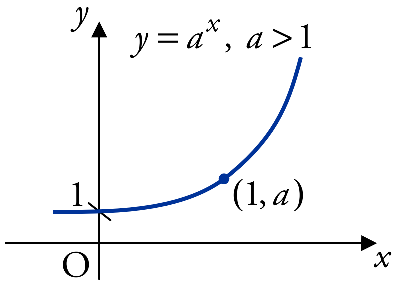
5 Exponential Functions

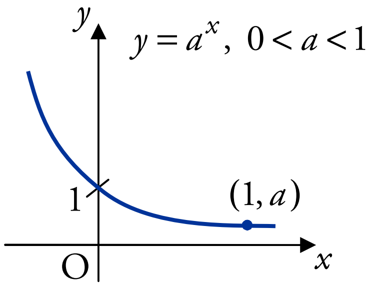
A function of the form $f(x) = a^x$ where $a, x \in \mathbb{R}$ and $a > 0$ is known as an **exponential** function to the base a.

We refer to x as the **power**, index or exponent.

Notice that when $x = 0$, $f(x) = a^0 = 1$. Also when $x = 1$, $f(x) = a^1 = a$.

Hence the graph of an exponential always passes through $(0,1)$ and $(1, a)$:

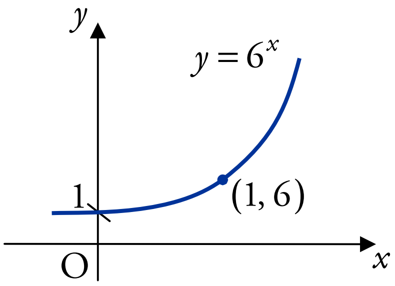




EXAMPLE

Sketch the curve with equation $y = 6^x$.

The curve passes through $(0,1)$ and $(1,6)$.



6 Introduction to Logarithms

Until now, we have only been able to solve problems involving exponentials when we know the index, and have to find the base. For example, we can solve $p^6 = 512$ by taking sixth roots to get $p = \sqrt[6]{512}$.

But what if we know the base and have to find the index?

To solve $6^q = 512$ for q, we need to find the power of 6 which gives 512. To save writing this each time, we use the notation $q = \log_6 512$, read as "log to the base 6 of 512". In general:

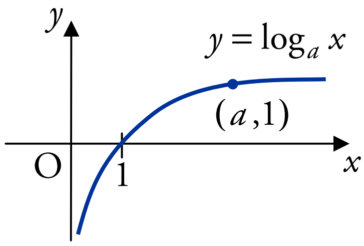
$\log_a x$ is the power of a which gives x.

The properties of logarithms will be covered in Unit 3 Outcome 3.

Logarithmic Functions

A **logarithmic** function is one in the form $f(x) = \log_a x$ where $a,\ x > 0$.

Logarithmic functions are inverses of exponentials, so to find the graph of $y = \log_a x$, we can reflect the graph of $y = a^x$ in the line $y = x$.

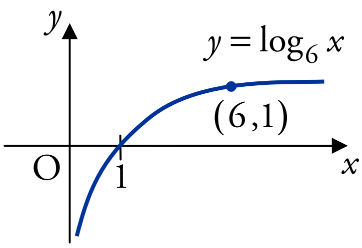


The graph of a logarithmic function always passes through $(1, 0)$ and $(a, 1)$.

EXAMPLE

Sketch the curve with equation $y = \log_6 x$.

The curve passes through $(1, 0)$ and $(6, 1)$.



7 Radians

Degrees are not the only units used to measure angles. The radian (RAD on the calculator) is an alternative measurement which is more useful in mathematics.

Degrees and radians bear the relationship:

$$\pi \text{ radians} = 180°.$$

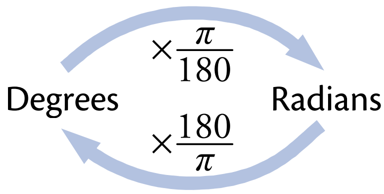
The other equivalences that you should become familiar with are:

$$30° = \tfrac{\pi}{6} \text{ radians} \qquad 45° = \tfrac{\pi}{4} \text{ radians} \qquad 60° = \tfrac{\pi}{3} \text{ radians}$$

$$90° = \tfrac{\pi}{2} \text{ radians} \qquad 135° = \tfrac{3\pi}{4} \text{ radians} \qquad 360° = 2\pi \text{ radians.}$$

Converting between degrees and radians is straightforward.

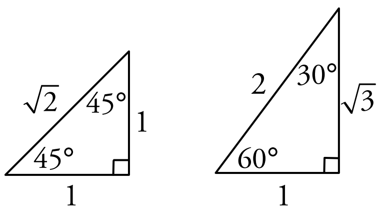
- To convert from degrees to radians, multiply by π and divide by 180.
- To convert from radians to degrees, multiply by 180 and divide by π.



For example, $50° = 50 \times \frac{\pi}{180} = \frac{5}{18}\pi$ radians.

8 Exact Values

The following exact values must be known. You can do this by either memorising the two triangles involved, or memorising the table.



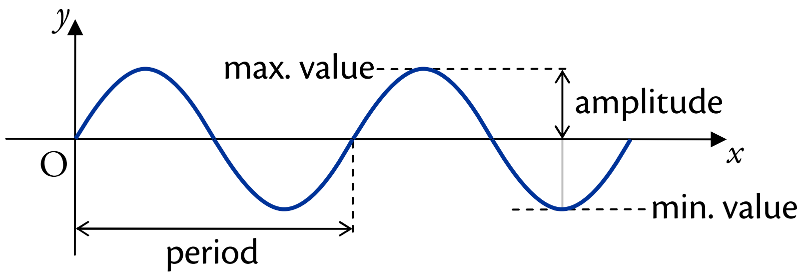
DEG	RAD	$\sin x$	$\cos x$	$\tan x$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	–

Tip
You'll probably find it easier to remember the triangles.

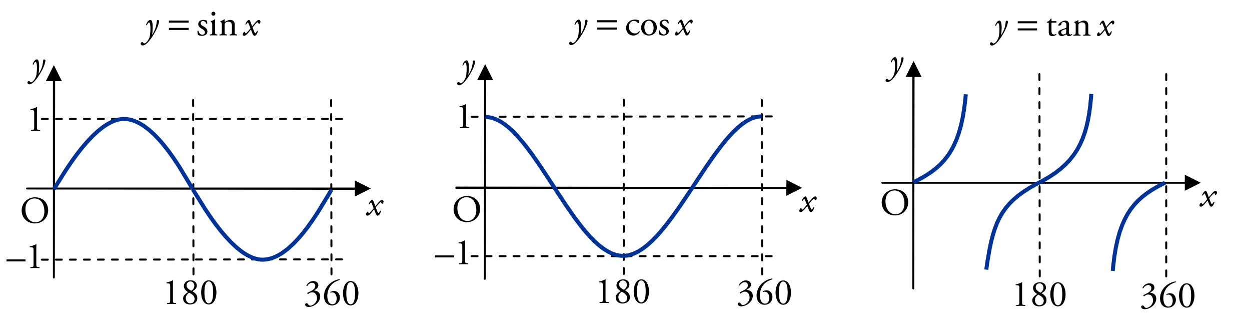
9 Trigonometric Functions

A function which has a repeating pattern in its graph is called **periodic**. The length of the smallest repeating pattern in the x-direction is called the **period**.

If the repeating pattern has a minimum and maximum value, then half of the difference between these values is called the **amplitude**.



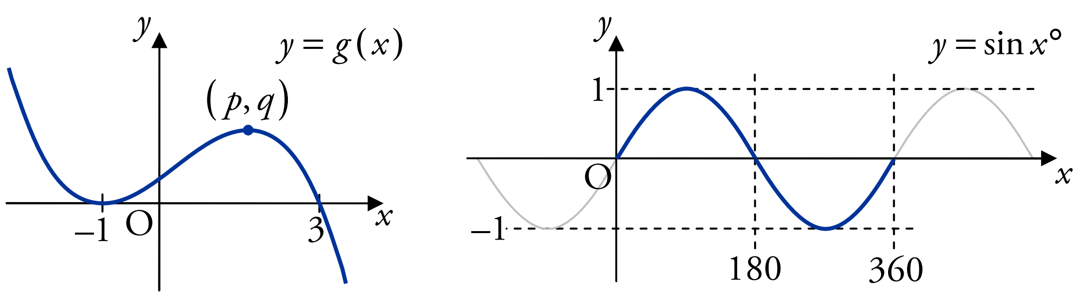
The three basic trigonometric functions (sine, cosine, and tangent) are periodic, and have graphs as shown below.



$y = \sin x$	$y = \cos x$	$y = \tan x$
Period $= 360° = 2\pi$ radians	Period $= 360° = 2\pi$ radians	Period $= 180° = \pi$ radians
Amplitude $= 1$	Amplitude $= 1$	Amplitude is undefined

10 Graph Transformations

The graphs below represent two functions. One is a cubic and the other is a sine wave, focusing on the region between 0 and 360.



In the following pages we will see the effects of three different "transformations" on these graphs: translation, reflection and scaling.

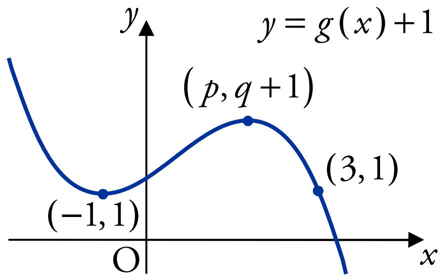
Translation

A **translation** moves every point on a graph a fixed distance in the same direction. The shape of the graph does not change.

