MATHEMATICS

STUDY MATERIAL

LIMITS AND DERIVATIVES

AIEEE

श्रमेव जयते

THE NARAYANA GROUP

NARAYANA INSTITUTE OF CORRESPONDENCE COURSES

FNS HOUSE, 63 KALU SARAI MARKET

SARVAPRIYA VIHAR, NEW DELHI-110016

PH.: (011) 32001131/32/50 • FAX : (011) 41828320

Website : www.narayanaicc.com

E-mail : info@narayanaicc.com

PREFACE

Dear Student,

Heartiest congratulations on making up your mind and deciding to be an engineer to serve the society.

As you are planning to take various Engineering Entrance Examinations, we are sure that this **STUDY PACKAGE** is going to be of immense help to you.

At NARAYANA we have taken special care to design this package according to the **Latest Pattern of AIEEE,** which will not only help but also guide you to compete for AIEEE & other State Level Engineering Entrance Examinations.

The salient features of this package include :

- Power packed division of units and chapters in a scientific way, with a correlation being there.
- Sufficient number of solved examples in Physics, Chemistry & Mathematics in all the chapters to motivate the students attempt all the questions.
- All the chapters are followed by various types of exercises (Level-I, Level-II, Level-III and Questions asked in AIEEE and other Engineering Exams).

These exercises are followed by answers in the last section of the chapter. *This package will help you to know* what to study, how to study, time management, your weaknesses and improve your performance.

We, at NARAYANA, strongly believe that quality of our package is such that the students who are not fortunate enough to attend to our Regular Classroom Programs, can still get the best of our quality through these packages.

We feel that there is always a scope for improvement. We would welcome your suggestions & feedback.

Wish you success in your future endeavours.

THE NARAYANA TEAM

ACKNOWLEDGEMENT

While preparing the study package, it has become a wonderful feeling for the NARAYANA TEAM to get the wholehearted support of our Staff Members including our Designers. They have made our job really easy through their untiring efforts and constant help at every stage.

We are thankful to all of them.

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CONTENTS

LIMITS AND DERIVATIVES

LIMITS AND DERIVATIVES

AIEEE Syllabus

Limits, Differentiation of the sum, difference, product and quotient of two functions, differentiation of trigonometric, inverse trigonometric, logarithmic, exponential, composite and implicit functions, derivatives of order up to two.

CONTENTS

INTRODUCTION

This chapter is an introduction to calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change. In this chapter we define limit and some algebra of limits. Also we study derivative and algebra of derivatives and derivatives of certain standard functions.

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1. DEFINITION

If y = f(x) is any function which is defined in a neighbourhood of a then for some '$\in$' greater than zero there exists a $\delta > 0$ such that $|f(x) - l| < \in \Rightarrow |x - a| < \delta$ then l is said to be limit of the function when x-approches a. It is symbolically written as $\underset{x\to a}{\text{Lt}}\, f(x) = l$

2. STANDARD FORMULA

$\underset{x\to a}{\text{Lt}} \dfrac{x^n - a^n}{x - a} = na^{n-1}$; $x \neq a$; n is a rational number or integer.

Remark : $\underset{x\to a}{\text{Lt}} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$

3. TRIGONOMETRIC LIMITS

(i) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

(ii) $\lim_{x\to 0} \cos x = 1$

(iii) $\lim_{x\to 0} \dfrac{\tan x}{x} = 1$

(iv) $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1$

(v) $\lim_{x\to 0} \dfrac{\tan^{-1} x}{x} = 1$

(vi) $\lim_{x\to 0} \dfrac{\sin x^{o}}{x} = \dfrac{\pi}{180}$

4. EXPONENTIAL AND LOGARITHMIC LIMITS

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - 1}{x} = 1$

(ii) $\underset{x\to 0}{\text{Lt}} \dfrac{a^x - 1}{x} = \log_e a$ (a > 0)

(iii) $\lim_{x\to 0} \dfrac{a^x - b^x}{x} = \log_e \left(\dfrac{a}{b}\right), a, b > 0$

(iv) $\lim_{x\to 0} \dfrac{(1+x)^n - 1}{x} = n$

(v) $\underset{x\to 0}{\text{Lt}} (1+x)^{1/x} = e = \underset{x\to\infty}{\text{Lt}} \left(1 + \dfrac{1}{x}\right)^x$

(vi) $\lim_{x\to 0} (1+ax)^{1/x} = e^a$

(vii) $\lim_{x\to\infty} \left(1 + \dfrac{a}{x}\right)^x = e^a$

(viii) $\lim_{x\to\infty} \left[1 + \dfrac{1}{f(x)}\right]^{f(x)} = e$, where $f(x) \to \infty$ as $x \to \infty$

(ix) $\lim_{x\to a} (1 + f(x))^{1/f(x)} = e$

(x) $\lim_{x\to\infty} \dfrac{\log x}{x^m} = 0$ (m > 0)

(xi) $\lim_{x\to 0} \dfrac{\log_a (1+x)}{x} = \log_a e \quad (a > 0, a \pm 1)$

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5. APPROXIMATIONS

(i) $\sin ax \simeq ax$

(ii) $\cos ax \simeq 1-\frac{a^2x^2}{2}$

(iii) $\tan ax \simeq ax$

(iv) $e^{ax} \simeq 1+ax$

(v) $e^{-ax} \simeq 1-ax$

(vi) $\log(1 + ax) \simeq ax$

(vii) $a^x \simeq 1 + (\log_e a)x$

(viii) $\sinh ax \simeq ax$

(ix) $\tanh ax \simeq ax$

(x) $\cosh ax \simeq 1+\frac{a^2x^2}{2}$

(xi) $\sqrt[n]{1\pm x} \simeq 1\pm\frac{x}{n}$, $|x| < 1$

6. SOME USEFUL EXPANSIONS

If $x \to 0$ and there is at least one function in the given expansion which can be expanded, then we express numerator and denominator in the ascending powers of x and remove the common factor there, the following expansions of some standard functions should be remembered.

(a) $e^x = 1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+......$

(b) $e^{-x} = 1-\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+.....$

(c) $a^x = 1+\frac{(\log a)x}{\underline{|1}}+\frac{(\log a)^2x^2}{\underline{|2}}+......$

(d) $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-.....$

(e) $\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+.....\right]$

(f) $\cos x = 1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-.......$

(g) $\sin x = x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-.......$

(h) $\sinh x = x+\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}+.......$

(i) $\tan x = x-\frac{x^3}{3}+\frac{2x^5}{15}-.......$

(j) $\tanh x = x+\frac{x^3}{3}+\frac{2x^5}{15}+.......$

(k) $\cosh x = 1+\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}+.......$

(l) $\cos^{-1} x = \frac{\pi}{2}-\left(x+\frac{x^3}{\underline{|3}}+\frac{9x^5}{\underline{|5}}+......\right)$

(m) $\sin^{-1} x = x+\frac{x^3}{\underline{|3}}+\frac{9x^5}{\underline{|5}}+......$

(n) $\tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+......$

(o) $(1+x)^n = 1+nx+\frac{n(n-1)}{\underline{|2}}x^2+.......$ where $n \in z^+$

(p) $\left(1+\frac{1}{x}\right)^x = e\left(1-\frac{x}{2}+\frac{11}{24}x^2+......\right)$

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7. INDETERMINATE FORMS

The forms which cannot be defined exactly are called indeterminate forms, they are

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, ∞^0 and 1^∞

L' HOSPITAL'S RULE

If $\underset{x\to a}{\mathrm{Lt}}\frac{f(x)}{g(x)}$ takes the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then the limit of the function is $\underset{x\to a}{\mathrm{Lt}}\frac{f'(x)}{g'(x)}$, if $\underset{x\to a}{\mathrm{Lt}}\frac{f'(x)}{g'(x)}$ itself takes the form again $\frac{0}{0}$, $\frac{\infty}{\infty}$ then the limit of the function is $\underset{x\to a}{\mathrm{Lt}}\frac{f''(x)}{g''(x)}$ and the process is continued till $\frac{0}{0}$, $\frac{\infty}{\infty}$ is eliminated then limit is obtained.

1. If $0\times\infty$ form is given, convert it in the form of $\frac{0}{0}$, $\frac{\infty}{\infty}$ by taking one term to the denominater then apply L'Hospital's Rule.

2. If $(\infty-\infty)$ form is given, take L.C.M convert it in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, then take the help of L'Hospital's Rule.

3. 0^0 and ∞^0 form is given take the help of logarithms convert the problem again in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then use L'Hospital's Rule.

4. If $\underset{x\to a}{\mathrm{Lt}}\left[f(x)\right]^{g(x)}$ takes the form of 1^∞ then write it as $\underset{x\to a}{\mathrm{Lt}}\left(f(x)\right)^{g(x)} = e^{\underset{x\to a}{\mathrm{Lt}}\, g(x)[f(x)-1]}$

8. LIMIT OF GREATEST INTEGER FUNCTION

Greatest integer function is denoted by [.]

Let $a\in R$ then two cases arise.