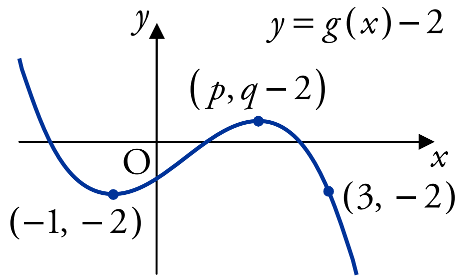
Translation parallel to the y-axis

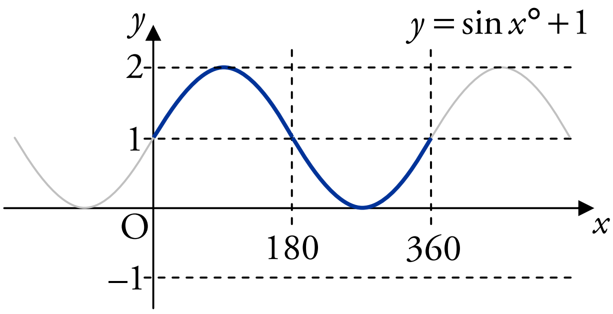
$f(x)+a$ moves the graph of $f(x)$ up or down. The graph is moved up if a is positive, and down if a is negative.

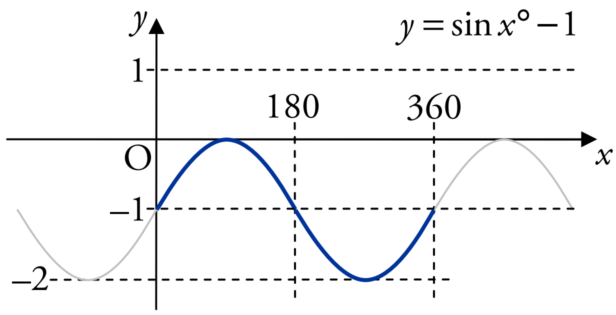
a is positive



a is negative



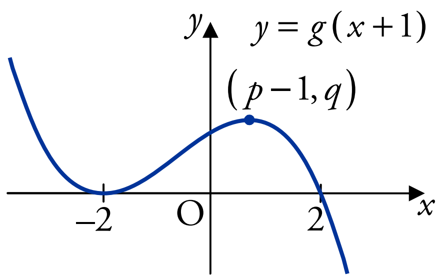




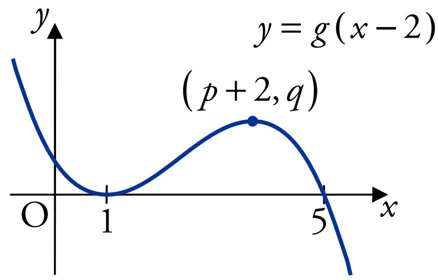
Translation parallel to the x-axis

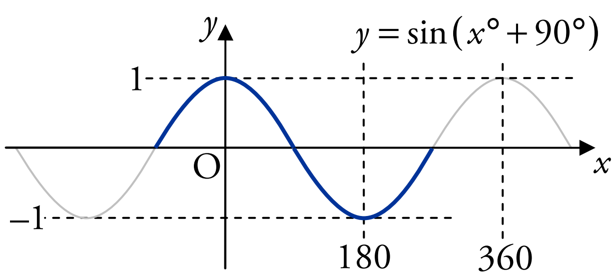
$f(x+a)$ moves the graph of $f(x)$ left or right. The graph is moved left if a is positive, and right if a is negative.

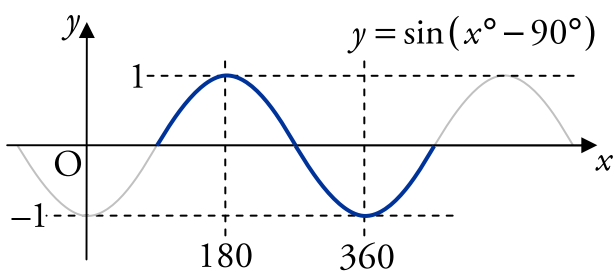
a is positive



a is negative







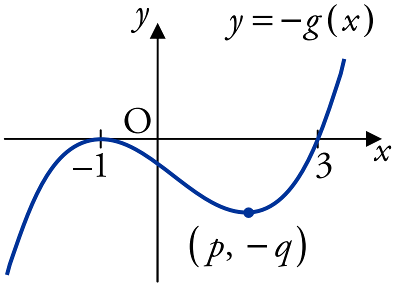
Reflection

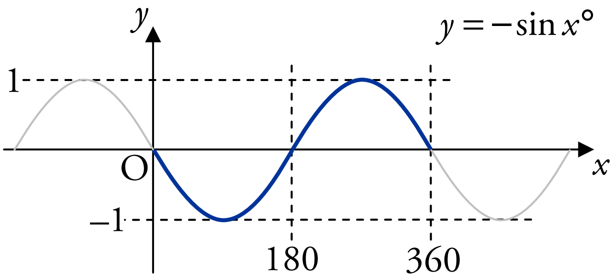
A **reflection** flips the graph about one of the axes.

When reflecting, the graph is flipped about one of the axes. It is important to apply this transformation *before* any translation.

Reflection in the x-axis

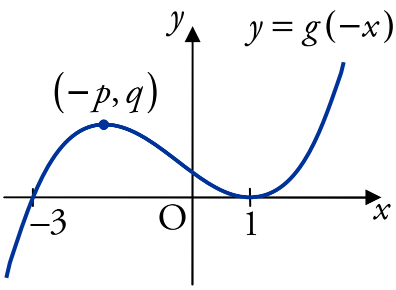
$-f(x)$ reflects the graph of $f(x)$ in the x-axis.

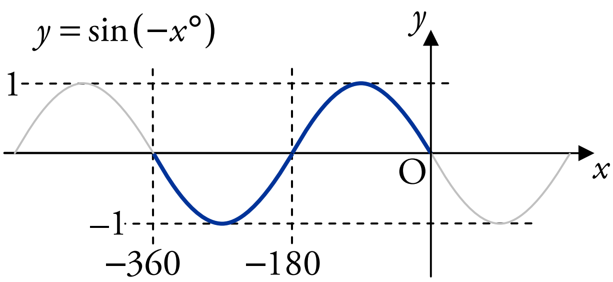




Reflection in the y-axis

$f(-x)$ reflects the graph of $f(x)$ in the y-axis.





From the graphs, $\sin(-x)° = -\sin x°$

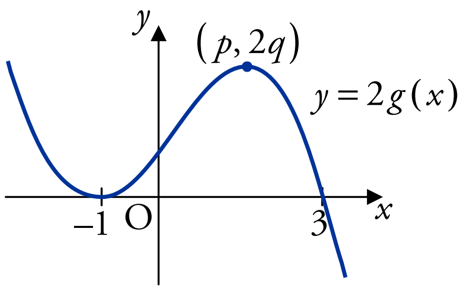
Scaling

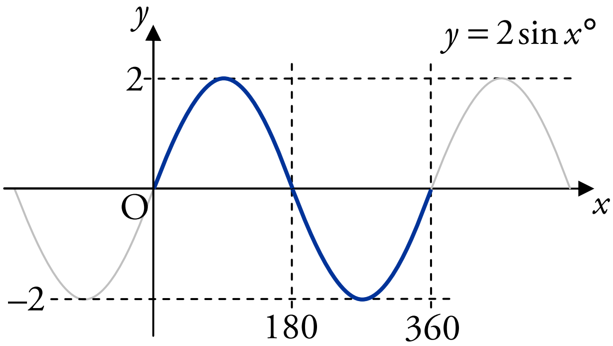
A **scaling** stretches or compresses the graph along one of the axes.

Scaling vertically

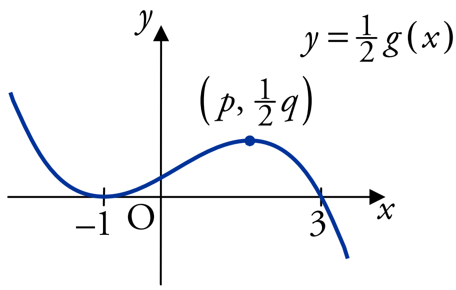
$kf(x)$ scales the graph of $f(x)$ in the vertical direction. The y-coordinate of each point on the graph is multiplied by k, roots are unaffected. These examples consider positive k.

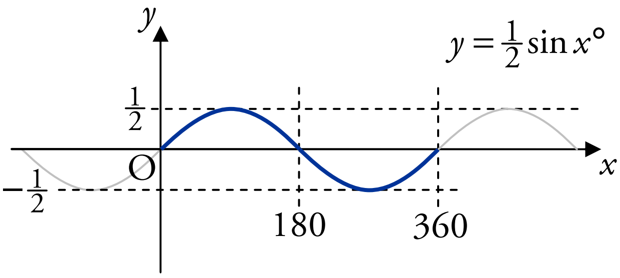
$k > 1$ stretches



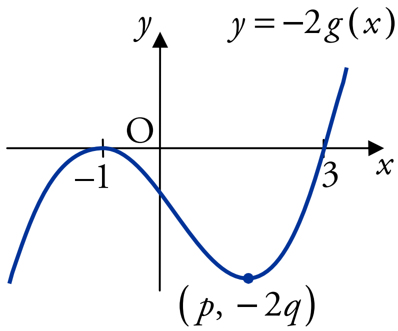


$0 < k < 1$ compresses





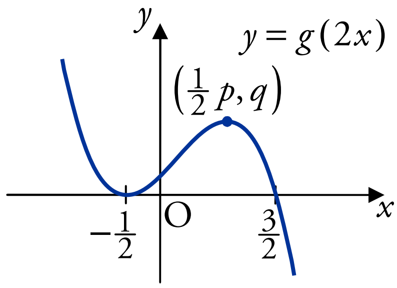
Negative k causes the same scaling, but the graph must then be reflected in the x-axis:



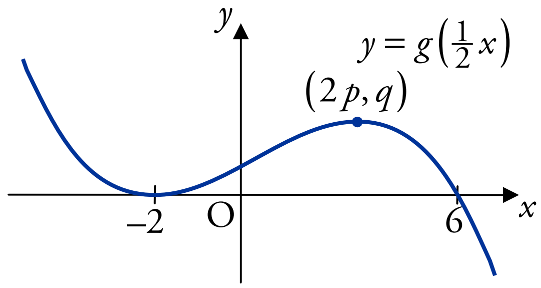
Scaling horizontally

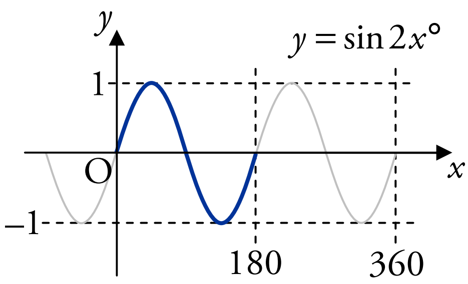
$f(kx)$ scales the graph of $f(x)$ in the horizontal direction. The coordinates of the y-axis intercept stay the same. The examples below consider positive k.