Case (1) if $a\in$ integer then we have

1. $\underset{x\to a^+}{\mathrm{Lt}}[x]=a$

2. $\underset{x\to a^-}{\mathrm{Lt}}[x]=a-1$

3. $\underset{x\to a}{\mathrm{Lt}}[x]$ does not exist

Case (2) If $a\notin$ integer then

$\underset{x\to c}{\mathrm{Lt}}[x]=c$

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Example : If $f(x)=\frac{\sin[x]}{[x]}, [x]\neq 0$

$= 0 \qquad [x] = 0$

where [x] denotes the greatest integer less than or equal to x, then find $\underset{x\to 0}{\text{Lt}} f(x)$

Solution : $\underset{h\to 0}{\text{Lt}} f(0-h)=\underset{h\to 0}{\text{Lt}} f(0+h)$

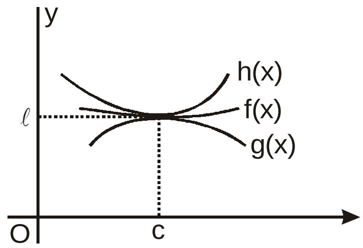
$$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\frac{\sin[-h]}{[-h]}=\underset{h\to 0}{\text{Lt}}\frac{\sin[h]}{[h]} \quad \Rightarrow \quad \sin 1\neq 1$$

9. SANDWICH THEOREM

Suppose that g(x) $\leq$ f(x) $\leq$ h(x) for all x in some open interval containing c, except possibly at 'c' itself.
Suppose also that

$$\underset{x\to c}{\text{Lt}}\, g(x)=\underset{x\to c}{\text{Lt}}\, h(x)=\ell \text{ then } \underset{x\to c}{\text{Lt}}\, f(x)=\ell$$

This is called sandwich theorem.



10. SPECIAL TYPES OF LIMITS

1. **Use of Leibnitz's formula for evaluating the limit**

Consider the integral

$$g(x)=\int_{\phi(x)}^{\psi(x)} f(t)\,dt \text{ then}$$

$$g'(x)=f[\psi(x)]\psi'(x)-f(\phi(x))\phi'(x)$$

2. **Summation of series using definite integral as the limit of a sum.**

It is used in the expression of the form

$$\underset{n\to\infty}{\text{Lt}}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)=\int_a^b f(x)\,dx$$

To Evaluate such limits we note the following

(a) $\sum$ is replaced by sign of integration

(b) $\frac{r}{n}\to x$ (r = x , n = 1)

(c) $\frac{1}{n}\to dx$

(d) Lower limit is always zero.

(e) Upper limit is Coefficient of n in the upper limit of Σ

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DERIVATIVES

11. DERIVATIVE OF A FUNCTION

Let y = f(x) be a function defined on an interval [a, b]. Let for a small increment δx in x, the corresponding increment in the value of y be δy. Then

$y = f(x)$ and $y + \delta y = f(x + \delta x)$

On subtraction, we get

$\delta y = f(x + \delta x) - f(x)$

or $\dfrac{\delta y}{\delta x} = \dfrac{f(x+\delta x) - f(x)}{\delta x}$

Taking limit on both sides when $\delta x \to 0$ we have,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

if this limit exists, is called the ***derivative or differential coefficient*** of y with respect to x and is written as $\dfrac{dy}{dx}$ or $f'(x)$. Thus

$\therefore \quad \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$. This is called Differentiation from first principle.

Derivative at a point:

The value of $f'(x)$ obtained by putting x = a, is called the derivative of f(x) at x = a and it is denoted by $f'(a)$ or $\left\{\dfrac{dy}{dx}\right\}_{x=a}$

Note : $\dfrac{dy}{dx}$ is $\dfrac{d}{dx}(y)$ in which $\dfrac{d}{dx}$ is simply a symbol of operation and not 'd' divided by dx.

12. SOME DIFFERENTIATION FORMULAE

(i) $\dfrac{d}{dx}(\text{constant}) = 0$

(ii) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\dfrac{d}{dx}(e^x) = e^x$

(iv) $\dfrac{d}{dx}(a^x) = a^x \log_e a$

(v) $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

(vi) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log_e a}$

(vii) $\dfrac{d}{dx}(\sin x) = \cos x$

(viii) $\dfrac{d}{dx}(\cos x) = -\sin x$

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(ix) $\frac{d}{dx}(\tan x) = \sec^2 x$

(x) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(xv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xvii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xviii) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

(xix) $\frac{d}{dx}(e^{ax}\sin bx) = e^{ax}(a \sin bx + b \cos bx) = \sqrt{a^2+b^2}\; e^{ax}\sin\left(bx + \tan^{-1}\frac{b}{a}\right)$

(xx) $\frac{d}{dx}(e^{ax}\cos bx) = e^{ax}(a \cos bx - b \sin bx) = \sqrt{a^2+b^2}\; e^{ax}\cos\left(bx + \tan^{-1}\frac{b}{a}\right)$

(xxi) $\frac{d}{dx}|x| = \frac{x}{|x|}$ or $\frac{|x|}{x}$: $x \neq 0$

(xxii) $\frac{d}{dx}\log|x| = \frac{1}{x}$

13. ALGEBRA OF DIFFERENTIATION

(i) Sum and difference rule

$$\frac{d}{dx}[f_1(x) \pm f_2(x)] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)]$$

(ii) Scalar multiple rule

$$\frac{d}{dx}[k\, f(x)] = k\frac{d}{dx}[f(x)]$$, where k is any constant

(iii) Product rule

$$\frac{d}{dx}[f_1(x).f_2(x)] = [f_1(x)]\frac{d}{dx}[f_2(x)] + [f_2(x)]\frac{d}{dx}[f_1(x)]$$

(iv) Quotient rule

$$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{f_2(x)\frac{d}{dx}[f_1(x)] - f_1(x)\frac{d}{dx}[f_2(x)]}{[f_2(x)]^2}$$

(v) Chain rule

if $y = f_1(u)$, $u = f_2(v)$ and $v = f_3(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$

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14. DERIVATIVE OF PARAMETRIC FUNCTIONS

Let x and y are two functions of variable 't' (parameter) such that x = f(t) and y = g(t). then

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{g'(t)}{f'(t)}$$

Example - 1 If x = a(cosθ + θsinθ), y = a(sinθ – θcosθ) then find $\frac{dy}{dx}$.

Solution : $\frac{dx}{d\theta} = a[-\sin\theta + \sin\theta + \theta\cos\theta] = a\theta\cos\theta$; $\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$

$\therefore \frac{dy}{dx} = \tan\theta$

15. DIFFERENTIATION OF IMPLICIT FUNCTIONS

If in an equation, x and y both occur together. i.e. f(x, y) = 0 or f(x, y) = c and this function can not be solved either for 'y' or 'x' then f(x, y) is called the implicit function of x (or y).

If $x^y + y^x = a^b$, then $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-(yx^{y-1} + y^x \log y)}{(x^y \log x + xy^{x-1})}$

15.1. WORKING RULE FOR FINDING THE DERIVATIVE

Method – 1

(i) Differentiate every term of f(x, y) = 0 with respect to 'x'.

(ii) Collect the coefficients of $\frac{dy}{dx}$ and obtain the value of $\frac{dy}{dx}$.

Method – 2

If f(x, y) = constant, then $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$ where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial differential coefficients of f(x, y) with respect to x and y respectively.

Note : *An implicit function can be differentiated either with respect to 'x' or with respect to 'y'*

16. DERIVATIVE OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

Let y = f(x) and z = g(x) be two functions of 'x' then the derivative of f(x) w.r.t g(x) or derivative of y is denoted by $\frac{dy}{dz}$

i.e. $$\frac{dy}{dz} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{f'(x)}{g'(x)}$$

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Example - 2 Find the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution : Let f(x) = $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x \quad \therefore \quad f'(x) = \frac{2}{1+x^2}$

Let $g(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x \quad \therefore \quad g'(x) = \frac{2}{1+x^2}$

Hence the derivative of f(x) with respect to g(x) is

$$\frac{f'(x)}{g'(x)} = \frac{2/1+x^2}{2/1+x^2} = 1$$

17. USE OF LOG IN FINDING DERIVATIVES OF THE FUNCTION OF TYPE $(\mathbf{f(x)})^{g(x)}$

Let $y = [f(x)]^{g(x)}$

Taking log on both sides we get log y = g(x) . log f(x)

Differentiating we get

$$\frac{dy}{dx} = [f(x)]^{g(x)} \cdot \left[g'(x)\log f(x) + g(x)\frac{f'(x)}{f(x)}\right]$$

Example - 3 $\frac{d}{dx}\{x^{\tan x}\} = x^{\tan x}\left[\sec^2 x \log x + \frac{\tan x}{x}\right]$

18. DIFFERENTIATION USING TRIGONOMETRICAL SUBSTITUTIONS

(i) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$

(ii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}[xy \mp \sqrt{(1-x^2)(1-y^2)}]$

(iii) $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\frac{x \pm y}{1 \mp xy}\right]$

(iv) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

(v) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

(vi) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(viii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(ix) $\frac{\pi}{4} - \tan^{-1}x = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

(x) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

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(xi) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(xii) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

(xiii) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(xiv) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(xv) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

19. SUITABLE SUBSTITUTION

(i) If the function involve the term $\sqrt{a^2 - x^2}$, then put $x = a\sin\theta$, or $x = a\cos\theta$

(ii) If the function involve the term $\sqrt{x^2 + a^2}$, then put $x = a\tan\theta$

(iii) If the function involve the term $\sqrt{x^2 - a^2}$, then put $x = a\sec\theta$

(iv) If the function involve the term $\left(\sqrt{\frac{a - x}{a + x}}\right)$, then put $x = a\cos\theta$

20. n^th DIFFERENTIATION OF SUITABLE FUNCTION

(1) $D^n(ax + b)^m = m(m - 1)(m - 2) \dots\dots\dots (m - n + 1)\, a^n(ax + b)^{m-n}$

(2) If $m \in N$ and $m > n$, then

$$D^n(ax + b)^m = \frac{m!}{(m - n)!}\, a^n(ax + b)^{m-n}$$

$$D^n(x^m) = \frac{m!}{(m - n)!}x^{m-n}$$

(3) $D^n(ax + b)^n = n!\, a^n$

$D^n(x^n) = n!$

(4) $D^n\left(\frac{1}{ax + b}\right) = \frac{(-1)^n n!\, a^n}{(ax + b)^{n+1}}$

$D^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$

(5) $D^n\{\log(ax + b)\} = \frac{(-1)^{n-1}(n - 1)!}{(ax + b)^n}a^n$

$D^n(\log x) = \frac{(-1)^{n-1}(n - 1)!}{x^n}$

(6) $D^n(e^{ax}) = a^n e^{ax}$

(7) $D^n(a^{mx}) = (\log a)^n a^{mx} . m^n$

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(8) $D^n\{\sin(ax + b)\} = a^n \sin(ax + b + n\frac{\pi}{2})$

$D^n(\sin x) = \sin(x + n\frac{\pi}{2})$

(9) $D^n\{\cos(ax + b)\} = a^n \cos(ax + b + n\frac{\pi}{2})$

$D^n(\cos x) = \cos(x + n\frac{\pi}{2})$

(10) $D^n\{e^{ax}\sin(bx + c)\} = (a^2 + b^2)^{n/2} e^{ax} \sin(bx + c + n\tan^{-1}\frac{b}{a})$

(11) $D^n\{e^{ax}\cos(bx + c)\} = (a^2 + b^2)^{n/2} e^{ax} \cos(bx + c + n\tan^{-1}\frac{b}{a})$

(12) $D^n(\tan^{-1}\frac{x}{a}) = \frac{(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta}{a^n}$

Where $\theta = \tan^{-1}(\frac{a}{x})$

(13) $D^n(\tan^{-1} x) = (-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$

Where $\theta = \tan^{-1}(\frac{1}{x})$

Example - 4 If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find $\frac{dy}{dx}$.

Solution : Since $x = a\cos^3\theta$

$\therefore \quad \frac{dx}{d\theta} = a\frac{d(\cos^3\theta)}{(\cos\theta)}.\frac{d(\cos\theta)}{d\theta}$ (Using chain rule)

$= 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$

and $y = a\sin^3\theta$

$\therefore \quad \frac{dy}{d\theta} = a\frac{d(\sin^3\theta)}{d(\sin\theta)}.\frac{d(\sin\theta)}{d\theta} = 3a\sin^2\theta.\cos\theta$

Now, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

21. HIGHER ORDER DIFFERENTIATION

If y = f(x) be a differentiable function of x, such that whose second, third............, n^{th} derivatives exist.

The first, second, third ...., n^{th} derivatives of y = f(x) are denoted respectively by $\frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}$

Also denoted by $y', y''........y^{(n)}$

$y_1, y_2, y_3..............,y_n$

$f', f''............f^n$

$dy, d^2y, d^3y,.........d^ny$

Example - 5 If $y = \frac{\ln x}{x}$ then find $\frac{d^2y}{dx^2}$

Solution : We have $y = \frac{\ln x}{x}$

$xy = \ln x$ .....(1)

Differentiating both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y.1 = \frac{1}{x}$$

$\Rightarrow \quad x^2 \frac{dy}{dx} + xy = 1 \quad \Rightarrow \quad x^2 \frac{dy}{dx} + \ln x = 1$ [From (1)] .....(2)

Again differentiating both sides w.r.t. x, we get

$$x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx}.2x + \frac{1}{x} = 0$$

$$\Rightarrow \quad x^3 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \quad x^3 \frac{d^2y}{dx^2} + 2(1 - \ln x) + 1 = 0$$ [from (2)]

Hence $x^3 \frac{d^2y}{dx^2} = (2\ln x - 3)$ or $\frac{d^2y}{dx^2} = \frac{2\ln x - 3}{x^3}$

22. DERIVATIVE OF INFINITE SERIES

If taking out one or more than one terms form an infinite series.

(A) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$ then $y = \sqrt{f(x) + y}$

$\Rightarrow \quad (y^2 - y) = f(x)$

Differentiating both sides w.r.t. x, we get $(2y - 1)\frac{dy}{dx} = f'(x)$

(B) If $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{.......\infty}}}$ then $y = \{f(x)\}^y$

$\Rightarrow \quad y = e^{y\,\ell n\, f(x)}$

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Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{y \ln f(x)} \left\{ y \cdot \frac{1}{f(x)} \cdot f'(x) + \ln f(x) \cdot \frac{dy}{dx} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \{f(x)\}^{y} \left\{ \frac{y}{f(x)} \cdot f'(x) + \ln f(x) \cdot \frac{dy}{dx} \right\}$$

$$\Rightarrow \quad \{1 - \{f(x)\}^{y} \ln f(x)\} \frac{dy}{dx} = y \{f(x)\}^{y-1} . f'(x)$$

(C) $y = f(x)^{f(x)}$ then $\frac{dy}{dx} = f(x)^{f(x)} \cdot f'(x)[1 + \log f(x)]$

$$\frac{d}{dx}(x^x) = x^x(1 + \log x)$$

$$d(\sin x^{\sin x}) = (\sin x)^{\sin x}.\cos x[1 + \log \sin x]$$

(D) $y = f(x)^{g(x)}$ then $\frac{dy}{dx} = f(x)^{g(x)} \left[g'(x)\log f(x) + g(x) \cdot \frac{f'(x)}{f(x)} \right]$

$$\frac{d}{dx}\left\{(\sin x)^{\log x}\right\} = (\sin x)^{\log x}\left[\frac{1}{x}\log(\sin x) + \log x.\cot x\right]$$

(E) $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + ...\infty}}$ then $\frac{dy}{dx} = \frac{y^2 f'(x)}{y^2 + 1}$

(F) $y = \log\left[\frac{1 + f(x)}{1 - f(x)}\right]$ then $\frac{dy}{dx} = \frac{2f'(x)}{1 - f^2(x)}$

$$y = \log\left(\frac{1 + \tan x}{1 - \tan x}\right) \text{ then } \frac{dy}{dx} = \frac{2\sec^2 x}{1 - \tan^2 x}$$

Example - 6 If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + ...\infty}}}}$ then find $\frac{dy}{dx}$

Solution : We have $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + ...\infty}}}}$

$$\Rightarrow \quad y = \sqrt{x + \sqrt{y + y}}$$

$$\Rightarrow \quad y^2 - x = \sqrt{2y}$$

$$\Rightarrow \quad (y^2 - x)^2 = 2y$$

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Differentiating both sides w.r.t. x, we get

$$2\,(y^2 - x)\left(2y\frac{dy}{dx} - 1\right) = 2\,\frac{dy}{dx}$$

$$\Rightarrow \quad (2y\,(y^2 - x) - 1)\,\frac{dy}{dx} = (y^2 - x)$$

Hence $\dfrac{dy}{dx} = \dfrac{(y^2 - x)}{(2y^3 - 2xy - 1)}$

23. DIFFERENTIATION OF A DETERMINANT FUNCTION

If $F(x) = \begin{vmatrix} f & g & h \\ \ell & m & n \\ u & v & w \end{vmatrix}$

Where f, g, h, ℓ, m, n, u, v, w are functions of x and differentiable then

$$F'(x) = \begin{vmatrix} f' & g' & h' \\ \ell & m & n \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ \ell' & m' & n' \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ \ell & m & n \\ u' & v' & w' \end{vmatrix}$$

or $$F'(x) = \begin{vmatrix} f' & g & h \\ \ell' & m & n \\ u' & v & w \end{vmatrix} + \begin{vmatrix} f & g' & h \\ \ell & m' & n \\ u & v' & w \end{vmatrix} + \begin{vmatrix} f & g & h' \\ \ell & m & n' \\ u & v & w' \end{vmatrix}$$

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SOLVED EXAMPLES

Example - 1 $\underset{x\to 0}{\text{Lt}}\ \frac{a^x - b^x}{\tan x} =$

(1) log(a/b) (2) log(b/a)

(3) logab (4) a/b

Solution : **Ans. (1)**

$$\underset{x\to 0}{\text{Lt}}\ \frac{(a^x - 1) - (b^x - 1)}{x}\left[\frac{x}{\tan x}\right]$$

$$\underset{x\to 0}{\text{Lt}}\ \frac{a^x - 1}{x} - \frac{b^x - 1}{x} \qquad (\because\ \underset{x\to 0}{\text{Lt}}\ \frac{\tan x}{x} = 1)$$

= loga – logb = log(a/b)

Example - 2 $\underset{x\to 0}{\text{Lt}}\ (\cos x + a\sin bx)^{1/x} =$

(1) e^{ab} (2) $e^{a/b}$

(3) log(a/b) (4) logab

Solution : **Ans. (1)**

Given limit is in the form 1^{∞}

$$e^{\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}[\cos x + a\sin bx - 1]} \qquad \left(\because\ \cos x \simeq 1 - \frac{x^2}{2}\right)$$

$$= e^{\underset{x\to 0}{\text{Lt}}\ \frac{\left(1 - \frac{x^2}{2} + abx - 1\right)}{x}} = e^{ab} \qquad (\sin bx \simeq bx)$$

Example - 3 $\underset{n\to\infty}{\text{Lt}}\left(\frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n+1)(2n+3)}\right) =$

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$

(3) $\frac{1}{6}$ (4) $\frac{1}{4}$

Solution : **Ans. (3)**

$$\underset{n\to\infty}{\text{Lt}}\ \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + ... + \frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

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Example - 4 $\underset{x\to\infty}{Lt}\dfrac{(x+1)^{10}+(x+4)^{10}+(x+9)^{10}+...+(x+100)^{10}}{x^{10}+100^{10}}=$

(1) 100 (2) 1000

(3) 10 (4) 1

Solution : **Ans. (3)**

$$\underset{x\to\infty}{Lt}\frac{x^{10}\left\{\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{4}{x}\right)^{10}+....+\left(1+\frac{100}{x}\right)^{10}\right\}}{x^{10}\left(1+\frac{100^{10}}{x^{10}}\right)}$$

= 1 + 1 + ... + 10 times

= 10

Note : If the degree of numerater and denomenater are equal, then the ratio of constant terms is the limit when $x\to 0$ and the ratio of coefficients of highest degree terms is the limit when $x\to\infty$.