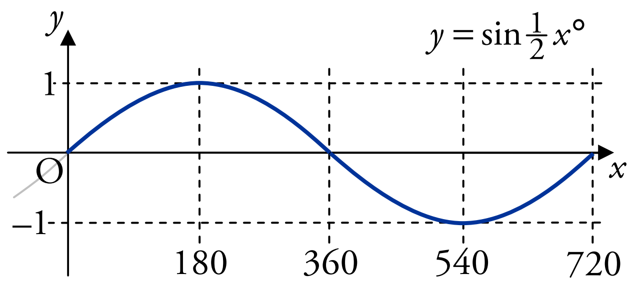
$k > 1$



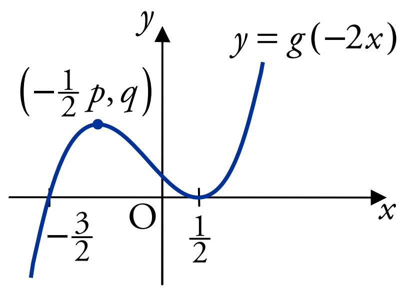
$0 < k < 1$





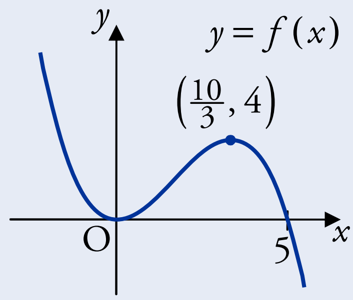


Negative k causes the same scaling, but the graph must then be reflected in the y-axis:



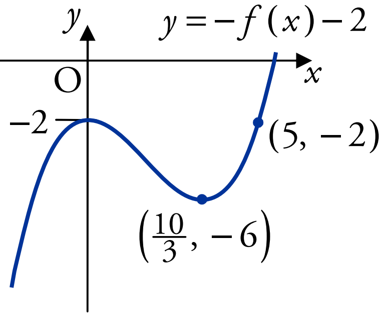
EXAMPLES

1. The graph of $y = f(x)$ is shown below.

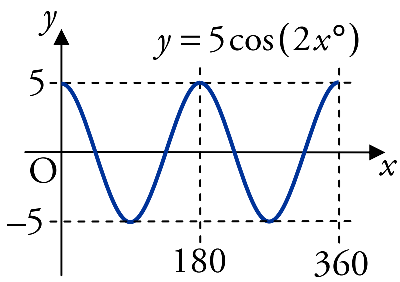


Sketch the graph of $y = -f(x) - 2$.

Reflect in the x-axis, then shift down by 2:

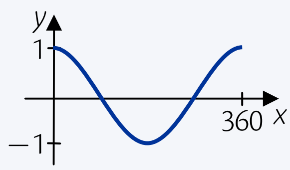


2. Sketch the graph of $y = 5\cos(2x°)$ where $0 \leq x \leq 360$.



Remember

The graph of $y = \cos x$:



OUTCOME 3

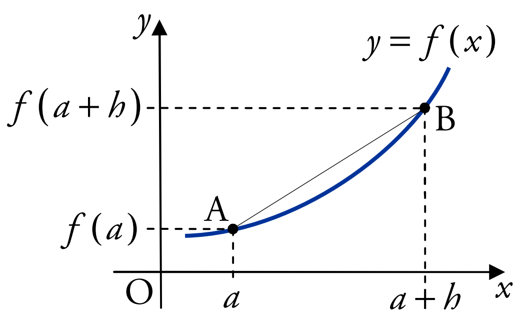
Differentiation

1 Introduction to Differentiation

From our work on Straight Lines, we saw that the gradient (or "steepness") of a line is constant. However, the "steepness" of other curves may not be the same at all points.

In order to measure the "steepness" of other curves, we can use lines which give an increasingly good approximation to the curve at a particular point.

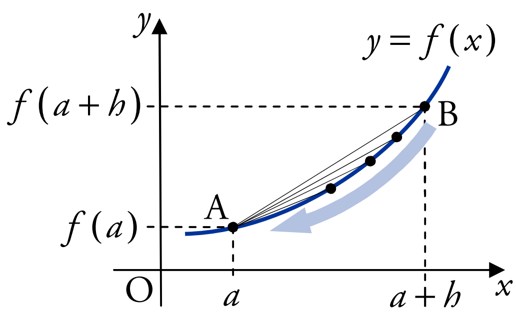
On the curve with equation $y = f(x)$, suppose point A has coordinates $(a, f(a))$.



At the point B where $x = a + h$, we have $y = f(a+h)$.

Thus the chord AB has gradient

$$m_{AB} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}.$$



If we let h get smaller and smaller, i.e. $h \to 0$, then B moves closer to A. This means that m_{AB} gives a better estimate of the "steepness" of the curve at the point A.

We use the notation $f'(a)$ for the "steepness" of the curve when $x = a$. So

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Given a curve with equation $y = f(x)$, an expression for $f'(x)$ is called the **derivative** and the process of finding this is called **differentiation**.

It is possible to use this definition directly to find derivates, but you will not be expected to do this. Instead, we will learn rules which allow us to quickly find derivatives for certain curves.

2 Finding the Derivative

The basic rule for differentiating $f(x)=x^n$, $n\in\mathbb{R}$, with respect to x is:

If $f(x)=x^n$ then $f'(x)=nx^{n-1}$.

Stated simply: the power (n) multiplies to the front of the x term, and the power lowers by one (giving $n-1$).

EXAMPLES

1. Given $f(x)=x^4$, find $f'(x)$.

$$f'(x)=4x^3.$$

2. Differentiate $f(x)=x^{-3}$, $x\neq 0$, with respect to x.

$$f'(x)=-3x^{-4}.$$

For an expression of the form $y=\dots$, we denote the derivative with respect to x by $\frac{dy}{dx}$.

EXAMPLE

3. Differentiate $y=x^{-\frac{1}{3}}$, $x\neq 0$, with respect to x.

$$\frac{dy}{dx}=-\tfrac{1}{3}x^{-\frac{4}{3}}.$$

When finding the derivative of an expression with respect to x, we use the notation $\frac{d}{dx}$.

EXAMPLE

4. Find the derivative of $x^{\frac{3}{2}}$, $x\geq 0$, with respect to x.

$$\frac{d}{dx}\left(x^{\frac{3}{2}}\right)=\tfrac{3}{2}x^{\frac{1}{2}}.$$

Preparing to differentiate

It is important that before you differentiate, all brackets are multiplied out and there are no fractions with an x term in the denominator (bottom line).

For example:

$$\frac{1}{x^3}=x^{-3} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}} \qquad \frac{1}{4x^5}=\tfrac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2}=\tfrac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Differentiate $\sqrt{x}$ with respect to x, where $x > 0$.

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Note
It is good practice to tidy up your answer.

2. Given $y = \frac{1}{x^2}$, where $x \neq 0$, find $\frac{dy}{dx}$.

$$y = x^{-2}$$

$$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}.$$

Terms with a coefficient

For any constant a,

$$\text{if } f(x) = a \times g(x) \text{ then } f'(x) = a \times g'(x).$$

Stated simply: constant coefficients are carried through when differentiating.

So if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$.

EXAMPLES

1. A function f is defined by $f(x) = 2x^3$. Find $f'(x)$.

$$f'(x) = 6x^2.$$

2. Differentiate $y = 4x^{-2}$ with respect to x, where $x \neq 0$.

$$\frac{dy}{dx} = -8x^{-3} = -\frac{8}{x^3}.$$

3. Differentiate $\frac{2}{x^3}$, $x \neq 0$, with respect to x.

$$\frac{d}{dx}\left(2x^{-3}\right) = -6x^{-4} = -\frac{6}{x^4}.$$

4. Given $y=\dfrac{3}{2\sqrt{x}}$, $x>0$, find $\dfrac{dy}{dx}$.

$$\begin{aligned}y&=\tfrac{3}{2}x^{-\frac{1}{2}}\\ \frac{dy}{dx}&=-\tfrac{3}{4}x^{-\frac{3}{2}}\\ &=-\frac{3}{4\sqrt{x}^{3}}.\end{aligned}$$

Differentiating more than one term

The following rule allows us to differentiate expressions with several terms.

If $f(x)=g(x)+h(x)$ then $f'(x)=g'(x)+h'(x)$.

Stated simply: differentiate each term separately.

EXAMPLES

1. A function f is defined for $x\in\mathbb{R}$ by $f(x)=3x^3-2x^2+5x$.
   Find $f'(x)$.

$$f'(x)=9x^2-4x+5.$$

2. Differentiate $y=2x^4-4x^3+3x^2+6x+2$ with respect to x.

$$\frac{dy}{dx}=8x^3-12x^2+6x+6.$$

Note

The derivative of an x term (e.g. $3x$, $\tfrac{1}{2}x$, $-\tfrac{3}{10}x$) is always a constant.

For example:

$$\tfrac{d}{dx}(6x)=6, \qquad \tfrac{d}{dx}\left(-\tfrac{1}{2}x\right)=-\tfrac{1}{2}.$$

The derivative of a constant (e.g. 3, 20, π) is always zero.