Example - 5 $\underset{x\to 0}{Lt}\dfrac{\int_0^{x^2}\sin\sqrt{t}\ dt}{x^3}=$

(1) $\frac{3}{2}$ (2) $\frac{1}{3}$

(3) $\frac{1}{2}$ (4) $\frac{2}{3}$

Solution : **Ans. (4)**

$$\underset{x\to 0}{Lt}\frac{\frac{d}{dx}\left\{\int_0^{x^2}\sin\sqrt{t}\,dt\right\}}{\frac{d}{dx}(x^3)}$$

$=\underset{x\to 0}{Lt}\dfrac{\sin x\,(2x)}{3x^2}$ (Use Leibnitz's rule and sinx $\simeq$ x)

$=\underset{x\to 0}{Lt}\dfrac{2x\cdot x}{3x^2}=\dfrac{2}{3}$

Example - 6 $\underset{x\to\infty}{Lt}\dfrac{[x]+[2x]+....+[nx]}{n^2}$ is, where [.] denotes the greatest integer function

(1) $\frac{x}{3}$ (2) $\frac{x}{6}$

(3) does not exist (4) $\frac{x}{2}$

Solution : **Ans. (4)**

Using the fact $nx-1<[nx]\le nx$

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$$\therefore \quad \sum (nx-1) < \sum [nx] \le nx$$

$$\Rightarrow \quad x\frac{n(n+1)}{2} - n < \sum_{r=1}^{n} [rn] \le x \cdot \frac{n(n+1)}{2}$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} \left(\frac{x}{2} \cdot \frac{n(n+1)}{n^2} - \frac{n}{n^2} \right) < \operatorname*{Lt}_{n\to\infty} \sum_{r=1}^{n} [rx] \le \operatorname*{Lt}_{n\to\infty} \frac{x.n(n+1)}{2n^2}$$

$$\Rightarrow \quad \frac{x}{2} \le \operatorname*{Lt}_{n\to\infty} \sum_{r=1}^{n} [rx] \le \frac{x}{2}$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} \sum_{r=1}^{n} [rx] = \frac{x}{2}$$ (Using sandwich theorem)

Example - 7 $\operatorname*{Lt}_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{\sinh^2 x} \right) =$

(1) $\frac{2}{3}$ (2) 0

(3) $\frac{1}{3}$ (4) $-\frac{2}{3}$

Solution : **Ans. (1)**

$$\operatorname*{Lt}_{x\to 0} \frac{\sinh^2 x - \sin^2 x}{\sinh^2 x \sin^2 x} = \operatorname*{Lt}_{x\to 0} \frac{\sinh^2 x - \sin^2 x}{x^4}$$

$$= \operatorname*{Lt}_{x\to 0} \frac{2\sinh x \cosh x - 2\sin x \cos x}{4x^3} \text{ (L.H.R)}$$

$$= \operatorname*{Lt}_{x\to 0} \frac{\sinh 2x - \sin 2x}{4x^3} = \operatorname*{Lt}_{x\to 0} \frac{\frac{e^{2x} - e^{-2x}}{2} - \sin 2x}{4x^3}$$

$$= \operatorname*{Lt}_{x\to 0} \frac{\frac{2^3 \cdot e^{2x} - (-2)^3 e^{-2x}}{2} - 2^3 \sin\left(\frac{3\pi}{2} + 2x\right)}{4.3!}$$

($\because$ the degree of denominator is 3, we take the 3rd order derivative)

$$= \frac{\frac{8+8}{2} + 8}{4.3.2} == \frac{16}{8.3} = \frac{2}{3}$$

Example - 8 Let a, b be the distinct roots of $ax^2 + bx + c = 0$, then $\operatorname*{Lt}_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

(1) $\frac{(\alpha-\beta)^2}{2}$ (2) $-\frac{a^2}{2}(\alpha-\beta)^2$

(3) 0 (4) $\frac{a^2}{2}(\alpha-\beta)^2$

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Solution : **Ans. (4)**

$$\underset{x\to\alpha}{\text{Lt}} \frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2} \qquad (\because\ ax^2+bx+c \equiv a(x-\alpha)(x-\beta))$$

$$\left(\cos ax \simeq 1-\frac{a^2x^2}{2}\right) = \underset{x\to\alpha}{\text{Lt}} \frac{a^2(x-\alpha)^2(x-\beta)^2}{2(x-\alpha)^2} = \frac{a^2(\alpha-\beta)^2}{2}$$

Example - 9 $\underset{x\to\frac{\pi}{2}}{\text{Lt}} \frac{(1-\tan x/2)(1-\sin x)}{(1+\tan x/2)(\pi-2x)^3} =$

(1) $\frac{1}{8}$ (2) 0

(3) $\frac{1}{32}$ (4) ∞

Solution : **Ans. (3)**

$$\lim_{x\to\frac{\pi}{2}} \frac{(1-\tan x/2)(1-\sin x)}{(1+\tan x/2)(\pi-2x)^3}$$

If $x \to \frac{\pi}{2}$ then $h \to 0$ put $x = \frac{\pi}{2}-h$

$$=\lim_{h\to 0} \frac{1-\tan\left(\frac{\pi}{4}-\frac{h}{2}\right)}{1+\tan\left(\frac{\pi}{4}-\frac{h}{2}\right)} \cdot \frac{1-\cos h}{(2h)^3} = \lim_{h\to 0} \tan\left(\frac{\pi}{4}+\frac{\pi}{4}+\frac{h}{2}\right) \cdot \frac{1-\cos h}{8h^3} \qquad \left(\cos h \simeq 1-\frac{h^2}{2}\right)$$

$$=\lim_{h\to 0} \frac{\tan\frac{h}{2}\cdot\frac{h^2}{2}}{8h^3} = \lim_{h\to 0} \frac{\frac{h}{2}\cdot\frac{h^2}{2}}{8h^3} = \frac{1}{32}$$

Example - 10 $\underset{x\to 0}{\text{Lt}} \frac{\tan[e^2]x^4 - \tan[-e^2]x^4}{\sin^4 x} =$

(1) 0 (2) 15

(3) 8 (4) 7

Solution : **Ans. (2)**

$e^2 = (2.718)^2 = 7.3875$

$[e^2] = 7$, $[-e^2] = -8$ $\qquad (\because \tan ax \simeq ax,\ \sin x \simeq x)$

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \frac{\tan 7x^4 + \tan 8x^4}{\sin^4 x} = \frac{15x^4}{x^4} = 15$$

Example - 11 $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

(1) $-\frac{1}{2}$ (2) 0

(3) $\frac{1}{2}$ (4) –1

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Solution : **Ans. (1)**

Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$.

$\therefore$ $y = \sin^2\left\{\cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right)\right\} = \sin^2 \cot^{-1} (\cot \theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$.

Example - 12 If $y = \sqrt{(a-x)(x-b)}$ - $(a-b)\tan^{-1}\sqrt{\left(\frac{a-x}{x-b}\right)}$ then find $\frac{dy}{dx}$.

(1) $\sqrt{\left(\frac{x-a}{x-b}\right)}$ (2) $\sqrt{\left(\frac{a-x}{x+b}\right)}$

(3) $\sqrt{\left(\frac{a-x}{x-b}\right)}$ (4) $\sqrt{\frac{x+a}{x+b}}$

Solution : **Ans. (3)**

Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore$ $a - x = a - a\cos^2\theta - b\sin^2\theta = (a-b)\sin^2\theta$ .....(1)

and $x - b = a\cos^2\theta + b\sin^2\theta - b = (a-b)\cos^2\theta$ .....(2)

$\therefore$ $y = (a-b)\sin\theta\cos\theta - (a-b)\tan^{-1}\tan\theta$

$= \frac{(a-b)}{2}\sin 2\theta - (a-b)\theta$

Then $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{(a-b)\cos 2\theta - (a-b)}{(b-a)\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

$= \sqrt{\left(\frac{a-x}{x-b}\right)}$ [From (1) and (2)]

Example - 13 Derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t $\sqrt{1+3x}$ at $x = \frac{-1}{3}$ is-

(1) 0 (2) $\frac{1}{2}$

(3) $\frac{1}{3}$ (4) $\frac{1}{6}$

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Solution : **Ans. (1)**

Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $z = \sqrt{1+3x}$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{(2x^2-1)\cdot 1}{\sqrt{\left(\frac{1}{2x^2-1}\right)^2-1}} \cdot \frac{\frac{-4x}{(2x^2-1)^2}}{\frac{3}{2\sqrt{1+3x}}} = \frac{-4x \times \frac{2}{3}\sqrt{1+3x}}{\sqrt{\left(\frac{1}{2x^2-1}\right)^2-1}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=\frac{-1}{3}} = 0 .$$

Example - 14 If $x = \theta - \frac{1}{\theta}$ and $y = \theta + \frac{1}{\theta}$, then $\frac{dy}{dx} =$

(1) $\frac{x}{y}$ (2) $\frac{y}{x}$

(3) $\frac{-x}{y}$ (4) $\frac{-x}{y}$

Solution : **Ans. (1)**

$x = \theta - \frac{1}{\theta} \quad \Rightarrow \quad \frac{dx}{d\theta} = 1 + \frac{1}{\theta^2}$,

$y = \theta + \frac{1}{\theta} \quad \Rightarrow \quad \frac{dy}{d\theta} = 1 - \frac{1}{\theta^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{1-\frac{1}{\theta^2}}{1+\frac{1}{\theta^2}} = \frac{\theta-\frac{1}{\theta}}{\theta+\frac{1}{\theta}} = \frac{x}{y}$$

Example - 15 If $y = \frac{2}{\sqrt{(a^2-b^2)}}.\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan\left(\frac{x}{2}\right)\right\}$ then find $\frac{d^2y}{dx^2}$.