For example:

$$\tfrac{d}{dx}(3)=0, \qquad \tfrac{d}{dx}\left(-\tfrac{1}{3}\right)=0.$$

Differentiating more complex expressions

We will now consider more complex examples where we will have to use several of the rules we have met.

EXAMPLES

1. Differentiate $y = \dfrac{1}{3x\sqrt{x}}$, $x > 0$, with respect to x.

$$y = \frac{1}{3x^{\frac{3}{2}}} = \tfrac{1}{3}x^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \tfrac{1}{3} \times -\tfrac{3}{2}x^{-\frac{5}{2}}$$

$$= -\tfrac{1}{2}x^{-\frac{5}{2}}$$

$$= -\frac{1}{2\sqrt{x}^{5}}.$$

Note
You need to be confident working with indices and fractions.

2. Find $\dfrac{dy}{dx}$ when $y = (x-3)(x+2)$.

$$y = (x-3)(x+2)$$

$$= x^2 + 2x - 3x - 6$$

$$= x^2 - x - 6$$

$$\frac{dy}{dx} = 2x - 1.$$

Remember
Before differentiating, the brackets must be multiplied out.

3. A function f is defined for $x \neq 0$ by $f(x) = \dfrac{x}{5} + \dfrac{1}{x^2}$. Find $f'(x)$.

$$f(x) = \tfrac{1}{5}x + x^{-2}$$

$$f'(x) = \tfrac{1}{5} - 2x^{-3}$$

$$= \tfrac{1}{5} - \frac{2}{x^3}.$$

4. Differentiate $\dfrac{x^4 - 3x^2}{5x}$ with respect to x, where $x \neq 0$.

$$\frac{x^4 - 3x^2}{5x} = \frac{x^4}{5x} - \frac{3x^2}{5x}$$

$$= \tfrac{1}{5}x^3 - \tfrac{3}{5}x$$

$$\frac{d}{dx}\left(\tfrac{1}{5}x^3 - \tfrac{3}{5}x\right) = \tfrac{3}{5}x^2 - \tfrac{3}{5}.$$

5. Differentiate $\frac{x^3+3x^2-6x}{\sqrt{x}}$, $x>0$, with respect to x.

$$\begin{aligned}\frac{x^3+3x^2-6x}{\sqrt{x}} &= \frac{x^3}{x^{\frac{1}{2}}}+\frac{3x^2}{x^{\frac{1}{2}}}-\frac{6x}{x^{\frac{1}{2}}}\\ &= x^{3-\frac{1}{2}}+3x^{2-\frac{1}{2}}-6x^{1-\frac{1}{2}}\\ &= x^{\frac{5}{2}}+3x^{\frac{3}{2}}-6x^{\frac{1}{2}}\end{aligned}$$

Remember
$\frac{x^a}{x^b}=x^{a-b}$.

$$\begin{aligned}\frac{d}{dx}\left(x^{\frac{5}{2}}+3x^{\frac{3}{2}}-6x^{\frac{1}{2}}\right) &= \tfrac{5}{2}x^{\frac{3}{2}}-\tfrac{9}{2}x^{\frac{1}{2}}-3x^{-\frac{1}{2}}\\ &= \tfrac{5}{2}\sqrt{x}^3-\tfrac{9}{2}\sqrt{x}-\frac{3}{\sqrt{x}}.\end{aligned}$$

6. Find the derivative of $y=\sqrt{x}\left(x^2+\sqrt[3]{x}\right)$, $x>0$, with respect to x.

$$y=x^{\frac{1}{2}}\left(x^2+x^{\frac{1}{3}}\right)=x^{\frac{5}{2}}+x^{\frac{5}{6}}$$

Remember
$x^a x^b=x^{a+b}$.

$$\begin{aligned}\frac{dy}{dx} &= \tfrac{5}{2}x^{\frac{3}{2}}+\tfrac{5}{6}x^{-\frac{1}{6}}\\ &= \tfrac{5}{2}\sqrt{x}^3+\frac{5}{6\sqrt[6]{x}}.\end{aligned}$$

3 Differentiating with Respect to Other Variables

So far we have differentiated functions and expressions with respect to x. However, the rules we have been using still apply if we differentiate with respect to any other variable. When modelling real-life problems we often use appropriate variable names, such as t for time and V for volume.

EXAMPLES

1. Differentiate $3t^2-2t$ with respect to t.

$$\tfrac{d}{dt}\left(3t^2-2t\right)=6t-2.$$

2. Given $A(r)=\pi r^2$, find $A'(r)$.

$$A(r)=\pi r^2$$
$$A'(r)=2\pi r.$$

Remember
π is just a constant.

When differentiating with respect to a certain variable, all other letters are treated as constants.

EXAMPLE

3. Differentiate px^2 with respect to p.

$$\tfrac{d}{dp}\left(px^2\right)=x^2.$$

Note
Since we are differentiating with respect to p, we treat x^2 as a constant.

4 Rates of Change

The derivative of a function describes its "rate of change". This can be evaluated for specific values by substituting them into the derivative.

EXAMPLES

1. Given $f(x)=2x^5$, find the rate of change of f when $x=3$.

$$f'(x)=10x^4$$
$$f'(3)=10(3)^4=10\times 81=810.$$

2. Given $y=\frac{1}{x^{\frac{2}{3}}}$ for $x\neq 0$, calculate the rate of change of y when $x=8$.

$$y=x^{-\frac{2}{3}}$$
$$\frac{dy}{dx}=-\tfrac{2}{3}x^{-\frac{5}{3}}$$
$$=-\frac{2}{3x^{\frac{5}{3}}}$$
$$=-\frac{2}{3\sqrt[3]{x}^5}.$$

At $x=8$, $\frac{dy}{dx}=-\frac{2}{3\sqrt[3]{8}^5}$
$$=-\frac{2}{3\times 2^5}$$
$$=-\tfrac{2}{96}$$
$$=-\tfrac{1}{48}.$$

Displacement, velocity and acceleration

The velocity v of an object is defined as the rate of change of displacement s with respect to time t. That is:

$$v=\frac{ds}{dt}.$$

Also, acceleration a is defined as the rate of change of velocity with respect to time:

$$a=\frac{dv}{dt}.$$

EXAMPLE

3. A ball is thrown so that its displacement s after t seconds is given by $s(t)=23t-5t^2$.

Find its velocity after 2 seconds.

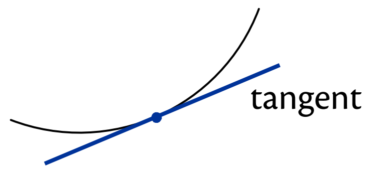
$$v(t)=s'(t)$$
$$=23-10t$$ by differentiating $s(t)=23t-5t^2$ with respect to t.

Substitute $t=2$ into $v(t)$:
$$v(2)=23-10(2)=3.$$

After 2 seconds, the ball has velocity 3 metres per second.

5 Equations of Tangents

As we already know, the gradient of a straight line is constant. We can determine the gradient of a curve, at a particular point, by considering a straight line which touches the curve at the point. This line is called a **tangent**.



The gradient of the tangent to a curve $y = f(x)$ at $x = a$ is given by $f'(a)$.

This is the same as finding the rate of change of f at a.

To work out the equation of a tangent we use $y - b = m(x - a)$. Therefore we need to know two things about the tangent:

- a point, of which at least one coordinate will be given;
- the gradient, which is calculated by differentiating and substituting in the value of x at the required point.

EXAMPLES

1. Find the equation of the tangent to the curve with equation $y = x^2 - 3$ at the point $(2, 1)$.

We know the tangent passes through $(2, 1)$.

To find its equation, we need the gradient at the point where $x = 2$:

$$y = x^2 - 3$$

$$\frac{dy}{dx} = 2x$$

At $x = 2$, $m = 2 \times 2 = 4$.

Now we have the point $(2, 1)$ and the gradient $m = 4$, so we can find the equation of the tangent:

$$y - b = m(x - a)$$
$$y - 1 = 4(x - 2)$$
$$y - 1 = 4x - 8$$
$$4x - y - 7 = 0.$$

2. Find the equation of the tangent to the curve with equation $y = x^3 - 2x$ at the point where $x = -1$.

We need a point on the tangent. Using the given x-coordinate, we can find the y-coordinate of the point on the curve:

$$\begin{aligned} y &= x^3 - 2x \\ &= (-1)^3 - 2(-1) \\ &= -1 + 2 \\ &= 1 \qquad \text{So the point is } (-1, -1). \end{aligned}$$

We also need the gradient at the point where $x = -1$:

$$y = x^3 - 2x$$

$$\frac{dy}{dx} = 3x^2 - 2$$

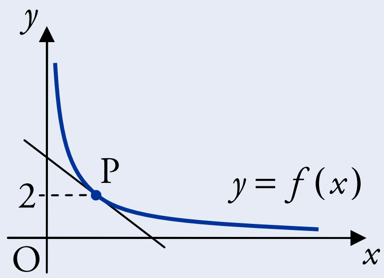
At $x = -1$, $m = 3(-1)^2 - 2 = 1$.

Now we have the point $(-1, 1)$ and the gradient $m = 1$, so the equation of the tangent is:

$$\begin{aligned} y - b &= m(x - a) \\ y - 1 &= 1(x + 1) \\ x - y + 2 &= 0. \end{aligned}$$

3. A function f is defined for $x > 0$ by $f(x) = \frac{1}{x}$.

Find the equation of the tangent to the curve $y = f(x)$ at P.



We need a point on the tangent. Using the given y-coordinate, we can find the x-coordinate of the point P:

$$\begin{aligned} f(x) &= 2 \\ \frac{1}{x} &= 2 \\ x &= \tfrac{1}{2} \qquad \text{So the point is } \left(\tfrac{1}{2}, 2\right). \end{aligned}$$

We also need the gradient at the point where $x = \frac{1}{2}$:

$$\begin{aligned} f(x) &= x^{-1} \\ f'(x) &= -x^{-2} \\ &= -\frac{1}{x^2}. \end{aligned}$$

At $x = \frac{1}{2}$, $m = -\dfrac{1}{\frac{1}{4}} = -4$.