(1) $\frac{b \cos x}{(a+b\cos x)^2}$ (2) $\frac{b \sin x}{(a+b\cos x)^2}$

(3) $\frac{b \sin x}{(b+a\cos x)^2}$ (4) $-\frac{b\cos x}{(a+b\cos x)^2}$

Solution : **Ans. (2)**

We have $y = \frac{2}{\sqrt{(a^2-b^2)}}.\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan\left(\frac{x}{2}\right)\right\}$

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Let $u = \sqrt{\left(\frac{a-b}{a+b}\right)}\ \tan\left(\frac{x}{2}\right)$ ....(1)

$\therefore \quad y = \frac{2}{\sqrt{(a^2-b^2)}}.\ \tan^{-1}u$

$\therefore \quad \frac{dy}{du} = \frac{2}{\sqrt{(a^2-b^2)}}.\frac{1}{1+u^2} = \frac{2}{\sqrt{(a^2-b^2)}}.\frac{1}{1+u^2}$

$= \frac{2}{\sqrt{(a^2-b^2)}}.\left\{\frac{1}{1+\left(\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)}\right\}$ [From (1)]

$= \frac{2}{\sqrt{(a^2-b^2)}}.\frac{1}{\left\{1+\left(\frac{a-b}{a+b}\right)\left(\frac{1-\cos x}{1+\cos x}\right)\right\}}$

$= \frac{2}{\sqrt{(a^2-b^2)}}.\frac{(a+b)(1+\cos x)}{\{(a+b)(1+\cos x)+(a-b)(1-\cos x)\}}$

$= \frac{2}{\sqrt{(a^2-b^2)}}.\frac{(a+b)(1+\cos x)}{(2a+2b\cos x)}$ .....(2)

and $\frac{du}{dx} = \sqrt{\left(\frac{a-b}{a+b}\right)}\ .\ \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \sqrt{\left(\frac{a-b}{a+b}\right)}\ .\frac{1}{(1+\cos x)}$ ....(3)

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$= \sqrt{\left(\frac{a+b}{a-b}\right)}\ .\ \frac{(1+\cos x)}{(a+b\cos x)}.\sqrt{\left(\frac{a-b}{a+b}\right)}$ [from (2) and (3)]

$= \frac{1}{a+b\cos x}$

$\frac{d^2y}{dx^2} = \frac{b\ \sin\ x}{(a+b\ \cos\ x)^2}$

Example - 16 If $x = e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$, then $\frac{dy}{dx}$ equals-

(1) x [1 + tan (log x) + $\sec^2$ x] (2) 2x [1 + tan (log x)] + $\sec^2$ x

(3) 2x [1 + tan (log x)] + sec x (4) none of these

Solution : **Ans. (4)**

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Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right)$$

$\Rightarrow$ $\tan(\log x) = (y - x^2) / x^2$

$\Rightarrow$ $y = x^2 + x^2 \tan(\log x)$

$\therefore$ $\frac{dy}{dx} = 2x + 2x\tan(\log x) + x\sec^2(\log x)$

$= 2x[1 + \tan(\log x)] + x\sec^2(\log x)$

Example - 17 If $x^2 e^y + 2xye^x + 13 = 0$, then $\frac{dy}{dx}$ equals

(1) $-\frac{2xe^{y-x} + 2y(x+1)}{x(xe^{y-x}+2)}$ (2) $\frac{2xe^{x-y} + 2y(x+1)}{x(xe^{y-x}+2)}$

(3) $-\frac{2xe^{x-y} + 2y(x+1)}{x(xe^{y-x}+2)}$ (4) none of these

Solution : **Ans. (1)**

Using partial derivatives, we have

$$\frac{dy}{dx} = -\left(\frac{2xe^y + 2ye^x + 2xye^x}{x^2e^y + 2xe^x}\right)$$

$$= -\left(\frac{2xe^{y-x} + 2y + 2xy}{x^2e^{y-x} + 2x}\right)$$

$$= -\left(\frac{2xe^{y-x} + 2y\ (x+1)}{x\ (xe^{y-x}+2)}\right)$$

Example - 18 If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{........}}}$, then $\frac{dy}{dx}$ equals-

(1) $\frac{\sin x}{2y+1}$ (2) $\frac{\cos x}{2y-1}$

(3) $\frac{\cos x}{2y+1}$ (4) none of these

Solution : **Ans. (2)**

$y = \sqrt{\sin x + y}$

$\Rightarrow$ $y^2 = \sin x + y$ $\Rightarrow$ $y^2 - y - \sin x = 0$

$\therefore$ $\frac{dy}{dx} = -\frac{-\cos x}{2y-1} = \frac{\cos x}{2y-1}$

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Example - 19 If $x^2 + y^2 = t - \frac{1}{t}$, $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $\frac{dy}{dx}$ equals-

(1) $\frac{1}{x^2 y}$ (2) $\frac{1}{x y^3}$

(3) $\frac{1}{x^3 y}$ (4) $-\frac{1}{x y^3}$

Solution : **Ans. (3)**

Squaring the first equation, we have

$$x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad t^2 + \frac{1}{t^2} + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2 \qquad \text{(from second equation)}$$

$$\Rightarrow \quad x^2y^2 = -1 \quad \Rightarrow \quad y^2 = -\frac{1}{x^2}$$

$$\therefore \quad 2y\frac{dy}{dx} = \frac{2}{x^3} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^3 y}$$

Example - 20 If $y^2 = p(x)$ is a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ is equal to-

(1) $p'''(x)\,p'(x)$ (2) $p''(x)\,p'''(x)$

(3) $p(x)\,p'''(x)$ (4) none of these

Solution : **Ans. (3)**

Given $y^2 = p(x)$ ...(i)

$p'(x) = 2yy'$

$p''(x) = 2yy'' + 2y'^2$

$p'''(x) = 2yy''' + 6y'y''$ ...(ii)

Also $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = 2\frac{d}{dx}(y^3y'')$

$= 2\,[y^3y''' + 3y^2y'y'']$

$= y^2\,[2yy''' + 6y'y'']$

$= p(x)\,p'''(x)$ from (i) and (ii)

Example - 21 If $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ $\forall\, x, y \in R$ and $f'(0) = -1$, $f(0) = 1$, then $f(2) =$

(1) $\frac{1}{2}$ (2) 1

(3) -1 (4) $-\frac{1}{2}$

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Solution : **Ans. (3)**

$$f(x) = f\left(\frac{2x+0}{2}\right) = \frac{f(2x)+f(0)}{2}$$

$$\therefore \quad f'(x) = \underset{h\to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{\text{Lt}} \frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \left\{\frac{f(2x)+f(2h)}{2} - \frac{f(2x)+f(0)}{2}\right\}/h$$

$$= \underset{2h\to 0}{\text{Lt}} \frac{f(2h)-f(0)}{2h} = f'(0)$$

$\therefore \quad f'(x) = f'(0) = -1 \Rightarrow f(x) = -x + c$

$\Rightarrow \quad f(0) = c \quad \Rightarrow \quad c = 1$

$\therefore \quad f(2) = -2 + 1 = -1$

Example - 22 If f(x + y) = f(x) f(y) and f(x) = 1 + xg(x) H(x) where $\underset{x\to 0}{\text{Lt}}\, g(x) = 2$ and $\underset{x\to 0}{\text{Lt}}\, H(x) = 3$ then $f'(x) =$

(1) f(x) (2) 2f(x)
(3) 3f(x) (4) 6f(x)

Solution : **Ans. (4)**

$$f'(x) = \underset{h\to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{\text{Lt}} \frac{f(x)f(h)-f(x)}{h} = f(x)\underset{h\to 0}{\text{Lt}} \frac{f(h)-1}{h}$$

$$= f(x)\underset{h\to 0}{\text{Lt}} \frac{1+hg(h)H(h)-1}{h} = f(x)\underset{h\to 0}{\text{Lt}}\, g(h)H(h)$$

= f(x) (2 × 3) = 6f(x)

Example - 23 If f(a) = 2, $f'(a) = 1$, g(a) = – 1, $g'(a) = 2$ then the value of $\underset{x\to a}{\text{Lt}} \frac{g(x)f(a)-g(a)f(x)}{x-a}$ is

(1) – 5 (2) $\frac{1}{5}$
(3) 5 (4) 4

Solution : **Ans. (3)**

Use L.H.R

$$\underset{x\to a}{\text{Lt}} \frac{g'(x)f(a)-g(a)f'(x)}{1} = g'(a)f(a)-g(a)f'(a)$$

= 2(2) – (–1) (1) = 4 + 1 = 5

Example - 24 If $g(x) = \frac{1}{x}\int_2^x \{3t - 2g'(t)\}\, dt$ then $g'(2) =$

(1) –2/3 (2) –3/2
(3) 2/3 (4) 3/2

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Solution : **Ans. (4)**

Given $g(x) = \frac{1}{x}\int_2^x \{3t - 2g'(t)\}\, dt$

$\Rightarrow \quad x\, g(x) = \int_2^x \{3t - 2g'(t)\}\, dt$

$\Rightarrow \quad g(x) + x g'(x) = \{3x - 2g'(x)\}\,(1) - 0$

$\Rightarrow \quad g(2) + 2\, g'(2) = 6 - 2g'(2)$

$4g'(2) = 6 - g(2) = 6 - 0 = 6$

$\therefore \quad g'(2) = \frac{3}{2}$

Example - 25 Let $f : R \to R$ be such that f(1) = 3 and $f'(1) = 6$ then $\underset{x\to 0}{\text{Lt}} \left\{\frac{f(1+x)}{f(1)}\right\}^{1/x} =$