Now we have the point $\left(\frac{1}{2}, 2\right)$ and the gradient $m = -4$, so the equation of the tangent is:

$$\begin{aligned} y - b &= m(x-a) \\ y - 2 &= -4\left(x - \tfrac{1}{2}\right) \\ y - 2 &= -4x + 2 \\ 4x + y - 4 &= 0. \end{aligned}$$

4. Find the equation of the tangent to the curve $y = \sqrt[3]{x}^2$ at the point where $x = -8$.

We need a point on the tangent. Using the given x-coordinate, we can work out the y-coordinate:

$$\begin{aligned} y &= \sqrt[3]{-8}^2 \\ &= (-2)^2 \\ &= 4 \end{aligned}$$

So the point is $(-8, 4)$.

We also need the gradient at the point where $x = -8$:

$$\begin{aligned} y &= \sqrt[3]{x}^2 = x^{\frac{2}{3}} \\ \frac{dy}{dx} &= \tfrac{2}{3}x^{-\frac{1}{3}} \\ &= \frac{2}{3\sqrt[3]{x}} \end{aligned}$$

At $x = -8$,

$$\begin{aligned} m &= \frac{2}{3\sqrt[3]{8}} \\ &= \frac{2}{3 \times 2} \\ &= \tfrac{1}{3}. \end{aligned}$$

Now we have the point $(-8, 4)$ and the gradient $m = \frac{1}{3}$, so the equation of the tangent is:

$$\begin{aligned} y - b &= m(x-a) \\ y - 4 &= \tfrac{1}{3}(x+8) \\ 3y - 12 &= x + 8 \\ x - 3y + 20 &= 0. \end{aligned}$$

5. A curve has equation $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + 5$.
   Find the coordinates of the points on the curve where the tangent has gradient 4.

The derivative gives the gradient of the tangent:

$$\frac{dy}{dx} = x^2 - x + 2.$$

We want to find where this is equal to 4:

$$\begin{aligned} x^2 - x + 2 &= 4 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x = -1 \quad &\text{or} \quad x = 2. \end{aligned}$$

Remember
Before solving a quadratic equation you need to rearrange to get "quadratic $= 0$".

Now we can find the y-coordinates by using the equation of the curve:

$$\begin{aligned} y &= \tfrac{1}{3}(-1)^3 - \tfrac{1}{2}(-1)^2 + 2(-1) + 5 \\ &= -\tfrac{1}{3} - \tfrac{1}{2} - 2 + 5 \\ &= 3 - \tfrac{5}{6} \\ &= \tfrac{13}{6} \end{aligned}$$

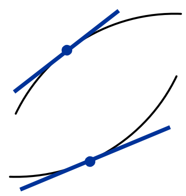
$$\begin{aligned} y &= \tfrac{1}{3}(2)^3 - \tfrac{1}{2}(2)^2 + 2(2) + 5 \\ &= \tfrac{8}{3} - \tfrac{4}{2} + 4 + 5 \\ &= 7 + \tfrac{8}{3} \\ &= \tfrac{29}{3}. \end{aligned}$$

So the points are $\left(-1, \frac{13}{6}\right)$ and $\left(2, \frac{29}{3}\right)$.

6 Increasing and Decreasing Curves

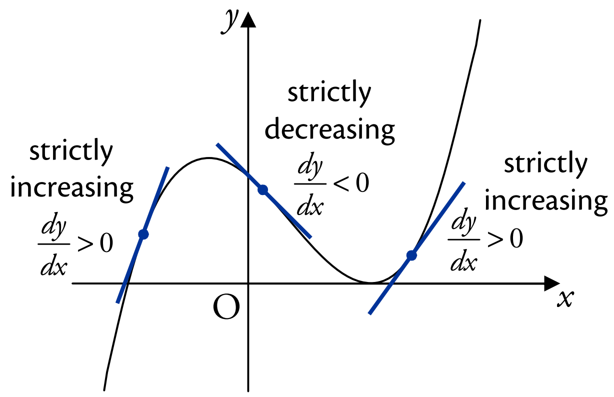
A curve is said to be **strictly increasing** when $\frac{dy}{dx} > 0$.

This is because when $\frac{dy}{dx} > 0$, tangents will slope upwards from left to right since their gradients are positive. This means the curve is also "moving upwards", i.e. strictly increasing.



Similarly:

A curve is said to be **strictly decreasing** when $\frac{dy}{dx} < 0$.



EXAMPLES

1. A curve has equation $y = 4x^2 + \frac{2}{\sqrt{x}}$.

   Determine whether the curve is increasing or decreasing at $x = 10$.

$$y = 4x^2 + 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 8x - x^{-\frac{3}{2}}$$

$$= 8x - \frac{1}{\sqrt{x}^3}.$$

When $x = 10$, $\frac{dy}{dx} = 8 \times 10 - \frac{1}{\sqrt{10}^3}$

$$= 80 - \frac{1}{10\sqrt{10}}$$

$$> 0.$$

Note

$\frac{1}{10\sqrt{10}} < 1.$

Since $\frac{dy}{dx} > 0$, the curve is increasing when $x = 10$.

2. Show that the curve $y = \frac{1}{3}x^3 + x^2 + x - 4$ is never decreasing.

$$\begin{aligned}\frac{dy}{dx} &= x^2 + 2x + 1 \\ &= (x+1)^2 \\ &\geq 0.\end{aligned}$$

Remember
The result of squaring any number is always greater than, or equal to, zero.

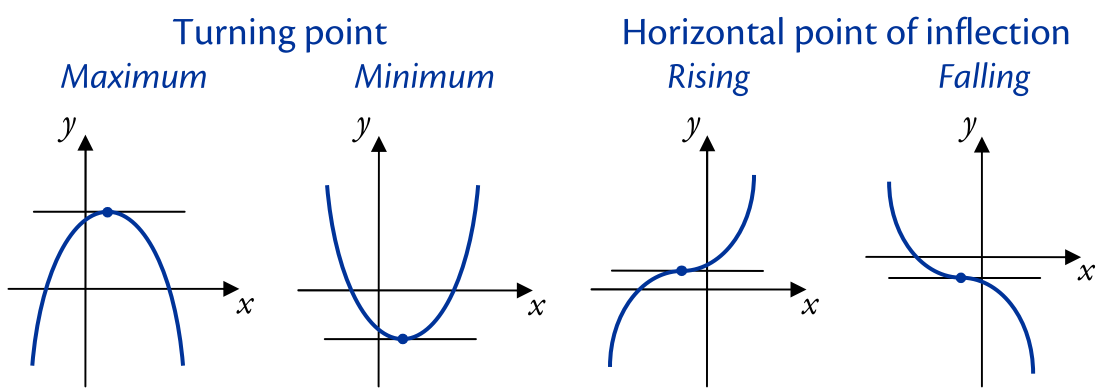
Since $\frac{dy}{dx}$ is never less than zero, the curve is never decreasing.

7 Stationary Points

At some points, a curve may be neither increasing nor decreasing – we say that the curve is **stationary** at these points.

This means that the gradient of the tangent to the curve is zero at stationary points, so we can find them by solving $f'(x) = 0$ or $\frac{dy}{dx} = 0$.

The four possible stationary points are:



A stationary point's nature (type) is determined by the behaviour of the graph to its left and right. This is often done using a "nature table".

8 Determining the Nature of Stationary Points

To illustrate the method used to find stationary points and determine their nature, we will do this for the graph of $f(x) = 2x^3 - 9x^2 + 12x + 4$.

Step 1

Differentiate the function.

$$f'(x) = 6x^2 - 18x + 12$$

Step 2

Find the stationary values by solving $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 6x^2 - 18x + 12 &= 0 \\ 6\left(x^2 - 3x + 2\right) &= 0 \quad (\div 6) \\ (x-1)(x-2) &= 0 \\ x = 1 \quad &\text{or} \quad x = 2 \end{aligned}$$

Step 3

Find the y-coordinates of the stationary points.

$f(1) = 9$ so $(1, 9)$ is a stat. pt.

$f(2) = 8$ so $(2, 8)$ is a stat. pt.

Step 4

Write the stationary values in the top row of the nature table, with arrows leading in and out of them.

x	$\rightarrow$	1	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$						
Graph						

Step 5

Calculate $f'(x)$ for the values in the table, and record the results. This gives the gradient at these x values, so zeros confirm that stationary points exist here.

x	$\rightarrow$	1	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$		0			0	
Graph						

Step 6

Calculate $f'(x)$ for values slightly lower and higher than the stationary values and record the sign in the second row, e.g.: $f'(0.8) > 0$ so enter $+$ in the first cell.

x	$\rightarrow$	1	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$	$+$	0	$-$	$-$	0	$+$
Graph						

Step 7

We can now sketch the graph near the stationary points:

$+$ means the graph is increasing and

$-$ means the graph is decreasing.

x	$\rightarrow$	1	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$	$+$	0	$-$	$-$	0	$+$
Graph	/	‾	\	\	_	/

Step 8

The nature of the stationary points can then be concluded from the sketch.

$(1, 9)$ is a max. turning point.

$(2, 8)$ is a min. turning point.

EXAMPLES

1. A curve has equation $y = x^3 - 6x^2 + 9x - 4$.
   Find the stationary points on the curve and determine their nature.

Given $y = x^3 - 6x^2 + 9x - 4$,

$$\frac{dy}{dx} = 3x^2 - 12x + 9.$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$3x^2 - 12x + 9 = 0$$
$$3(x^2 - 4x + 3) = 0 \quad (\div 3)$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 1 \qquad\qquad x = 3.$$

When $x = 1$,

$$y = (1)^3 - 6(1)^2 + 9(1) - 4$$
$$= 1 - 6 + 9 - 4$$
$$= 0.$$

Therefore the point is $(1,0)$.