(1) 1
(2) $e^{1/2}$
(3) e^2
(4) e^3

Solution : **Ans. (3)**

Let $y = \left\{\frac{f(1+x)}{f(1)}\right\}^{1/x}$

$\Rightarrow \quad \log y = \frac{1}{x}\{\log f(1+x) - \log f(1)\}$

$\Rightarrow \quad \log\left(\underset{x\to 0}{\text{Lt}}\,(y)\right) = \underset{x\to 0}{\text{Lt}}\, \frac{f'(1+x)}{f(1+x)} = \frac{f'(1)}{f(1)} = \frac{6}{3} = 2$

$\therefore \quad \underset{x\to 0}{\text{Lt}}\,(y) = e^2$

EXERCISES

LEVEL - I

1. $\underset{x\to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}} =$

(1) $\frac{1}{e^2}$ (2) $\frac{1}{e}$

(3) e^2 (4) e

2. $\underset{n\to\infty}{\text{Lt}}\frac{1+\frac{1}{2}+\frac{1}{2^2}+....+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+.....+\frac{1}{3^n}} =$

(1) $\frac{4}{3}$ (2) $\frac{2}{3}$

(3) $\frac{1}{3}$ (4) $\frac{1}{2}$

3. $\underset{h\to 0}{\text{Lt}}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h} =$

(1) $a^2\cos a + 2a\sin a$ (2) $a(\cos a + 2\sin a)$

(3) $a^2(\cos a + 2\sin a)$ (4) 0

4. If α, β be the roots of $ax^2 + bx + c = 0$, then $\underset{x\to\alpha}{\text{Lt}}(1+ax^2+bx+c)^{\frac{1}{x-\alpha}}$ is

(1) $a(\alpha-\beta)$ (2) $\log|a(\alpha-\beta)|$

(3) $e^{a(\alpha-\beta)}$ (4) $e^{a(\alpha+\beta)}$

5. $\underset{x\to 0}{\text{Lt}}\frac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}} = 2$ Then

(1) a = 1, b = 2 (2) a = 1, $b = -3^{1/3}$

(3) a = 1, b = –1/2 (4) a = 2, $b \in R$

6. $\underset{n\to\infty}{\text{Lt}}\frac{\sin n\theta}{\sqrt{n}} =$

(1) 0 (2) ∞

(3) 1 (4) $\sqrt{n}$

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7. $\underset{n\to\infty}{\text{Lt}}\left\{\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{4}\right)\cdot\cos\left(\frac{x}{8}\right)\cdots\cos\left(\frac{x}{2^n}\right)\right\} =$

(1) 1 (2) $\frac{\sin x}{x}$

(3) $\frac{x}{\sin x}$ (4) –1

8. $\underset{n\to\infty}{\text{Lt}}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+.....+\frac{n}{1-n^2}\right)=$

(1) 0 (2) $-\frac{1}{2}$

(3) $\frac{1}{2}$ (4) $\frac{1}{3}$

9. If $\underset{x\to 0}{\text{Lt}}\frac{\sin 2x + a\sin x}{x^3}$ be finite, then the value of ‘a’ and the limit are given by

(1) –2, 1 (2) – 2, –1

(3) 2, 1 (4) 2, –1

10. $\underset{x\to 0}{\text{Lt}}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x} =$

(1) $\frac{1}{6}$ (2) $\frac{1}{3}$

(3) $-\frac{1}{12}$ (4) $-\frac{1}{14}$

11. $\underset{x\to 1}{\text{Lt}}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2} =$

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$

(3) $-\frac{1}{4}$ (4) $\frac{1}{3}$

12. If [x] denotes the greatest integer $\le x$, then $\underset{n\to\infty}{\text{Lt}}\frac{\{[1^2x]+[2^2x]+....[n^2x]\}}{n^3}$

(1) $\frac{x}{2}$ (2) $\frac{x}{3}$

(3) $\frac{x}{6}$ (4) 0

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13. $\frac{d}{dx}\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=$

(1) $\frac{1}{1+x^2}$

(2) $\frac{-1}{1+x^2}$

(3) $\frac{2}{1+x^2}$

(4) $\frac{-2}{1+x^2}$

14. If $y=e^{x+e^{x+e^{x+.....\infty}}}$, then $\frac{dy}{dx}=$

(1) $\frac{y}{1-y}$

(2) $\frac{1}{1-y}$

(3) $\frac{y}{1+y}$

(4) $\frac{y}{y-1}$

15. If $y = 1 + x + \frac{x^2}{2!}+\frac{x^3}{3!} + ... \infty$, then $\frac{dy}{dx}=$

(1) y

(2) y – 1

(3) y + 1

(4) none of these

16. Differential coefficient of $\sec^{-1}\frac{1}{2x^2-1}$ w.r.t. $\sqrt{1-x^2}$ at $x=\frac{1}{2}$ is-

(1) 2

(2) 4

(3) 6

(4) 1

17. If $y=\left(1+\frac{1}{x}\right)^x$, then $\frac{dy}{dx}=$

(1) $\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$

(2) $\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)\right]$

(3) $\left(x+\frac{1}{x}\right)^x\left[\log(x-1)-\frac{x}{x+1}\right]$

(4) $\left(x+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)+\frac{1}{1+x}\right]$

18. If $f'(x) = \sin(\log x)$ and $y=f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}=$

(1) $\frac{9\cos(\log x)}{x(3-2x)^2}$

(2) $\frac{9\cos\left(\log\frac{2x+3}{3-2x}\right)}{x(3-2x)^2}$

(3) $\frac{9\sin\left(\log\frac{2x+3}{3-2x^2}\right)}{(3-2x)^2}$

(4) $\frac{12}{(3-2x)^2}\sin\left(\log\left(\frac{2x+3}{3-2x}\right)\right)$

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19. If $y = \cot^{-1} (\cos 2x)^{1/2}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$ will be-

(1) $\left(\frac{2}{3}\right)^{1/2}$ (2) $\left(\frac{1}{3}\right)^{1/2}$

(3) $(3)^{1/2}$ (4) $(6)^{1/2}$

20. If $y = \log_{\cos x} \sin x$, then $\frac{dy}{dx}$ is equal to-

(1) $\frac{\cot x \log \cos x + \tan x \log \sin x}{(\log \cos x)^2}$ (2) $\frac{\tan x \log \cos x + \cot x \log \sin x}{(\log \cos x)^2}$

(3) $\frac{\cot x \log \cos x + \tan x \log \sin x}{(\log \sin x)^2}$ (4) $\frac{\cot x}{(\log \sin x)^2}$

21. If $y = \sin^n x \cos nx$, then $\frac{dy}{dx}$ equals

(1) $n \sin^{n-1} x \cos(n+1)x$ (2) $n \sin^{n-1} x \sin(n+1)x$

(3) $n \sin^{n-1} x \cos(n-1)x$ (4) $n \sin^{n-1} x \cos nx$

22. If $f(x) = \cos x \cos 2x \cos 4x \cos 8x \cos 16x$ then $f'\left(\frac{\pi}{4}\right)$ is

(1) $\sqrt{2}$ (2) $\frac{1}{\sqrt{2}}$

(3) 1 (4) None of these

23. The value of the derivative of $|x - 2| + |x - 3|$ at $x = 3$ is

(1) 2 (2) –2

(3) 0 (4) 1

24. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is constant, then $\frac{d^3}{dx^3}(f(x))$ at x = 0 is

(1) p (2) $p + p^2$

(3) $p + p^3$ (4) Independent of p

25. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$ then $f^2(19) + g^2(19)$ is

(1) 16 (2) 32

(3) 64 (4) 8

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LEVEL - II

1. $\underset{x\to a}{\text{Lt}} \dfrac{x}{x-a}\int_a^x f(x)\,dx =$

(1) 2 f(a) (2) f(a)
(3) a f(a) (4) 0

2. $\underset{n\to\infty}{\text{Lt}} \left(\dfrac{1}{n^3+1}+\dfrac{4}{n^3+1}+\dfrac{9}{n^3+1}+....+\dfrac{n^2}{n^3+1}\right)=$

(1) 1 (2) 2/3
(3) 1/3 (4) 0

3. $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1+\{x\})}{\{x\}} =$ (where {x} denotes the fractional part of x)

(1) 1 (2) 0
(3) 2 (4) does not exist

4. $\underset{x\to 0}{\text{Lt}} \dfrac{a^{\sqrt{x}}-a^{1/\sqrt{x}}}{a^{\sqrt{x}}+a^{1/\sqrt{x}}}$, a > 1 is

(1) 4 (2) 2
(3) –1 (4) 0

5. $\underset{n\to\infty}{\text{Lt}} \dfrac{a^n+b^n}{a^n-b^n}$, where a > b > 1, is equal to

(1) –1 (2) 1
(3) 0 (4) none of these

6. $\underset{x\to 0}{\text{Lt}} \left\{1^{1/\sin^2 x}+2^{1/\sin^2 x}+....+n^{1/\sin^2 x}\right\}^{\sin^2 x}$ is

(1) ∞ (2) 0
(3) $\dfrac{n(n+1)}{2}$ (4) n

7. If $G(x)=-\sqrt{25-x^2}$, then $\underset{x\to 1}{\text{Lt}} \dfrac{G(x)-G(1)}{x-1}$ is

(1) $\dfrac{1}{24}$ (2) $\dfrac{1}{5}$
(3) $-\sqrt{24}$ (4) $\dfrac{1}{\sqrt{24}}$

8. If f(9) = 9, $f'(9) = 4$, then $\underset{x\to 9}{\text{Lt}} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3} =$

(1) 3 (2) 4

(3) 6 (4) 8

9. $\underset{x\to 0}{\text{Lt}} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ is

(1) 2 (2) –2

(3) 1/2 (4) –1/2

10. $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{\sin x - (\sin x)^{\sin x}}{1-\sin x + \log(\sin x)} =$

(1) 1 (2) 2

(3) 3 (4) 4

11. Given $f'(2) = 6$, and $f'(1) = 4$, $\underset{h\to 0}{\text{Lt}} \frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$ is

(1) 3/2 (2) 3

(3) 5/2 (4) –3

12. Let g(x) = $\sin^2[\pi^3]x^2 - \sin^2[-\pi^3]x^2$, [.] represents greatest integer function then $\underset{x\to 0}{\text{Lt}} \frac{g(x)}{\sin^4 x} =$

(1) –63 (2) 63

(3) 1 (4) –1

13. If y = x |x| then $\frac{dy}{dx} =$

(1) |x| (2) $\frac{|x|}{x}$

(3) x|x| (4) 2 |x|

14. If $2^x + 2^y = 2^{x+y}$ then $\frac{dy}{dx} =$

(1) 2^{x-y} (2) 2^{y-x}

(3) -2^{x-y} (4) -2^{y-x}

15. If $x^3y^2 = (x+y)^5$ then $\frac{dy}{dx} =$

(1) $\frac{x}{y}$ (2) $\frac{y}{x}$

(3) $-\frac{x}{y}$ (4) logx

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16. $\frac{d}{dx}\left\{(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)\right\}=$

(1) $\frac{15x^{16}-16x^{15}a+a^{16}}{(x-a)^2}$

(2) $\frac{x^{16}-x^{15}a+a^{16}}{(x-a)^2}$

(3) $\frac{x^{16}-a^{16}}{x-a}$

(4) $16x^{15}-a^{16}$

17. If $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$ then $f'(x)=$

(1) 0

(2) 1

(3) x^2

(4) $6x^2$

18. If f(x) = $\cos^2 x + \cos^3\left(\frac{\pi}{3}+x\right) - \sin x \sin\left(x+\frac{\pi}{3}\right)$ and γ (5/4) = 3 then (gof)(x) =

(1) 0

(2) 1

(3) $\cos x + \cos\left(\frac{\pi}{3}+x\right)$

(4) 3

19. Let f(x) be a polynomial function of second degree if f(1) = f(–1) and a, b, c are in A.P then $f'(a), f'(b), f'(c)$ are in

(1) A.P

(2) G.P

(3) H.P

(4) A.G.P

20. If $\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2^2}\right)\cdot\cos\left(\frac{x}{2^3}\right)$.........to $\infty = \frac{\sin x}{x}$ then

(1) $\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{2^2}\tan\left(\frac{x}{2^2}\right)+\frac{1}{2^3}\tan\left(\frac{x}{2^3}\right)+......\infty = -\cot x + \frac{1}{x}$

(2) $\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{2^2}\tan\left(\frac{x}{2^2}\right)+\frac{1}{2^3}\tan\left(\frac{x}{2^3}\right)+......\infty = \cot x - \frac{1}{x}$

(3) $\frac{1}{2^2}\sec^2\left(\frac{x}{2}\right)+\frac{1}{2^4}\sec^2\left(\frac{x}{2^2}\right)+......+\infty = \operatorname{cosec}^2 x - \frac{1}{x^2}$

(4) $\frac{1}{2^2}\sec^2\left(\frac{x}{2}\right)+\frac{1}{2^4}\sec^2\left(\frac{x}{2^2}\right)+......\infty = \operatorname{cosec}^2 x + \frac{1}{x^2} + \frac{1}{x^3}$

21. A triangle has two of its vertices at P(a, 0), Q(0, b) and the third vertex R(x, y) is moving along the st.line y = x, if A be the area of the Δ, then $\frac{dA}{dx}=$

(1) $\frac{a-b}{2}$

(2) $\frac{a-b}{4}$

(3) $\frac{a+b}{2}$

(4) $\frac{a+b}{4}$

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22. If $f'(x) = \phi(x)$ and $\phi'(x) = f(x)\ \forall x$, also f(3) = 5 and $f'(3) = 4$, then the value of $[f(10)]^2 - [\phi(10)]^2 =$

(1) 0
(2) 9
(3) 41
(4) 25

23. If $f(x) = \sin\left(\frac{\pi}{2}[x] - x^5\right)$, 1 < x < 2 and [x] denotes the greatest integer less than or equal to x, then $f'\left(\sqrt[5]{\frac{\pi}{2}}\right) =$

(1) $5\left(\frac{\pi}{2}\right)^{4/5}$
(2) $-5\left(\frac{\pi}{2}\right)^{4/5}$
(3) 0
(4) $3\left(\frac{\pi}{2}\right)^{4/5}$

24 If $y = \tan^{-1}\left(\frac{a\sin x + b\cos x}{a\cos x - b\sin x}\right)$ then $\frac{dy}{dx} =$

(1) 1
(2) –1
(3) 0
(4) $\frac{a}{a\cos x - b\sin x}$

25. If $y = \cfrac{x}{a + \cfrac{x}{b + \cfrac{x}{a + \cfrac{x}{b + \ldots\ldots}}}}$ then $\frac{dy}{dx} =$

(1) $\frac{b}{a(b+2y)}$
(2) $\frac{b}{b+2y}$
(3) $\frac{a}{b(b+2y)}$
(4) $\frac{ab}{a+b2y}$

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LEVEL - III

1. $\underset{x\to\infty}{\text{Lt}}\left\{x\cos\left(\frac{\pi}{4x}\right)\sin\left(\frac{\pi}{4x}\right)\right\}=$

(1) $\frac{1}{\pi}$ (2) $\frac{\pi}{4}$

(3) 1 (4) π

2. $\underset{x\to\infty}{\text{Lt}}\frac{[x]}{x}=$

(1) 1 (2) 3

(3) –1 (4) does not exist

3. $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\frac{\left(\frac{\pi}{2}-x\right)}{\sec x-\tan x}=$

(1) 1 (2) –1

(3) 2 (4) 3

4. $\underset{x\to 0}{\text{Lt}}\frac{xe^{3x}-x}{\sqrt{1+x^2}-1}=$

(1) 3 (2) 6

(3) 4 (4) 1

5. $\underset{x\to 0}{\text{Lt}}\frac{e^{|x|}-1}{x}=$

(1) 1 (2) –1

(3) does not exist (4) 0

6. $\underset{x\to\infty}{\text{Lt}}\left\{\frac{1^{1/x}+2^{1/x}+....+n^{1/x}}{n}\right\}^{nx}$ is

(1) $\underline{|n}$ (2) n

(3) $\underline{|n-1}$ (4) 0

7. $\underset{x\to\infty}{\text{Lt}}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]=$

(1) $\frac{1}{2}$ (2) 1

(3) 0 (4) does not exist

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8. $\underset{x \to 2}{\text{Lt}} \dfrac{2^x + 2^{3-x} - 6}{2^{-x/2} - 2^{1-x}} =$

(1) 8 (2) 4
(3) 2 (4) 6

9. If $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$ then $\underset{n \to \infty}{\text{Lt}} a_n =$

(1) 0 (2) –2
(3) $\sqrt{2}$ (4) $-\sqrt{2}$

10. If $f(x) = x\left\{\dfrac{1}{1+x} + \dfrac{1}{(1+x)(1+2x)} + ... + \dfrac{1}{(1+2x)(1+3x)} + ...n\text{terms}\right\}$ x > 0, then $\underset{n \to \infty}{\text{Lt}} f(x) =$

(1) $\dfrac{1}{1-x}$ (2) $\dfrac{1}{1+x}$
(3) 1 (4) 0

11. The value of $\lim\limits_{x \to -\infty} \left[\dfrac{x^4 \sin(1/x) + x^2}{1+|x|^3}\right]$ is

(1) 1 (2) –1
(3) 0 (4) ∞

12. The value of $\lim\limits_{x \to \frac{\pi}{2}} \left[1^{1/\cos^2 x} + 2^{1/\cos^2 x} + ... + n^{1/\cos^2 x}\right]^{\cos^2 x}$ is

(1) 0 (2) n
(3) ∞ (4) $\dfrac{n(n+1)}{2}$

13. If siny = xcos(a + y) then $\dfrac{dy}{dx} =$

(1) $\dfrac{\sin^2(a+y)}{\sin a}$ (2) $\dfrac{\cos^2(a+y)}{\cos a}$

(3) $\dfrac{\cos^2(a-y)}{\cos a}$ (4) $\dfrac{\cos^2(a+y)}{\sin a}$

14. If $\sin^2 mx + \cos^2 ny = a^2$ then $\dfrac{dy}{dx} =$

(1) $\dfrac{m\sin 2mx}{n\sin 2ny}$ (2) $\dfrac{n\sin 2ny}{m\sin 2mx}$

(3) $\dfrac{n\sin 2mx}{m\sin 2ny}$ (4) $-\dfrac{m\sin 2mx}{n\sin 2ny}$

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15. If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$ then $f'(1) =$

(1) –1 (2) 1
(3) log2 (4) –log2

16. If $y = \log_{\cot x} \tan x \cdot \log_{\tan x} \cot x + \tan^{-1}\left(\frac{4x}{4 - x^2}\right)$ then $\frac{dy}{dx} =$

(1) $\frac{1}{4+x^2}$ (2) $\frac{4}{4+x^2}$

(3) $\frac{1}{4-x^2}$ (4) $\frac{4}{4-x^2}$

17. If 3f(cosx) + 2f(sinx) = 5x then $f'(\cos x) =$

(1) $-\frac{5}{\cos x}$ (2) $\frac{5}{\cos x}$

(3) $-\frac{5}{\sin x}$ (4) $\frac{5}{\sin x}$

18. If f(x) = (cosx + sinx) (cos3x + isin3x) .... (cos[(2n – 1)x] + isin(2n – 1)x] then $f''(x) =$

(1) $n^2 f(x)$ (2) $-n^4 f(x)$
(3) $-n^2 f(x)$ (4) $n^3 f(x)$

19. Let $\phi(x)$ be the inverse of the function f(x) and $f'(x) = \frac{1}{1+x^5}$ then $\frac{d}{dx}[\phi(x)] =$

(1) $\frac{1}{1+(\phi(x))^5}$ (2) $\frac{1}{1+(f(x))^5}$

(3) $1+(\phi(x))^5$ (4) $1+(f(x))^5$

20. If $\sqrt{x^2 + y^2} = ae^{\tan^{-1}(y/x)}, a > 0$ then $y''(0)$ is

(1) $\frac{a}{2}e^{-\pi/2}$ (2) $ae^{\pi/2}$

(3) $-\frac{2}{a}e^{-\pi/2}$ (4) $\frac{2}{a}e^{\pi/2}$

21. If y = a cos(logx) + b sin(logx) then $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} =$

(1) 0 (2) y
(3) –y (4) 1

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22. If siny = x sin(a + y) and $\frac{dy}{dx} = \frac{A}{1 + x^2 - 2x\cos a}$ then the value of A is