When $x = 3$,

$$y = (3)^3 - 6(3)^2 + 9(3) - 4$$
$$= 27 - 54 + 27 - 4$$
$$= -4.$$

Therefore the point is $(3,-4)$.

Nature:

x	$\rightarrow$	1	$\rightarrow$	$\rightarrow$	3	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$-$	$-$	0	$+$
Graph	/	—	\	\	_	/

So $(1, 0)$ is a maximum turning point,
$(3, -4)$ is a minimum turning point.

2. Find the stationary points of $y = 4x^3 - 2x^4$ and determine their nature.

Given $y = 4x^3 - 2x^4$,

$$\frac{dy}{dx} = 12x^2 - 8x^3.$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$12x^2 - 8x^3 = 0$$

$$4x^2(3 - 2x) = 0$$

$4x^2 = 0$ or $3 - 2x = 0$

$x = 0$ $\quad x = \frac{3}{2}$.

When $x = 0$,

$$y = 4(0)^3 - 2(0)^4 = 0.$$

Therefore the point is $(0, 0)$.

When $x = \frac{3}{2}$,

$$\begin{aligned} y &= 4\left(\tfrac{3}{2}\right)^3 - 2\left(\tfrac{3}{2}\right)^4 \\ &= \tfrac{27}{2} - \tfrac{81}{8} \\ &= \tfrac{27}{8}. \end{aligned}$$

Therefore the point is $\left(\frac{3}{2}, \frac{27}{8}\right)$.

Nature:

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	$\frac{3}{2}$	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$+$	$+$	0	$-$
Graph	/	—	/	/	—	\

So $(0, 0)$ is a rising point of inflection,

$\left(\frac{3}{2}, \frac{27}{8}\right)$ is a maximum turning point.

3. A curve has equation $y = 2x + \frac{1}{x}$ for $x \neq 0$. Find the x-coordinates of the stationary points on the curve and determine their nature.

Given $y = 2x + x^{-1}$,

$$\begin{aligned}\frac{dy}{dx} &= 2 - x^{-2} \\ &= 2 - \frac{1}{x^2}.\end{aligned}$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$\begin{aligned}2 - \frac{1}{x^2} &= 0 \\ 2x^2 &= 1 \\ x^2 &= \tfrac{1}{2} \\ x &= \pm\tfrac{1}{\sqrt{2}}.\end{aligned}$$

Nature:

x	$\rightarrow$	$-\frac{1}{\sqrt{2}}$	$\rightarrow$	$\rightarrow$	$\frac{1}{\sqrt{2}}$	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$-$	$-$	0	$+$
Graph	/	—	\	\	—	/

So the point where $x = -\frac{1}{\sqrt{2}}$ is a maximum turning point and the point where $x = \frac{1}{\sqrt{2}}$ is a minimum turning point.

9 Curve Sketching

In order to sketch a curve, we first need to find the following:

- x-axis intercepts (roots) – solve $y = 0$;
- y-axis intercept – find y for $x = 0$;
- stationary points and their nature.

EXAMPLE

Sketch the curve with equation $y = 2x^3 - 3x^2$.

y-axis intercept, i.e. $x = 0$:

$$y = 2(0)^3 - 3(0)^2 = 0.$$

Therefore the point is $(0, 0)$.

x-axis intercepts i.e. $y = 0$:

$$2x^3 - 3x^2 = 0$$
$$x^2(2x - 3) = 0$$

$x^2 = 0$ or $2x - 3 = 0$

$x = 0$ $\qquad$ $x = \frac{3}{2}$

$(0, 0)$ $\qquad$ $\left(\frac{3}{2}, 0\right)$.

Given $y = 2x^3 - 3x^2$,

$$\frac{dy}{dx} = 6x^2 - 6x.$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$6x^2 - 6x = 0$$
$$6x(x - 1) = 0$$

$6x = 0$ or $x - 1 = 0$

$x = 0$ $\qquad$ $x = 1$.

When $x = 0$,

$$y = 2(0)^3 - 3(0)^2 = 0.$$

Therefore the point is $(0, 0)$.

When $x = 1$,

$$y = 2(1)^3 - 3(1)^2 = 2 - 3 = -1.$$

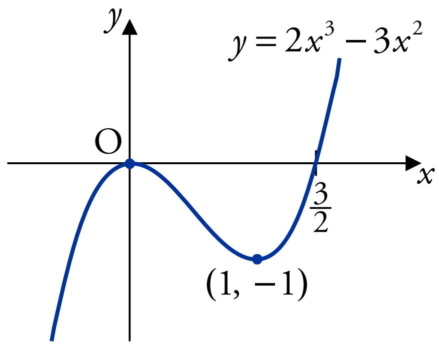
Therefore the point is $(1, -1)$.

Nature:

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	1	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$-$	$-$	0	$+$
Graph	/	—	\	\	_	/

$(0, 0)$ is a maximum turning point.

$(1, -1)$ is a minimum turning point.

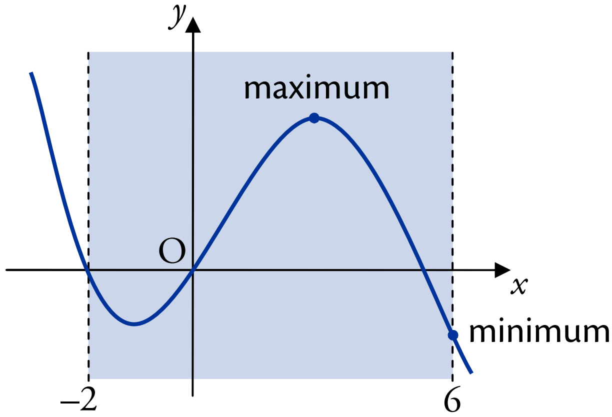


10 Closed Intervals

Sometimes it is necessary to restrict the part of the graph we are looking at using a **closed interval** (also called a restricted domain).

The maximum and minimum values of a function can either be at its stationary points *or* at the end points of a closed interval.

Below is a sketch of a curve with the closed interval $-2 \leq x \leq 6$ shaded.



Notice that the minimum value occurs at one of the end points in this example. It is important to check for this whenever we are dealing with a closed interval.

EXAMPLE

A function f is defined for $-1 \leq x \leq 4$ by $f(x) = 2x^3 - 5x^2 - 4x + 1$.

Find the maximum and minimum value of $f(x)$.

Given $f(x) = 2x^3 - 5x^2 - 4x + 1$,

$$f'(x) = 6x^2 - 10x - 4.$$

Stationary points exist where $f'(x) = 0$:

$$6x^2 - 10x - 4 = 0$$

$$2(3x^2 - 5x - 2) = 0$$

$$(x-2)(3x+1) = 0$$

$$x - 2 = 0 \quad \text{or} \quad 3x + 1 = 0$$

$$x = 2 \qquad\qquad x = -\tfrac{1}{3}.$$

To find coordinates of stationary points:

$$\begin{aligned} f(2) &= 2(2)^3 - 5(2)^2 - 4(2) + 1 \\ &= 16 - 20 - 8 + 1 \\ &= -11. \end{aligned}$$

Therefore the point is $(2, -11)$.

$$\begin{aligned} f\left(-\tfrac{1}{3}\right) &= 2\left(-\tfrac{1}{3}\right)^3 - 5\left(-\tfrac{1}{3}\right)^2 - 4\left(-\tfrac{1}{3}\right) + 1 \\ &= 2\left(-\tfrac{1}{27}\right) - 5\left(\tfrac{1}{9}\right) - 4\left(\tfrac{1}{3}\right) + 1 \\ &= -\tfrac{2}{27} - \tfrac{5}{9} + \tfrac{4}{3} + 1 \\ &= \tfrac{46}{27}. \end{aligned}$$

Therefore the point is $\left(-\tfrac{1}{3}, \tfrac{46}{27}\right)$.

Nature:

x	$\rightarrow$	$-\frac{1}{3}$	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$	$+$	0	$-$	$-$	0	$+$
Graph	/	‾	\	\	_	/

$\left(-\tfrac{1}{3}, \tfrac{46}{27}\right)$ is a max. turning point.

$(2, -11)$ is a min. turning point.

Points at extremities of closed interval:

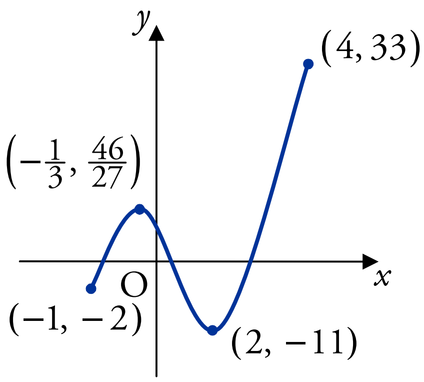
$$\begin{aligned} f(-1) &= 2(-1)^3 - 5(-1)^2 - 4(-1) + 1 \\ &= -2 - 5 + 4 + 1 \\ &= -2. \end{aligned}$$

Therefore the point is $(-1, -2)$.

$$\begin{aligned} f(4) &= 2(4)^3 - 5(4)^2 - 4(4) + 1 \\ &= 128 - 80 - 16 + 1 \\ &= 33. \end{aligned}$$

Therefore the point is $(4, 33)$.

Now we can make a sketch:



Note
A sketch may help you to decide on the correct answer, but it is not required in the exam.

The maximum value is 33 which occurs when $x = 4$.