(1) 2 (2) cosa

(3) sina (4) –2

23. If $y = \frac{ax + b}{x^2 + c}$, where a, b, c are constants then $(2xy' + y)y'''$ is equal to

(1) $3(xy'' + y')y''$ (2) $3(xy' + y'')y''$

(3) $3(xy'' + y')y'$ (4) none of these

24. If $\sqrt{1 - x^6} + \sqrt{1 - y^6} = a^3(x^3 - y^3)$, then $\frac{dy}{dx}$ is equal to

(1) $\frac{x^2}{y^2}\sqrt{\frac{1 - y^6}{1 - x^6}}$ (2) $\frac{y^2}{x^2}\sqrt{\frac{1 - y^6}{1 - x^6}}$

(3) $\frac{x^2}{y^2}\sqrt{\frac{1 - x^6}{1 - y^6}}$ (4) none of these

25. Let $f(x) = \log\left\{\frac{u(x)}{v(x)}\right\}$, $u'(2) = 4$, $v'(2) = 2$, u(2) = 2, v(2) = 1, then $f'(2)$ is

(1) 0 (2) 1

(3) -1 (4) 2

QUESTIONS ASKED IN AIEEE & OTHER ENGINEERING EXAMS

1. If $x^m \cdot y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is

(1) $\frac{y}{x}$ (2) $\frac{x+y}{xy}$

(3) xy (4) $\frac{x}{y}$ **[AIEEE - 2006]**

2. $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+....+\frac{1}{n}\sec^2 1\right]$ is

(1) $\frac{1}{2}\sec 1$ (2) $\frac{1}{2}\operatorname{cosec} 1$

(3) $\tan 1$ (4) $\frac{1}{2}\tan 1$ **[AIEEE - 2005]**

3. Let α and β be the distinct roots of $ax^2 + bx + c = 0$ then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-a)^2}$ is equal to

(1) $\frac{a^2}{2}(\alpha-\beta)^2$ (2) 0

(3) $\frac{-a^2}{2}(\alpha-\beta)^2$ (4) $\frac{1}{2}(\alpha-\beta)^2$ **[AIEEE - 2005]**

4. If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b, are

(1) $a \in R, b = 2$ (2) $a = 1, b \in R$

(3) $a \in R, b \in R$ (4) a = 1 and b = 2. **[AIEEE - 2004]**

5. $\lim_{x\to\pi/2}\frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is

(1) $\frac{1}{8}$ (2) 0

(3) $\frac{1}{32}$ (4) ∞ **[AIEEE - 2003]**

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6. If $\lim_{x\to 0} \frac{\log(3+x)-\log(3-x)}{x} = k$, the value of k is

(1) 0
(2) $-\frac{1}{3}$
(3) $\frac{2}{3}$
(4) $-\frac{2}{3}$ **[AIEEE - 2003]**

7. Let f(a) = g(a) = k and their nth derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if $\lim_{x\to a} \frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)} = 4$, then the value of k is

(1) 4
(2) 2
(3) 1
(4) 0 **[AIEEE - 2003]**

8. If $f(x) = x^n$, then the value of $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + ... + \frac{(-1)^n f^n(1)}{n!}$ is

(1) 2^n
(2) 2^{n-1}
(3) 0
(4) 1 **[AIEEE - 2003]**

9. If $x = e^{y+e^{y+....to\ \infty}}$, x > 0 then $\frac{dy}{dx}$ is

(1) $\frac{1-x}{x}$
(2) $\frac{1}{x}$
(3) $\frac{x}{1+x}$
(4) $\frac{1+x}{x}$ **[AIEEE - 2003]**

10. $\lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{\sqrt{2}\,x}$ is

(1) λ
(2) –1
(3) zero
(4) does not exist **[AIEEE - 2002]**

11. $\lim_{x\to\infty} \left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$ is

(1) e^4
(2) e^2
(3) e^3
(4) e **[AIEEE - 2002]**

12. For $x \in R$, $\lim_{x\to\infty} \left(\frac{x-3}{x+2}\right)^x$ is

(1) e
(2) e^{-1}
(3) e^{-5}
(4) e^5 **[AIEEE - 2002]**

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13. Let f(2) = 4 and $f'(2) = 4$. Then $\lim_{x\to 2} \frac{xf(2) - 2f(x)}{x-2}$ is given by

(1) 2
(2) –2
(3) –4
(4) 3 **[AIEEE - 2002]**

14. If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is

(1) n^2y
(2) $-n^2y$
(3) –y
(4) $2x^2y$ **[AIEEE - 2002]**

15. If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}$ is

(1) $\frac{\sin a}{\sin a \sin^2(a+y)}$
(2) $\frac{\sin^2(a+y)}{\sin a}$
(3) $\sin a \sin^2(a+y)$
(4) $\frac{\sin^2(a-y)}{\sin a}$ **[AIEEE - 2002]**

16. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is

(1) $\frac{1+x}{1+\log x}$
(2) $\frac{1-\log x}{1+\log x}$
(3) not defined
(4) $\frac{\log x}{(1+\log x)^2}$ **[AIEEE - 2002]**

17. $\lim_{x\to 0}\left[\frac{e^x - e^{\sin x}}{x - \sin x}\right]$ is equal to

(1) – 1
(2) 0
(3) 1
(4) none of these **[UPSEAT - 2004]**

18. $\lim_{x\to 0}\frac{\cos 2x^3 - 1}{\sin^6 2x}$ is equal to

(1) $\frac{1}{16}$
(2) $-\frac{1}{16}$ **[CEET (Haryana) - 2004]**
(3) $\frac{1}{32}$
(4) $-\frac{1}{32}$

19. $\lim_{x\to\infty}\left(1 - \frac{4}{x-1}\right)^{3x-1}$ is equal to:

(1) e^{12}
(2) e^{-12}
(3) e^4
(4) e^3 **[CET (Karnataka) - 2004]**

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20. $\lim_{n\to\infty}\left(1+\sin\frac{a}{n}\right)^n$ equals to:

(1) e^a (2) e **[CEE (Delhi) - 2004]**
(3) e^{2a} (4) 0

21. The differential coefficient of f(sinx) with respect to x where f(x) = logx is

(1) tanx (2) cotx
(3) f(cosx) (4) $\frac{1}{x}$ **[CET (Karnataka) - 2004]**

22. If x = A cos 4t + B sin4t, then $\frac{d^2x}{dt^2}$ is equal to

(1) –16x (2) 16x
(3) x (4) –x **[CET (Karnataka) - 2004]**

23. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are given, then

(1) $\Delta_1 = 3(\Delta_2)^2$ (2) $\frac{d}{dx}\Delta_1 = 3\Delta_2$
(3) $\frac{d}{dx}\Delta_1 = 3(\Delta_2)^2$ (4) $\Delta_1 = 3(\Delta_2)^{3/2}$ **[UPSEAT - 2000]**

24. If $y = \sqrt{x\log_e x}$ then $\frac{dy}{dx}$ at x = e is

(1) $\frac{1}{e}$ (2) $\frac{1}{\sqrt{e}}$
(3) $\sqrt{e}$ (4) none of these **[BIT (Mesra) - 2000]**

25. If $y = \tan^{-1}\left[\frac{\log(e/x^2)}{\log(ex^2)}\right] + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then $\frac{d^2y}{dx^2} =$

(1) 2 (2) 1
(3) 0 (4) –1 **[CEE (Delhi) - 2004]**

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ANSWERS

EXERCISES

LEVEL - I

1. (2)	2. (1)	3. (1)	4. (3)	5. (2)
6. (1)	7. (2)	8. (2)	9. (2)	10. (3)
11. (1)	12. (2)	13. (4)	14. (1)	15. (1)
16. (1)	17. (1)	18. (4)	19. (1)	20. (1)
21. (1)	22. (1)	23. (3)	24. (4)	25. (2)

LEVEL - II

1. (3)	2. (3)	3. (4)	4. (3)	5. (2)
6. (3)	7. (4)	8. (2)	9. (3)	10. (2)
11. (2)	12. (2)	13. (4)	14. (4)	15. (2)
16. (1)	17. (4)	18. (4)	19. (1)	20. (1)
21. (3)	22. (2)	23. (2)	24. (1)	25. (1)

LEVEL - III

1. (2)	2. (1)	3. (3)	4. (2)	5. (3)
6. (1)	7. (1)	8. (1)	9. (3)	10. (3)
11. (2)	12. (2)	13. (2)	14. (1)	15. (1)
16. (2)	17. (3)	18. (2)	19. (3)	20. (3)
21. (3)	22. (3)	23. (1)	24. (1)	25. (2)

QUESTIONS ASKED IN AIEEE & OTHER ENGINEERING EXAMS

1. (2)	2. (4)	3. (1)	4. (2)	5. (3)
6. (3)	7. (1)	8. (3)	9. (1)	10. (4)
11. (1)	12. (3)	13. (3)	14. (1)	15. (2)
16. (4)	17. (3)	18. (4)	19. (2)	20. (1)
21. (2)	22. (1)	23. (2)	24. (2)	25. (3)

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