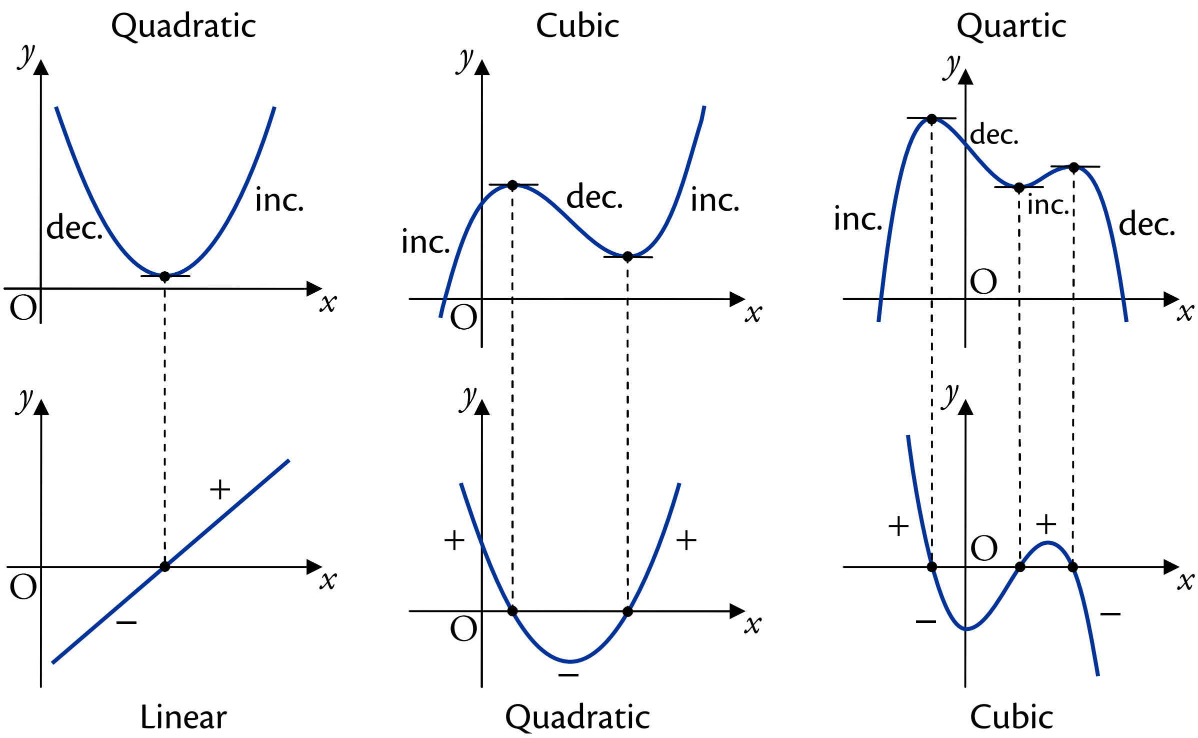
The minimum value is -11 which occurs when $x = 2$.

11 Graphs of Derivatives

The derivative of an x^n term is an x^{n-1} term – the power lowers by one. For example, the derivative of a cubic (where x^3 is the highest power of x) is a quadratic (where x^2 is the highest power of x).

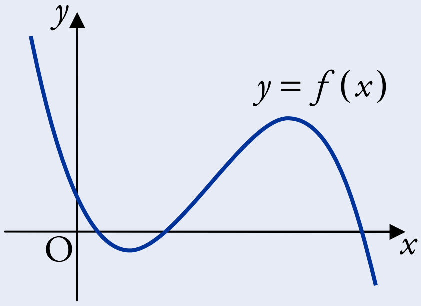
When drawing a derived graph:

- All stationary points of the original curve become roots (i.e. lie on the x-axis) on the graph of the derivative.
- Wherever the curve is strictly decreasing, the derivative is negative. So the graph of the derivative will lie below the x-axis – it will take negative values.
- Wherever the curve is strictly increasing, the derivative is positive. So the graph of the derivative will lie above the x-axis – it will take positive values.



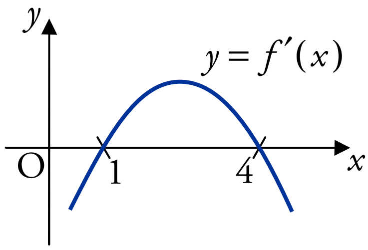
EXAMPLE

The curve $y = f(x)$ shown below is a cubic. It has stationary points where $x = 1$ and $x = 4$.



Sketch the graph of $y = f'(x)$.

Since $y = f(x)$ has stationary points at $x = 1$ and $x = 4$, the graph of $y = f'(x)$ crosses the x-axis at $x = 1$ and $x = 4$.



Note
The curve is increasing between the stationary points so the derivative is positive there.

12 Optimisation

In the section on closed intervals, we saw that it is possible to find maximum and minimum values of a function.

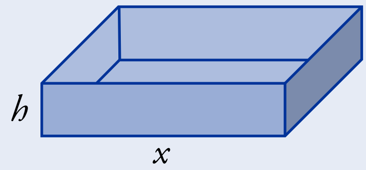
This is often useful in applications; for example a company may have a function $P(x)$ which predicts the profit if £x is spent on raw materials – the management would be very interested in finding the value of x which gave the maximum value of $P(x)$.

The process of finding these optimal values is called **optimisation**.

Sometimes you will have to find the appropriate function before you can start optimisation.

EXAMPLE

1. Small plastic trays, with open tops and square bases, are being designed. They must have a volume of 108 cubic centimetres.



The internal length of one side of the base is x centimetres, and the internal height of the tray is h centimetres.

(a) Show that the total internal surface area A of one tray is given by

$$A = x^2 + \frac{432}{x}.$$

(b) Find the dimensions of the tray using the least amount of plastic.

(a)
$$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height}\\ &= x^2 h.\end{aligned}$$

We are told that the volume is 108 cm^3, so:

$$\begin{aligned}\text{Volume} &= 108\\ x^2 h &= 108\\ h &= \frac{108}{x^2}.\end{aligned}$$

Let A be the surface area for a particular value of x:

$$A = x^2 + 4xh.$$

We have $h = \frac{108}{x^2}$, so:

$$\begin{aligned}A &= x^2 + 4x\left(\frac{108}{x^2}\right)\\ &= x^2 + \frac{432}{x}.\end{aligned}$$

(b) The smallest amount of plastic is used when the surface area is minimised.

$$\frac{dA}{dx} = 2x - \frac{432}{x^2}.$$

Stationary points occur when $\frac{dA}{dx} = 0$:

$$\begin{aligned}2x - \frac{432}{x^2} &= 0\\ x^3 &= 216\\ x &= 6.\end{aligned}$$

Nature:

x	$\rightarrow$	6	$\rightarrow$
$\frac{dA}{dx}$	$-$	0	$+$
Graph	$\diagdown$	$\_$	$\diagup$

So the minimum surface area occurs when $x = 6$. For this value of x:

$$h = \frac{108}{6^2} = 3.$$

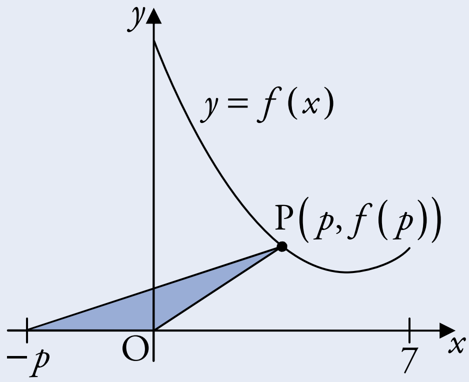
So a length and depth of 6 cm and a height of 3 cm uses the least amount of plastic.

Optimisation with closed intervals

In practical situations, there may be bounds on the values we can use. For example, the company from before might only have £100 000 available to spend on raw materials. We would need to take this into account when optimising.

Recall from the section on Closed Intervals that the maximum and minimum values of a function can occur at turning points *or* the endpoints of a closed interval.

2. The point P lies on the graph of $f(x) = x^2 - 12x + 45$, between $x = 0$ and $x = 7$.



A triangle is formed with vertices at the origin, P and $(-p, 0)$.

(a) Show that the area, A square units, of this triangle is given by

$$A = \tfrac{1}{2}p^3 - 6p^2 + \tfrac{45}{2}p.$$

(b) Find the greatest possible value of A and the corresponding value of p for which it occurs.

(a) The area of the triangle is

$$\begin{aligned} A &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times p \times f(p) \\ &= \tfrac{1}{2}p\left(p^2 - 12p + 45\right) \\ &= \tfrac{1}{2}p^3 - 6p^2 + \tfrac{45}{2}p. \end{aligned}$$

(b) The greatest value occurs at a stationary point or an endpoint.

At stationary points $\frac{dA}{dp} = 0$:

$$\begin{aligned}
\frac{dA}{dp} = \tfrac{3}{2}p^2 - 12p + \tfrac{45}{2} &= 0 \\
3p^2 - 24p + 45 &= 0 \\
p^2 - 8p + 15 &= 0 \\
(p-3)(p-5) &= 0 \\
p = 3 \quad \text{or} \quad p &= 5.
\end{aligned}$$

Now evaluate A at the stationary points and endpoints:

- when $p = 0$, $A = 0$;
- when $p = 3$, $A = \tfrac{1}{2}\times 3^3 - 6\times 3^2 + \tfrac{45}{2}\times 3 = 27$;
- when $p = 5$, $A = \tfrac{1}{2}\times 5^3 - 6\times 5^2 + \tfrac{45}{2}\times 5 = 25$;
- when $p = 7$, $A = \tfrac{1}{2}\times 7^3 - 6\times 7^2 + \tfrac{45}{2}\times 7 = 35$.

So the greatest possible value of A is 35, which occurs when $p = 7$.

OUTCOME 4

Sequences

1 Introduction to Sequences

A **sequence** is an ordered list of objects (usually numbers).

Usually we are interested in sequences which follow a particular pattern. For example, $1, 2, 3, 4, 5, 6, \ldots$ is a sequence of numbers – the "…" just indicates that the list keeps going forever.

Writing a sequence in this way assumes that you can tell what pattern the numbers are following but this is not always clear, e.g.

$$28,\ 22,\ 19,\ 17\tfrac{1}{2},\ \ldots.$$

For this reason, we prefer to have a formula or rule which explicitly defines the terms of the sequence.

It is common to use subscript numbers to label the terms, e.g.

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots$$

so that we can use u_n to represent the nth term.

We can then define sequences with a formula for the nth term. For example:

Formula	List of terms
$u_n = n$	$1,\ 2,\ 3,\ 4,\ \ldots$
$u_n = 2n$	$2,\ 4,\ 6,\ 8,\ \ldots$
$u_n = \frac{1}{2}n(n+1)$	$1,\ 3,\ 6,\ 10,\ \ldots$
$u_n = \cos\left(\frac{n\pi}{2}\right)$	$0,\ -1,\ 0,\ 1,\ \ldots$

Notice that if we have a formula for u_n, it is possible to work out *any* term in the sequence. For example, you could easily find u_{1000} for any of the sequences above without having to list all the previous terms.

Recurrence Relations

Another way to define a sequence is with a **recurrence relation**. This is a rule which defines each term of a sequence using previous terms.

For example:

$$u_{n+1} = u_n + 2, \; u_0 = 4$$

says "the first term (u_0) is 4, and each other term is 2 more than the previous one", giving the sequence $4, 6, 8, 10, 12, 14, \ldots$.

Notice that with a recurrence relation, we need to work out all earlier terms in the sequence before we can find a particular term. It would take a long time to find u_{1000}.

Another example is interest on a bank account. If we deposit £100 and get 4% interest per year, the balance at the end of each year will be 104% of what it was at the start of the year.

$$\begin{aligned} u_0 &= 100 \\ u_1 &= 104\% \text{ of } 100 = 1{\cdot}04 \times 100 = 104 \\ u_2 &= 104\% \text{ of } 104 = 1{\cdot}04 \times 104 = 108{\cdot}16 \\ &\vdots \end{aligned}$$

The complete sequence is given by the recurrence relation

$$u_{n+1} = 1{\cdot}04 u_n \text{ with } u_0 = 100,$$

where u_n is the amount in the bank account after n years.

EXAMPLE

The value of an endowment policy increases at the rate of 5% per annum. The initial value is £7000.

(a) Write down a recurrence relation for the policy's value after n years.

(b) Calculate the value of the policy after 4 years.

(a) Let u_n be the value of the policy after n years.

So $u_{n+1} = 1{\cdot}05 u_n$ with $u_0 = 7000$.

(b)
$$\begin{aligned} u_0 &= 7000 \\ u_1 &= 1{\cdot}05 \times 7000 = 7350 \\ u_2 &= 1{\cdot}05 \times 7350 = 7717{\cdot}5 \\ u_3 &= 1{\cdot}05 \times 7717{\cdot}5 = 8103{\cdot}375 \\ u_4 &= 1{\cdot}05 \times 8103{\cdot}375 = 8508{\cdot}54375 \end{aligned}$$

After 4 years, the policy is worth £8508·54.

2 Linear Recurrence Relations

In Higher, we will deal with recurrence relations of the form

$$u_{n+1} = au_n + b$$

where a and b are any real numbers and u_0 is specified. These are called **linear recurrence relations** of order one.

Note

To properly define a sequence using a recurrence relation, we must specify the initial value u_0.

EXAMPLES

1. A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream. To compensate, a further 25 ml dose is given every 8 hours.

   (a) Find a recurrence relation for the amount of drug in his bloodstream.

   (b) Calculate the amount of drug remaining after 24 hours.

(a) Let u_n be the amount of drug in his bloodstream after $8n$ hours.

$u_{n+1} = 0{\cdot}78u_n + 25$ with $u_0 = 156$

(b) $u_0 = 156$

$u_1 = 0{\cdot}78 \times 156 + 25 = 146{\cdot}68$

$u_2 = 0{\cdot}78 \times 146{\cdot}68 + 25 = 139{\cdot}4104$

$u_3 = 0{\cdot}78 \times 139{\cdot}4104 + 25 = 133{\cdot}7401$

After 24 hours, he will have 133.74 ml of drug in his bloodstream.

2. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}6u_n + 4$ with $u_0 = 7$.

   Calculate the value of u_3 and the smallest value of n for which $u_n > 9{\cdot}7$.

$u_0 = 7$

$u_1 = 0{\cdot}6 \times 7 + 4 = 8{\cdot}2$

$u_2 = 0{\cdot}6 \times 8{\cdot}2 + 4 = 8{\cdot}92$

$u_3 = 0{\cdot}6 \times 8{\cdot}92 + 4 = 9{\cdot}352$

The value of u_3 is $9{\cdot}352$.

$u_4 = 9{\cdot}6112$

$u_5 = 9{\cdot}76672$

The smallest value of n for which $u_n > 9{\cdot}7$ is 5.

Using a Calculator

Using the ANS button on the calculator, we can carry out the above calculation more efficiently.

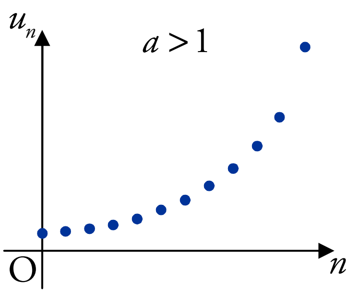
						7	=
0	.	6	×	ANS	+	4	=
							=
							=

3 Divergence and Convergence

If we plot the graphs of some of the sequences that we have been dealing with, then some similarities will occur.

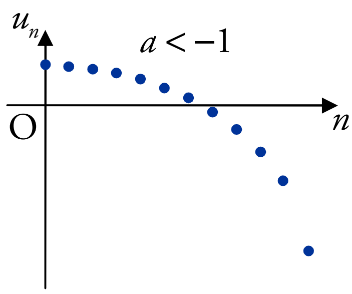
Divergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $a < -1$ or $a > 1$, will have a graph like this:



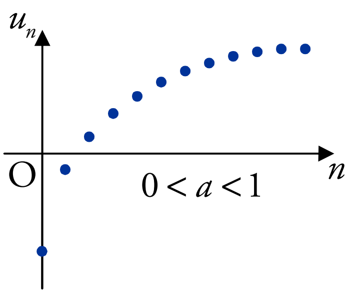
Sequences like this will continue to increase or decrease forever.

They are said to **diverge**.



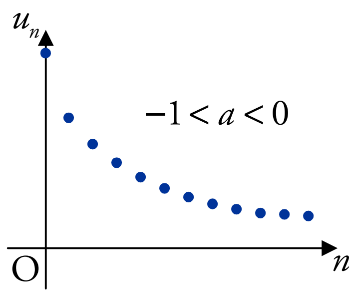
Convergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $-1 < a < 1$, will have a graph like this:



Sequences like this "tend to a limit".

They are said to **converge**.



4 The Limit of a Sequence

We saw that sequences defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ "tend to a limit". In fact, it is possible to work out this limit just from knowing a and b.

The sequence defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ tends to a limit l as $n \to \infty$ (i.e. as n gets larger and larger) given by

$$l = \frac{b}{1-a}.$$

You will need to know this formula, as it is not given in the exam.

EXAMPLES

1. The deer population in a forest is estimated to drop by 7·3% each year. Each year, 20 deer are introduced to the forest. The initial deer population is 200.

   (a) How many deer will there be in the forest after 3 years?

   (b) What is the long term effect on the population?

(a) $u_{n+1} = 0{\cdot}927u_n + 20$

$$u_0 = 200$$
$$u_1 = 0{\cdot}927 \times 200 + 20 = 205{\cdot}4$$
$$u_2 = 0{\cdot}927 \times 205{\cdot}4 + 20 = 210{\cdot}4058$$
$$u_3 = 0{\cdot}927 \times 210{\cdot}4058 + 20 = 215{\cdot}0461$$

Therefore there are 215 deer living in the forest after 3 years.

(b) A limit exists, since $-1 < 0{\cdot}927 < 1$.

$$l = \frac{b}{1-a} \qquad \text{where } a = 0{\cdot}927 \text{ and } b = 20$$
$$= \frac{20}{1-0{\cdot}927}$$
$$= 273{\cdot}97 \text{ (to 2 d.p.)}.$$

Therefore the number of deer in the forest will settle around 273.

Note
Whenever you calculate a limit using this method, you must state that "A limit exists since $-1 < a < 1$".

2. A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 2k$ and the first term is u_0.
   Given that the limit of the sequence is 27, find the value of k.

The limit is given by $\frac{b}{1-a} = \frac{2k}{1-k}$, and so

$$\begin{aligned} \frac{2k}{1-k} &= 27 \\ 27(1-k) &= 2k \\ 29k &= 27 \\ k &= \tfrac{27}{29}. \end{aligned}$$

5 Finding a Recurrence Relation for a Sequence

If we know that a sequence is defined by a linear recurrence relation of the form $u_{n+1} = au_n + b$, and we know three consecutive terms of the sequence, then we can find the values of a and b.

This can be done easily by forming two equations and solving them simultaneously.

EXAMPLE

A sequence is defined by $u_{n+1} = au_n + b$ with $u_1 = 4$, $u_2 = 3{\cdot}6$ and $u_3 = 2{\cdot}04$.

Find the values of a and b.

Form two equations using the given terms of the sequence:

$$u_2 = au_1 + b \qquad \text{and} \qquad u_3 = au_2 + b$$
$$3{\cdot}6 = 4a + b \quad ① \qquad\qquad 2{\cdot}04 = 3{\cdot}6a + b \quad ②.$$

Eliminate b:

$$\begin{aligned} ①-②:\quad 1{\cdot}56 &= 0{\cdot}4a \\ a &= \frac{1{\cdot}56}{0{\cdot}4} \\ &= 3{\cdot}9. \end{aligned}$$

Put $a = 3{\cdot}9$ into ①:

$$\begin{aligned} 4 \times 3{\cdot}9 + b &= 3{\cdot}6 \\ b &= 3{\cdot}6 - 15{\cdot}6 \\ b &= -12. \end{aligned}$$

So $a = 3{\cdot}9$ and $b = -12$.